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E. Taflin

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We prove that the *KdV* equation is linearized by an analytic function, which is projectively analytically invertible. The Cauchy problem for the *KdV* equation is entirely solved by this fact. The non-linear superposition principle is a trivial consequence of convexity for the image of the linearization operator.

1. Introduction. Since the discovery [7] of the inverse scattering formalism for the *KdV* equation

$$\frac{\partial}{\partial t}u(t,x)+\frac{\partial^3}{\partial x^3}u(t,x)-6u(t,x)\frac{\partial}{\partial x}u(tx)=0, \tag{1.1}$$

$$t,x,u(t,x)\in\mathbf{R},$$

it is known, given a certain class of solution for the linear equation

$$\frac{\partial}{\partial t}v(t,x)+\frac{\partial^3}{\partial x^3}v(t,x)=0$$

how to construct solutions of equation (1.1). However, it is not clear how this reduces the Cauchy problem for the *KdV* equation, on a given space of initial conditions, into that of the above linear equation. The Cauchy problem for (1.1) has been solved by direct functional analysis methods on the Sobolev space H^3, ([**11**]), and on H^2, ([**1**]). The inverse scattering formalism has been used to solve the Cauchy problem on $S(\mathbf{R})$, ([**12**]) and for sufficiently rapidly decreasing C^3 initial conditions ([**3**]), where in both cases, two linear problems are associated with the *KdV* equation.

To formulate the problem of linearization of the non linear Cauchy problem, it is convenient to give topological vectorspaces of initial conditions and of solutions for the non linear resp. for the linear problem. In this context it is possible to give a precise meaning to the concept of linearization (see [**5**]). What we want to show in this paper is that the linearization program defined in [**6**], entirely goes through and solves the initial value problem for the *KdV* equation. We stress the fact that this approach is straightforward (in contrast to the inverse scattering formalism), when the spaces of initial conditions are given. The inverse scattering formalism is recovered (c.f. [**8**]). This is not surprising as, with the particular choice of spaces we have done, the linearization is unique (for fixed first term).

The choice of spaces is of course important for the properties of the linearization mapping (if it exists). It is easy to illustrate this fact for the *KdV* equation. Let E_1 and E_2 be two TVS of functions f: $\mathbf{R} \to \mathbf{R}$ and F: $E_1 \to E_2$ a C^1, one to one map, which linearizes the C^0 vector fields on E_1:

$$X_0(u) = \partial u, \qquad X_1(u) = -\partial^3 u + 6u\partial u,$$

$$[\partial u](x) = \frac{\partial}{\partial x} u(x), \qquad u \in E_1,$$

i.e. F is a translation invariant linearization of the *Kdv* equation. Denoting D the Frechet derivative, this means that

$$DF.X_0 = X_0^1 \circ F, \qquad DF.X_1 = X_1^1 \circ F,$$

where $X_1^1(u) = -\partial^3 u$ and $X_0^1(u) = X_0(u)$. Let $u_0 \neq 0$ be the initial condition for a soliton, i.e.

$$kX_0(u_0) = X_1(u_0) \quad \text{for some } k < 0.$$

Then

$$0 = DF.(kX_0 - X_1)(u_0) = \left(kX_0^1 - X_1^1\right) \circ F(u_0)$$

i.e. $(k\partial + \partial^3)v_0 = 0$ for $v_0 = F(u_0)$

The non-constant solutions of this equation in $\mathcal{D}$ are exponentials. Thus if F is one to one and is defined on the solitons then E_2 has to contain exponentials. Further if F^{-1} exists as a C^1 function on some open set $O \ni 0$ in E_2 then $DF(0)$ and $DF(0)^{-1}$ are continuous linear mappings. So E_1 and E_2 are topologically isomorphic. One can then as well choose $E_1 = E_2 = E$. Suppose that $C^\infty(\mathbf{R}) \supset E \supset S(\mathbf{R})$ (the Schwartz space of test functions) and that E contains at least the exponentials e^{ax}, $a > 0$. If F is C^∞, one finds by calculating (in a very formal way) some of the derivatives $D^n(F^{-1})$, (see [**8**] for explicit formulas) that the operator $f \to \partial f$ has to be invertible on E, i.e. $f(x) \to 0$ fast as $x \to -\infty$.

These qualitative remarks show that it is a quite natural choice to treat the linearization problem for the *KdV* equation on a space like the TVS of all functions $f \in C^\infty(\mathbf{R})$ for which the seminorms

$$\|f\|_N = \sup_{\substack{x \in]-\infty, N] \\ 0 \le k \le N}} \left|(1 + |x|)^N \partial^k f(x)\right|, \qquad N = 0, 1, \ldots,$$

are finite. On this space, denoted S_b, we prove (Corollary 3.3 and Proposition 3.4) that there is a unique translation invariant formal linearization A (taking the non linear equation into the linear) of the *KdV* equation (in the sense of [5]), i.e. the linearization operator exists as a formal power series where each term is finite and commute with the space translations. A converges to an entire analytic function $\hat{A}$: $S_b \to S_b$ (Corollary 4.2). The power series A^{-1} defines a C^∞ mapping $\hat{A}^{-1}$: $\hat{A}[S_b] \to S_b$, which is "projectively analytic" (Corollary 4.4 and Proposition 4.6). The set $\hat{A}[S_b]$ is convex, which gives the non-linear superposition principle for solutions of the *KdV* equation. The Cauchy problem for the *KdV* equation can now be solved entirely by linearization (Proposition 5.2 and Remark 5.3).

