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Helson and Lowdenslager extended the classical F. and M. Riesz theorem as follows:

Let G be a compact abelian group with the ordered dual $\hat{G}$. Let μ be a bounded regular measure on G which is of analytic type. Then μ_a and μ_s are of analytic type.

Doss extended this theorem for a LCA group with the algebraically ordered dual. On the other hand, deLeeuw and Glicksberg obtained an analogous result for a compact abelian group G such that there exists a nontrivial homomorphism from $\hat{G}$ into R. In this paper, we prove that the theorem of Helson and Lowdenslager is satisfied for a LCA group with partially ordered dual.

1. Introduction. Let G be a LCA group with the dual group $\hat{G}$. We denote by m_G the Haar measure on G. Let $M(G)$ be the Banach algebra of bounded regular measures on G under convolution multiplication and the total variation norm. $M_s(G)$ and $L^1(G)$ denote the closed subspace of $M(G)