2. Formal linearization. Let us first introduce some necessary notation. Given two *TVS*, X and Y. Denote $F(X, Y)$ (resp. $F_s(X, Y)$) the space of formal power series from X to Y of the form $f = \sum_{n\geq 1} f^n$, where $f^n \in \mathcal{L}_n(X, Y)$ (resp. $\mathcal{L}_n^s(X, Y)$), the space of n-linear continuous (resp. continuous symmetric) mappings from X to Y. $f_s \in F_s(X, Y)$ will denote the symmetrization of $f \in F(X, Y)$. If X, Y and Z are *TVS* then the product

$$F_s(Y, Z) \times F_s(X, Y) \ni (A, B) \mapsto A*B \in F_s(X, Z)$$

is defined by (see [5]):

$$A * B = \sum_{n\geq 1} \left(\sum_{1\leq p\leq n} A^p \left(\sum_{0\leq q\leq p-1} I_q \otimes B^{n-p+1} \otimes I_{p-q-1} \right) \sigma_n \right)$$

where I_q is the identity mapping on $X \hat{\otimes}_s \cdots \hat{\otimes}_s X$ (q-times) and σ_n is the normalized symmetrization mapping on $\hat{\otimes}^n X$. $\hat{\otimes}_s$ is the symmetric projective tensor product. $F(X, X)$ is denoted $F(X)$, etc. If $A, B \in F_s(X)$ then the bracket $[A, B]_* = A * B - B * A$ is defined.

REMARK 2.1. If $A, B \in F_s(X)$ define entire functions $\hat{A}$, $\hat{B}$: $X \to X$, where

$$\hat{A}(u) = \sum_{n=1}^{\infty} \hat{A}^n(u), \qquad \hat{A}^n(u) = A^n(u,\ldots,u),$$

then

$$D\hat{A}.\hat{B} = \widehat{A * B} \text{ and } [\hat{A}, \hat{B}] = \widehat{[A, B]}_*.$$

Here D is the Frechet derivative and $[\hat{A}, \hat{B}] = D\hat{A}.\hat{B} - D\hat{B}.\hat{A}$ the usual vector field bracket.

We denote by S_b the Frechet space of all C^∞ functions f from $\mathbf{R}$ to $\mathbf{R}$ for which $\|f\|_N < \infty$, $N = 0, 1, \ldots$. The seminorms in S_b are given by

$$\|f\|_N = \sup_{\substack{x\in]-\infty,N] \\ 0\le k\le N}} \left|(1+|x|)^N \partial^k f(x)\right|, \qquad [\partial f](x) = \frac{\partial}{\partial x} f(x).$$

S_b is the projective limit of the spaces $S(n)$, $n = 0, 1, \ldots$ where $S(n)$ is the subset of all $f \in C^\infty(]-\infty, n])$ such that

$$\sup_{\substack{x\in]-\infty,n] \\ 0\le \alpha\le k}} \left|(1+|x|)^k \partial^\alpha f(x)\right| < \infty, \qquad k = 0, 1, \ldots,$$

PROPOSITION 2.1. *The Frechet space S_b is a nuclear Montel space.*

Proof. S_b is barreled as it is Frechet. Let $B \subset S_b$ be a bounded closed subset, i.e. $\|B\|_N = \sup_{f\in B} \|f\|_N < \infty$ for each $N \ge 0$. It follows from the Ascoli-Arzela theorem that if $\|B\|_{N+1} < \infty$, then for each sequence $\{f_n\}_{n=0}^\infty$ in B there exists a subsequence $\{f_{i_n}\}_{n=0}^\infty$ such that f_{i_n} converges in the $\|\ \|_N$ norm. By a diagonalisation argument there is then a subsequence $\{f_{j_n}\}_{n=0}^\infty$ of $\{f_n\}_{n=0}^\infty$ which converges in the norm $\|\ \|_N$ for each $N \ge 0$. This proves that S_b is a Montel space.

That S_b is a nuclear space from the fact that S is nuclear. In fact since S is nuclear, the space $S(n) = S/E_n$, $n \ge 0$, is nuclear (c.f. [9] Theorem III.7.4), where E_n is the closed subspace of functions $f \in S$ such that $\operatorname{supp} f \subset [n, \infty[$. S_b, being the projective limit $\varprojlim S(n)$, is then nuclear (cf. [9] corollary of Theorem III.7.4). □

Let E_N be the closed subspace of S' of distributions with support in $]-\infty, N]$.

PROPOSITION 2.2. *S_b' (the strong dual of S_b) is a nuclear Montel space, and is isomorphic to the strict inductive limit of $\{E_N\}_{N=0}^\infty$.*

Proof. As S is dense in S_b, $S_b' \subset S'$ (set theoretically). Further if $T \in S'$ has a continuous extension to S_b then $\operatorname{supp} T \subset]-\infty, N]$ for some $N \ge 0$. Each semi norm in S_b' has the form $S_b' \ni F \to q_B(F) = \sup_{u\in B} |F(u)|$, where B is a bounded set in S_b. The inductive limit of E_N is then identic with S_b' as topological vectorspace. Hence S_b' is nuclear as E_N is nuclear ([9] Theorem III.7.4 and corollary). Further S_b' is Montel as S_b is Montel (cf. [2], IV, §3, Prop. 7). □

3. Formal linearlization of the *KDV*-equation. Let t^2 be the two-dimensional commutative Lie algebra. The *KdV*-equation is defined in the representation

$$(3.1)\qquad t^2 \ni (a, b) \overset{T}{\mapsto} aT_0 + bT_1 \equiv T_{(a,b)} \in F_s(S_b),$$

where

$$T_0 = T_0^1, \qquad T_1 = T_1^1 + T_1^2, \quad T_0^1(u) = \partial u,$$

$$T_1^1(u) = -\partial^3 u \quad \text{and} \quad T_1^2(u_1, u_2) = 3(u_1 \partial u_2 + u_1 \partial u_1),$$

for u, u_1, $u_2 \in S_b$. The representation is formally linearizable [5] on S_b into S_b if there exists an element $C \in F(S_b)$, with C^1 continuously invertible and

$$(3.2)\qquad T_X C = C * T_X^1, \quad \forall X \in t^2.$$

(C takes then the space where the linear equation is defined into the space where the non-linear equation is defined.) The nth order of equation (3.2) reads

$$(3.3)\quad T_X^1 C^n - C^n * T_X^1 = -T_X^2 \Big(\sum_{1 \le p \le n-1} C^p \otimes C^{n-p} \Big) \sigma_n, \forall X \in t^2.$$

For $X = (1, 0)$ we get explicitly

$$(3.4)\quad \partial C^n(\varphi_2 \otimes \cdots \otimes \varphi_n) = \sum_{1 \le i \le n} C^n(\varphi_1 \otimes \cdots \otimes \partial\varphi_i \otimes \cdots \otimes \varphi_n), \qquad n \ge 1.$$

LEMMA 3.1. *If* $C^n \in \mathfrak{L}_n^s(S_b)$ *and* C^n *satisfies* (3.4) *then there exists a unique* $F^n \in S'(\mathbf{R}^n)$, *symmetric with* $\operatorname{supp} F_n \subset X^n]-\infty, k]$ *for some* $k \ge 0$ *and*

$$(3.5)\qquad C^n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \big[\check{F}^n \circledast (\varphi_1 \otimes \cdots \otimes \varphi_n)\big] \circ i_n$$

where $i_n(x) = (x, \ldots, x)$ *(n-times),* $\check{}$ *denotes space inversion and* $\circledast$ *convolution. Conversely given any* $F^n \in S'(\mathbf{R}^n)$ *with* $\operatorname{supp} F^n \subset X^n]-\infty, k]$, $k \ge 0$, *then* C^n *so constructed satisfies equation* (3.4).

Proof. If $C^n \in \mathfrak{L}_n^s(S_b)$, then

$$F^n(\phi_1 \otimes \cdots \otimes \phi_n) = \big[C^n(\phi_1 \otimes \cdots \otimes \phi_n)(0)$$

defines a symmetric element $F_n \in S'(\mathbf{R}^n)$ and $\operatorname{supp} F^n \subset X^n]-\infty, k]$ for some $k \geq 0$ (Proposition 2.2). Let τ_a denote the operator of translation by a on S_b. Then (3.4) gives

$$\tau_a C^n(\varphi_1 \otimes \cdots \otimes \varphi_n) = F^n(\tau_a\varphi_1 \otimes \cdots \otimes \tau_a\varphi_n) \quad \forall a \in \mathbf{R}, \varphi_i \in S_b.$$

Hence

$$\begin{aligned}[C^n(\varphi_1 \otimes \cdots \otimes \varphi_n)](x) &= F^n(\tau_x\varphi_1 \otimes \cdots \otimes \tau_x\varphi_n) \\ &\equiv [\check{F}^n \circledast (\varphi_1 \otimes \cdots \varphi_n)] \circ i_n(x).\end{aligned}$$

The converse is obvious. □

We next prove that a cohomology space $H^0(t^2, \mathfrak{L}_n^s(S_b)) = 0$.

PROPOSITION 3.2. *For $n \geq 2$, the equation*

$$T_X^1 C^n - C^n * T_X^1 = 0, \quad \forall x \in t^2$$

has the unique solution $C^n = 0$ in $\mathfrak{L}_n^s(S_b)$.

Proof. By Lemma 3.1 each solution C^n has the form

$$C^n(\varphi_1 \otimes \cdots \otimes \varphi_n) = [\check{F}^n \circledast (\varphi_1 \otimes \cdots \otimes \varphi_n)] \circ i_n \qquad (F \text{ symmetric})$$

Take $X = (0, 1)$. Then, we get

$$-\partial^3 C^n(\varphi_1 \otimes \cdots \otimes \varphi_n) + \sum_{1 \leq i \leq n} C^n(\varphi_1 \otimes \cdots \otimes \partial^3\varphi_i \otimes \cdots \otimes \varphi_n) = 0.$$

Thus

$$[(\partial_1 + \cdots + \partial_n)^3 - (\partial_1^3 + \cdots + \partial_n^3)]F^n = 0.$$

After Fourier transformation[1]

$$P_n \tilde{F}^n = 0, \qquad P_n(k_1, \ldots, k_n) = (k_1 + \cdots + k_n)^3 - (k_1^3 + \cdots + k_n^3),$$

where F^n is analytic in the domain $\pi_+^n = \{k \mid \operatorname{Re} k_i > 0, 1 \leq i \leq n\}$. (Proposition 2.2). But $P_n \neq 0$ for $n \geq 2$ and the ring of analytic functions on π_+^n is an integral domain so $F^n = 0$ is the only solution. □

COROLLARY 3.3. *Equation* (3.2) *has at most one solution for a given C^1.*

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[1] Fourier transformation in $S(\mathbf{R}^n)$ is defined by $\tilde{f}(k) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} dx\, e^{-ikx} f(x)$.

To solve equation (3.3) we introduce the holomorphic functions $\tilde{G}^n$: $\pi_+^n \to \mathbf{C}$,

$$(3.6)\qquad \tilde{G}^1(k) = ik(2\pi)^{-1/2}, \qquad \tilde{G}^2(k_2, k_2) = (2\pi)^{-1},$$

$$\tilde{G}^n(k_1,\dots,k_n) = \frac{i(k_1 + \cdots + k_n)}{(k_1 + k_2)\cdots(k_{n-1} + k_n)}(-i)^{n-1}(2\pi)^{-n/2}, \qquad n \geq 3.$$

We introduce also the holomorphic function $\tilde{F}^n$: $\pi_+^n \to \mathbf{C}$,

$$\tilde{F}^n(k_1,\dots,k_n) = \frac{1}{n!}\sum_{i\in\mathcal{P}_n}\tilde{G}^n(k_{i_1},\dots,k_{i_n}),$$

where $\mathcal{P}_n$ is the group of permutations of n elements. The holomorphic functions $\tilde{G}^n$ (resp. $\tilde{F}^n$) define uniquely (cf. **[10]**), by Fourier-Laplace transformation, distributions G^n (resp. F^n) $\in S'(\mathbf{R}^n)$ with supp G^n (resp. F^n) $\in x^n]-\infty, k]$ for some $k \geq 0$. Lemma 3.1 can now be applied to construct an element $C \in F(S_b)$ (resp. $C_s \in F_s(S_b)$) by the distributions G^n (resp. F^n). The algebraic expressions for $\tilde{G}^n$ coincide with them in **[8]**.

PROPOSITION 3.4. *$C_s \in F_s(S_b)$ constructed by* (3.5) *and* (3.6) *is the unique translation invariant symmetric formal linearization on S_b of the KdV-equation, i.e. $T_X C_s = C_s * T_X^1$ for each $X \in t^2$, with $C^1 = \partial$.*

Proof. C_s is unique if it exists (Corollary 3.3). By (3.3) and (3.5) the linearization of T by C_s is equivalent to

$$\left[(\partial_1 + \cdots + \partial_n)^3 - (\partial_1^3 + \cdots + \partial_n^3)\right]F^n = 3\sum_{1\leq p\leq n-1}(\partial_1 + \cdots + \partial_n)(F^p \otimes F^{n-p})\circ\sigma_n, \qquad n \geq 2.$$

This is by Fourier-Laplace transformation equivalent to

$$\left[(k_1 + \cdots + k_n)^3 - (k_1^3 + \cdots + k_n^3)\right]\tilde{F}^k(k_1,\dots,k_n) = -3(k_1 + \cdots + k_n)\sum_{1\leq p\leq n-1}\frac{1}{n!}\sum_{i\in\mathcal{P}_n}(\tilde{F}^p \otimes \tilde{F}^{n-p})(k_{i_1},\dots,k_{i_n}),$$

$$n \geq 2.$$

$\tilde{F}^n$ is a solution of this equation if

$$\left[(k_1 + \cdots + k_n)^3 - (k_1^3 + \cdots + k_n^3)\right]\tilde{G}^n(k_1,\dots,k_n) = -3(k_1 + \cdots + k_n)\sum_{1\leq p\leq n-1}(\tilde{G}^p \otimes \tilde{G}^{n-p})(k_1,\dots,k_n), \qquad n \geq 2.$$

Direct substitution as in [8] proves that $\tilde{G}^n$ defined by (3.6) satisfies this equation. □

The inverse Fourier transformation G^n of $\tilde{G}^n$, defined by

$$G^n(\tilde{\varphi}_1 \otimes \cdots \otimes \tilde{\varphi}_n) = \tilde{G}^n(\varphi_1 \otimes \cdots \otimes \varphi_n), \qquad \varphi_1, \ldots, \varphi_n \in S(\mathbf{R}),$$

is explicitly

$$(3.7) \quad G^n(u_1 \otimes \cdots \otimes u_n) = (-1)^n \int_{-\infty}^{0} dy_1 \cdots \int_{-\infty}^{0} dy_{n-1} u_1(y_1) u_2(y_1 + y_2) \cdots u_{n-1}(y_{n-2} + y_{n-1}) u_n(y_{n-1}).$$

Let U be the Frechet space of all functions $f \in C^\infty(\mathbf{R} \times \mathbf{R}^-, \mathbf{R})$ for which the seninorms M_n are finite:

$$(3.8) \quad M_n(f) = \sup_{\substack{0 \le x+y \le n \\ |\alpha| \le n}} |\partial^\alpha f(x,y)| + \sup_{\substack{x+y \le 0 \\ |\alpha| \le n}} \left|(1 + |x+y|)^n \partial^\alpha f(x,y)\right|,$$

$n = 0, 1, \ldots$. Here $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ and $|\alpha| = \alpha_1 = \alpha_2$. We introduce for $u \in S_b$ the commonly used (see [2], cf. [8], [12]) continuous integral operator $\Omega(u)\colon U \to U$:

$$(3.9) \quad [\Omega(u)f](x,y) = \int_{-\infty}^{0} u(x+y+t) f(x,t)\, dt, \qquad (x,y) \in \mathbf{R} \times \mathbf{R}^-, \quad f \in U.$$

(3.5), (3.7) and (3.9) give the following explicit expression for $\hat{C}^n$ in Proposition 3.4:

$$(3.10) \quad [\hat{C}^n(\varphi)](x) = [\partial_1 \hat{B}^n(\varphi)](x, 0)$$

where

$$(3.11) \quad B^n(\varphi_1 \otimes \cdots \otimes \varphi_n) = (-1)^n \Omega(\varphi_1) \cdots \Omega(\varphi_{n-1}) a(\varphi_n),$$
$$[a(\varphi_n)](x,y) = \varphi_n(x+y), \qquad \varphi_1, \ldots, \varphi_n \in S_b.$$

(3.9) and (3.11) give

$$(3.12) \quad \hat{B}(\varphi) + \Omega(\varphi)\hat{B}(\varphi) + a(\varphi) = 0, \qquad B = \sum_{n \ge 1} B^n.$$

The inverse $A = C^{-1}$ is needed for solving the Cauchy problem for the KdV equation. As is seen directly from (3.12), the power series B is easily inverted (on its image):

If $B(\phi) = \psi$, then $\phi = P(\psi)$, where

$$(3.13) \quad \hat{P}(\psi) + \mathfrak{B}(\psi)\hat{P}(\psi) + \psi(\cdot, 0) = 0$$

and

$$[\mathcal{B}(f)u](x) = \int_{-\infty}^{0} f(x,t)u(x+t)dt \tag{3.14}$$

defines a continuous integral operator $\mathcal{B}(f)\colon S_b \to S_b$ for each $f \in U$.

It seems difficult to invert C directly. However there is a unique power series $Q_s \in F_s(S_b, U)$ such that $\hat{Q}_s \circ \hat{C}_s = \hat{B}_s$. To find the inverse of C we first construct a $Q \in F(S_b, U)$. The expressions (3.10) and (3.11) give

$$\sum_{\substack{1 \le k \le N \\ n_1 + \cdots + n_k = N}} Q^k \circ \left(\hat{C}^{n_1} \otimes \cdots \otimes \hat{C}^{n_k}\right) = B^N, \qquad N = 1, 2, \ldots.$$

Multiplication with $\Omega(\varphi)$ gives by (3.11)

$$- \sum_{\substack{1 \le k \le N \\ n_1 + \cdots + n_k = N}} \Omega(\varphi) Q^k\left(\hat{C}^{n_1}(\varphi) \otimes \cdots \otimes \hat{C}^{n_k}(\varphi)\right)$$
$$= \sum_{\substack{1 \le k \le N+1 \\ n_1 + \cdots + n_k = N+1}} Q^k\left(\hat{C}^{n_1}(\varphi) \otimes \cdots \otimes \hat{C}^{n_k}(\varphi)\right).$$

Finally by (3.10) and (3.11)

$$Q^{N+1}(\partial\varphi \otimes \cdots \otimes \partial\varphi)$$
$$= \sum_{\substack{1 \le k \le N \\ n_1 + \cdots + n_k = N}} \Omega(\varphi) Q^k\left([\partial\Omega^{n_1 - 1}(\varphi)a(\varphi)](\cdot, 0) \otimes \cdots \otimes \left[\partial\Omega^{n_k - 1}(\varphi)a(\varphi)\right](\cdot, 0)\right)$$
$$- \sum_{\substack{1 \le k \le N,\ m_i \ge 1 \\ m_1 + \cdots + m_k = N+1}} Q^k\left([\partial\Omega^{m_1 - 1}(\varphi)a(\varphi)](\cdot, 0) \otimes \cdots \otimes \left[\partial\Omega^{m_k - 1}(\varphi)a(\varphi)\right](\cdot, 0)\right).$$

A lengthy but straight forward calculation gives that

$$\hat{Q}^N(\partial\varphi) = \mathcal{C}^{N-1}(\partial\varphi)a(\varphi), \qquad N = 1, 2, \ldots,$$

where

$$[\mathcal{C}(u)f](x,y) = \int_y^0 dz \int_{-\infty}^{x+y-z} dt\, u(t) f(t,z) \tag{3.15}$$

defines for each $u \in S_b$ a continuous operator $U \to U$. (The operator $\mathcal{C}$ is well-known, cf. [4].) It follows then that

$$\hat{Q}^N(\varphi) = \mathcal{C}(\varphi)^{N-1} a(\partial^{-1}\varphi), \qquad N = 1, 2, \ldots, \tag{3.16}$$

and that $\hat{Q}(\phi)$ satisfies the integral equation

$$\hat{Q}(\varphi) - \mathcal{C}(\varphi)\hat{Q}(\varphi) = a(\partial^{-1}\varphi). \tag{3.17}$$

4. Convergence properties of the formal power series. We study in this paragraph the existence and the properties of the functions defined by the formal power series Q, P, B and by them A and A^{-1}. $\hat{Q}$, $\hat{P}$, $\hat{B}$ are uniquely defined by the integral equations (3.17), (3.13) resp. (3.12):

$$\hat{Q}(\varphi) - \mathcal{C}(\varphi)\hat{Q}(\varphi) = a(\partial^{-1}\varphi), \qquad Q \in F(S_b, U), \varphi \in S_b, \tag{4.1}$$

$$\hat{P}(f) + \mathcal{B}(f)\hat{P}(f) = -f(\cdot, 0), \qquad P \in F(U, S_b), f \in U, \tag{4.2}$$

$$\hat{B}(\varphi) + \Omega(\varphi)\hat{B}(\varphi) = -a(\varphi), \qquad B \in F(S_b, U), \varphi \in S_b. \tag{4.3}$$

PROPOSITION 4.1. *The formal power series Q* (resp. P) *converges to an entire function $\hat{Q}$: $S_b \to U$* (resp. $\hat{P}$: $U \to S_b$). $\hat{Q}$(resp. $\hat{P}$) *is the unique solution of* (4.1) (*resp.* (4.2)).

Proof. Using that

$$[\partial_1 \mathcal{C}(\varphi)f](x, y) = \int_y^0 dy_1 \varphi(x + y - y_1) f(x + y - y_1, y_1)$$

and that

$$[(\partial_1 - \partial_2)\mathcal{C}(\varphi)f](x, y) = \int_{-\infty}^x dx_1 \varphi(x_1) f(x_1, y)$$

one deduces the existence of seminorms $p_0 \leq p_1 \leq \cdots$ for each given $K \geq 0$ such for $\alpha_1 \geq 1$:

$$\sup_{|\alpha| \leq N} |\partial^\alpha \mathcal{C}(\varphi) f|(x, y) \leq p_N(\varphi) \sup_{\substack{|\alpha| \leq N \\ \alpha_1 \neq N \\ (x_1, y_1) \in E(x, y)}} |\partial^\alpha f(x_1, y_1)|, \tag{4.4}$$

$$\forall (x, y) \in]-\infty, K] \times \mathbf{R}^-,$$

and for $\alpha_1 = 0$:

$$\sup_{0 \leq \alpha_2 \leq N} |\partial_2^{\alpha_2} \mathcal{C}(\varphi) f|(x, y) \leq p_N(\varphi) \sup_{\substack{|\alpha| \leq N-1 \\ (x_1, y_1) \in E(x, y)}} |\partial^\alpha f(x_1, y_1)|, \tag{4.5}$$

$$\forall (x, y) \in]-\infty, K] \times \mathbf{R}^-,$$

where $E(x, y) = \{(x_1, y_1) \in \mathbf{R} \times \mathbf{R}^- | x_1 + y_1 \le x + y\}$. We have here used the inequality (cf. [12])

$$(4.6) \quad |\mathcal{Q}(\varphi)f|(x, y) \le \pi \int_{-\infty}^{x} dt(1 + t^2)|\varphi(t)| \sup_{(x_1, y_1) \in E(x, y)} |f(x_1, y_1)|.$$

Denote

$$\eta_x(\varphi) = \pi \int_{-\infty}^{x} dt(1 + t^2)|\varphi(t)|.$$

The explicit form of $\mathcal{Q}(\phi)$ and (4.4), (4.5) (4.6) give for $M \ge N + 1$ and $(x, y) \in]-\infty, K] \times \mathbf{R}^-$:

$$(4.7) \quad \sup_{|\alpha| \le N} |\partial^\alpha \mathcal{Q}^{M-1}(\varphi) a(\partial^{-1}\varphi)|(x, y)$$
$$\le [(M - 1 - N)!]^{-1} p_N(\varphi) \eta_x(\varphi)^{M-1-N} \sup_{t \le x+y} |\partial^{-1}\varphi(t)|.$$

Finally we find that (see (3.8), (3.16))

$$(4.8) \quad M_N(\hat{Q}^n(\varphi)) \le p_N(\varphi)^{N+1} \eta_k(\varphi)^{n-1-N}[(n - 1 - N)!]^{-1},$$

where $n \ge N + 1$. p_N has (for convenience) been chosen sufficiently large and $K \ge N$. (4.8) proves that the series $\hat{Q}(\phi)$ converges for each $\phi \in S_b$. Hence $\hat{Q}$: $S_b \to U$ is entire analytic.

To prove the second statement, we first remark that

$$(4.9) \quad [\mathcal{B}(\psi)\varphi](x) = \int_{-\infty}^{x} dt \psi(x, t - x)\varphi(t)$$

gives

$$(4.10) \quad [\partial \mathcal{B}(\psi)\varphi](x) = \psi(x,0)\varphi(x) + \mathcal{B}((\partial_1 - \partial_2)\psi)\varphi.$$

Formula (4.9) and (4.10) give the estimate, for $n \ge N + 1$,

$$(4.11) \quad |\partial^N \mathcal{B}^{n-1}(\psi)\varphi|(x)$$
$$\le q_N(\psi)^N[(n - 1 - N)!]^{-1}\left(\int_{-\infty}^{x} dt|\psi(x, t - x)|\right)^{n-1-N} \sup_{t \le x} |\varphi(t)|,$$

where q_N is a sufficiently large seminorm on U. Then for p_N sufficiently large

$$\|\mathcal{B}^{n-1}(\psi)\varphi\|_N \le p_N(\psi)^{n-1}\|\varphi\|_N[(n - 1 - N)!]^{-1}.$$

Thus, once more if p_N is sufficiently large

$$(4.12) \qquad \|\hat{P}^n(\psi)\|_N \leq p_N(\psi)^n[(n-1-N)!]^{-1},$$

which proves that $\hat{P}$ is entire analytic. □

COROLLARY 4.2. *Denote $A = C^{-1} \in F(S_b)$. Then $\hat{A}$ is an entire analytic function.*

Proof. $\hat{A} = \hat{P} \circ \hat{Q}$ is entire analytic by Proposition 4.1. □

We now turn to the question of the existence of $\hat{A}^{-1}$. We prove the convergence of $C = A^{-1}$ and find the maximal connected domain of analyticity (containing $u = 0$) for $\hat{C}$.

Introduce the operator (see [**4**]) $\Omega_x(u)$: $L^2(]-\infty, 0]) \to L^2(]-\infty, 0])$ for $x \in \mathbf{R}$, $u \in S_b$:

$$(4.13) \qquad [\Omega_x(u)h](y) = \int_{-\infty}^{0} u(x+y+t)h(t)\,dt.$$

$\Omega_x(u)$ is a self-adjoint Hilbert-Schmidt operator with

$$(4.14) \qquad \|\Omega_x(u)\|_{\text{H.S.}} \leq \left(\frac{\pi}{2}\right)^{1/2} \sup_{t \leq 0} |(1+t^2)u(x+t)|.$$

Let $\sigma(\Omega_x(u))$ be the spectrum of $\Omega_x(u)$, let $U(n)$, $n = 0, 1, \ldots,$ be the factorspace of U and the closed subspace of functions f with $\operatorname{supp} f \subset [n, \infty[\times \mathbf{R}^-$. $B \in F(S_b, U)$ defines an element in $F(S(n), U(n))$ which also will be denoted B. The seminorms on $U(n)$ induced by the complete set of seminorms (3.8) on U will be denoted M_N^n, $N = 0, 1, \ldots.$

PROPOSITION 4.3. *For each $n \in \mathbf{N}$, B defines an analytic function $\hat{B}$: $O_n \supset S(n) \to U(n)$, where O_n is the set of all $u \in S(n)$ such that $\Omega_x(u) > -I$, $\forall x \in]-\infty, n]$. O_n is open in $S(n)$ and*

$$(4.15) \qquad [\hat{B}(u)](x, y) = -\left[(I + \Omega_x(u))^{-1}[a(u)](x, \cdot)\right](y).$$

(*Analytic means here convergence on a neighbourhood of each point in O_n*).

Proof. Let $u \in O_n$. Then by estimate (4.14) there exists an neighbourhood V of u such that, for each $v \in V$ and $x \in]-\infty, n]$, $\sigma(\Omega_x(v)) \subset [-1+\varepsilon, \infty[$ for some $0 < \varepsilon$. Hence O_n is open in $S(n)$.

It follows from Lemma III.9 in **[12]**, (where it is only used that $\sigma(\Omega_x(u)) \subset [-1+\varepsilon, \infty[$ for each $x \in]-\infty, n]$), that $\hat{B}(u) \in U(n)$ and that $[I + \Omega_x(u)]^{-1} f(x, \cdot)$ defines an element in $U(n)$ for each $f \in U(n)$. Let u, $u + \phi \in O_n$. Then at the point $u + \phi$, the r.h.s. of (4.15) can be written

$$-(I + \Omega(u+\varphi))^{-1} a(u+\varphi)$$
$$= -\left[I + \Omega(\varphi)(I + \Omega(u))^{-1}\right]^{-1}(I + \Omega(u))^{-1} a(u+\varphi).$$

We prove the convergence of the series

$$(4.16) \qquad \sum_{0 \le k} (-1)^k \left[\Omega(\varphi)(I + \Omega(u))^{-1}\right]^k (I + \Omega(u))^{-1} a(u+\varphi)$$

in $U(n)$ for all ϕ in some $S(n)$ neighbourhood of u. Let $\phi_1, \ldots, \phi_{k-1} \in S(n)$, $u_1 \ldots, u_k \in O_n$, $k \ge 2$, $f \in U(n)$ and define the functions F^k: $S(n)^{k-1} \times O_n^k \times U(n) \to U(n)$:

$$(4.17) \quad F^k(\varphi_1, \ldots, \varphi_{k-1}; u_1, \ldots, u_{k-1}; f)$$
$$= (-1)^k \Omega(\varphi_1)(I + \Omega(u_1))^{-1} \Omega(\varphi_2)(I + \Omega(u_2))^{-1}$$
$$\cdots \Omega(\varphi_{k-1})(I + \Omega(u_{k-1}))^{-1} f.$$

The following estimates are obtained directly from formula (3.9) and the identity $-(I + \Omega(u))^{-1} f - \Omega(u)(I + \Omega(u))^{-1} f + f = 0$, $u \in O_n$, $f \in U(n)$:

$$(4.18) \quad \sup_{\substack{x \le n \\ y \le 0}} \left|(1 + |x| + |y|)^N [\Omega(\varphi) f](x, y)\right|$$
$$\le p_N(\varphi) \sup_{\substack{x \le n \\ y \le 0}} |f(x, y)|, \quad \forall \varphi \in S(n), f \in U(n), N \in \mathbf{N},$$

where $p_0 \le \cdots \le p_N \le \cdots$ are sufficiently large seminorms on $S(n)$ and

$$(4.19) \quad \left|f(x, y) - \left[(I + \Omega(u))^{-1} f\right](x, y)\right|$$
$$\le \left\|(I + \Omega_x(u))^{-1}\right\|_{L^2} \cdot \sup_{t \le 0} \left|(1-t)^2 u(x+y+t)\right|$$
$$\cdot \sup_{t \le 0} \left|(1-t)^2 f(x, t)\right|.$$

(4.19) gives:

$$(4.20) \quad \sup_{\substack{x \le n \\ y \le 0}} \left|\left[(I + \Omega(u))^{-1} f\right](x, y)\right| \le C_u \sup_{\substack{t \le 0 \\ x \le n}} \left|(1-t)^2 f(x, t)\right|,$$

where (for given n) C_u is a positive constant depending on $u \in O_n$. (4.18) and (4.20) give for some sufficiently large seminorm q on $S(n)$:

$$(4.21)\quad \sup_{\substack{x\leq n\\ y\leq 0}} \left|\left[(I+\Omega(u))^{-1}\Omega(\varphi)f\right](x,y)\right| \leq C_u q(\varphi) \sup_{\substack{x\leq n\\ y\leq 0}} |f(x,y)|, \quad \forall u \in O_n,\ \varphi \in S(n), f \in U(n).$$

By (4.18) (4.21) and then (4.20) an estimation for F^k is obtained:

$$(4.22)\quad \sup_{\substack{x\leq n\\ y\leq 0}} \left|(1+|x|+|y|)^N\left[F^k(\varphi_1,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};f)\right](x,y)\right| \leq p_N(\varphi_1)C_{u_1}\cdots C_{u_{k-1}}q(\varphi_2)\cdots q(\varphi_{k-1}) \sup_{\substack{t\leq 0\\ x\leq n}} \left|(1-t)^2 f(x,t)\right|.$$

The identity (for a given $x \in\,]-\infty, n]$) for $\varepsilon \leq n-x$

$$-(I+\Omega_{x+\varepsilon}(u))^{-1} - \Omega_{x+\varepsilon}(u)(I+\Omega_{x+\varepsilon}(u))^{-1} + I = 0$$

gives

$$(4.23)\quad \frac{d}{d\varepsilon}(I+\Omega_{x+\varepsilon}(u))^{-1}\Big|_{\varepsilon=0} = (I+\Omega_x(u))^{-1}\Omega_x(\partial u)(I+\Omega_x(u))^{-1}$$

(This is well-defined in $U(n)$, cf. **[9]**.)

Explicit expressions for the derivatives of $F^k(\varphi; u; f)$ are obtained from (4.17), (3.9) and (4.23):

$$(4.24)\quad \partial_1 F^k(\varphi_1,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};f) = \sum_{1\leq i\leq k-1}\{F^k(\varphi_1,\ldots,\partial\varphi_i,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};f) + F^{k+1}(\varphi_1,\ldots,\varphi_i,\partial u_i,\varphi_{i+1},\ldots,\varphi_{k-1}; u_1,\ldots,u_i,u_i,u_{i+1},\ldots,u_{k-1};f)\} + F^k(\varphi_1,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};\partial_1 f)$$

and

$$(4.25)\quad \partial_2 F^k(\varphi_1,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};f) = F^k(\partial\varphi_1,\varphi_2,\ldots,\varphi_{k-1};u_1,\ldots,u_{k-1};f).$$

(4.22), (4.24) and (4.25) give the estimate for $k \geq N+1$ (q_N is a sufficiently large seminorm):

(4.26)
$$M_N\big(F^k(\varphi,\ldots,\varphi;\, u,\ldots,u;\, f\big)\big) \leq (2(k+N)-1)^N q_N(\varphi)^N (C_u)^{k-N-1} \times \sup_{\substack{x\leq n\\ t\leq 0\\ 0\leq\alpha\leq N}} \left|(1-t)^2\partial_1^\alpha f(x,t)\right| q(\varphi)^{k-N-1}.$$

Thus, for a given $u \in O_n$ the series

$$\sum_{k\geq 2} F^k(\varphi,\ldots,\varphi;\, u,\ldots,u;\, f)$$

converges for all $f \in U(n)$ and all $\phi \in S(n)$ with $q(\phi)C_u < 1$. $f = (I+\Omega(u))^{-1}a(u+\phi)$ is linear (and continuous) in ϕ, which proves that the series (4.16) converges on a neighbourhood of u in $S(n)$. □

Corollary 4.4. *The mapping* $O_n \ni u \mapsto [\partial_t \hat{B}(u)](\cdot, \circ) \equiv \hat{C}(u) \in S(n)$ *is analytic and this mapping composed with* $\hat{A}$ *is* id_{O_n}.

Proof. The first statement is a trivial consequence of Proposition 4.3. Secondly denote $\hat{F}$: $O_n \to S(n)$ the composite map $\hat{A} \circ \hat{C}$. $\hat{F}$ is analytic (Corollary 4.2 and Proposition 4.3) and the formal power series F on $S(n)$ is the identity (§3). □

Remark 4.5. Let $\{u_l\}_{l=1}^\infty \subset O_n$ be a convergent sequence in $S(n)$ with limit v, such that $-1 \in \sigma(\Omega_x(v))$ for some $x \in\,]-\infty, n]$. Then by (4.15) $\hat{B}(u_l)$ will likely (if there is no cancellation) develop singularities as $l \to \infty$, and v would not be in $\hat{A}[S(n)]$. Further if for $u \in S(n)$, $-1 \notin \sigma(\Omega_x(u))$ for $x \in\,]-\infty, n]$, then by the continuity of $x \to \Omega_x(u)$, $\sigma(\Omega_x(u)) \subset [-1+\varepsilon, \infty[$ for each $x \in\,]-\infty, n]$ for some $\varepsilon > 0$. Thus by Corollary 4.4. it seems reasonable to try to prove that $\Omega_x(u) > -I$ for each $u \in \hat{A}[S(n)]$ and $x \in\,]-\infty, n]$. The result is well-known, (see **[4]**).

Proposition 4.6. *The image* $\hat{A}[S_b]$ *is exactly the subset* $O \subset S_b$, *where* O *is the projective limit of* O_n, $n \in \mathbf{N}$, *i.e.* $u \in O$ iff

(4.27)
$$\Omega_x(u) > I \quad \forall x \in \mathbf{R}.$$

Proof. For given $x \in \mathbf{R}$ choose $n > x$. Then as $\hat{A}$: $S_b \to S_b$ defines a mapping $\hat{A}$: $S(n) \to S(n)$

$$[\hat{A}(u)](x) = [\hat{A}(\theta_{n+1}u)](x), \quad \forall u \in S_b,$$

where $\theta_{n+1} \in C^\infty(\mathbf{R})$. $\theta_{n+1}(t) = 1$ for $t \in]-\infty, n]$ and $\theta_{n+1}(t) = 0$ for $t \geq n + 1$. But $\theta u \in S(\mathbf{R})$, so $\Omega_x(\hat{A}(\theta u)) > -I$, $\forall x \in]-\infty, n]$, (see [**4**], cf. [**12**]). □

REMARK 4.7. The condition (4.27) resembles to that obtained in linearizing Burger's equation in [**13**].

5. Solution of the non-linear initial value problem and superposition of solutions. Corollaries 4.2 and 4.4 reduce the solution of the *KdV* equation in S_b:

$$(5.1) \qquad \frac{d}{dt}u(t) = -\partial^3 u(t) + 6u(t)\partial u(t), \qquad u(0) = u_0 \in S_b$$

to the linear equation

$$(5.2) \qquad \frac{d}{dt}v(t) = -\partial^3 v(t), \qquad v(0) = v_0 = \hat{A}(u_0) \in 0$$

together with the condition in Proposition 4.6. But before stating formally the result we prove the following lemma:

LEMMA 5.1. *Let V be a n-dimensional C^∞ differentiable manifold and F: $V \to O$ a C^∞ mapping then*

$$f = \hat{A}^{-1} \circ F: V \to S_b \text{ is } C^\infty.$$

Proof. The mapping $V \ni \xi \mapsto [F(\xi)]\restriction_{]-\infty,n]} \in O_n$ is C^∞ and $\hat{A}^{-1}$: $O_n \to S(n)$ is analytic (Corollary 4.4). Then the mapping $V \ni \xi \mapsto [p_n \circ \hat{A}^{-1} \circ F](\xi) \in S(n)$ is C^∞, where p_n: $S_b \to S(n)$ is the canonical projection. Thus $V \ni \xi \mapsto [\hat{A}^{-1} \circ F](\xi) \in S_b$ is C^∞ as $S_b = \varprojlim S(n)$. □

PROPOSITION 5.2. *Let $u_0 \in S_b$. Then equation* (5.1) *has a C^1 solution for $t \in]a_1, a_2[$, $-\infty \leq a_1 \leq 0 \leq a_2 \leq \infty$,* iff *the equation* (5.2) *with $v_0 = \hat{A}(u_0)$ has a C^1 solution $v(t) \in O$ for $t \in]a_1, a_2[$. On the interval of existence $u(t) = \hat{A}^{-1}(v(t))$ and $t \mapsto u(t)$ is C^∞.*

REMARK. Instead of C^1 solutions one can write C^0 solution of the integrated equations.

Proof. $v_0 \in O$ (Proposition 4.6). There exists a maximal interval $]b_1, b_2[$, $-\infty \leq b_1 \leq 0 \leq b_2 \leq \infty$, where equation (5.2) has a C^1 solution in S_b. Hence the solution $u(t) = \hat{A}^{-1}(v(t))$ of equation (5.1) exists on a maximal interval $t \in]a_1, a_2[$, $b_1 \leq a_1 \leq 0 \leq a_2 \leq b_2$, on which $v(t) \in O$.

$t \mapsto v(t)$ is C^∞ if it is C^1. Lemma 5.1 gives now the "if part" of the proposition. Conversely if $u(t)$ is a C^1-solution on a maximal interval of existence $]a_1, a_2[$ then by Corollary (4.2) and Proposition 4.6 $\hat{A}(u(t)) \in O$ is a C^1 solution of equation 5.2 on $]a_1, a_2[$. Further $t \mapsto \hat{A}(u(t))$ is then C^∞, so by Lemma 5.1 $u(t)$ is C^∞. □

REMARK 5.3.
(a) $u(t)$ can blow up in finite time ($t = a$) for two reasons:
(i) $v(t) \in$ does not converge in S_b as $t \to a$;
(ii) $v(a) \exists$ in S_b but $v(a) \notin O$.
(b) If $u_0 \in S(\mathbf{R})$ then $u(t) \in S(\mathbf{R})$ $\forall t \in \mathbf{R}$ as is seen from space-time inversion of equation 5.1 on $S(\mathbf{R})$ and linearization by A. $\hat{A}(S(\mathbf{R}))$ is invariant under the linear evolution, [**12**].

The non-linear superposition principle is a trivial consequence of the following Corollary:

COROLLARY 5.4. *O is a convex set.*

Proof. See Proposition 4.6.

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UNIVERSITÉ DE GENÈVE
1211 GENÈVE 4, SWITZERLAND

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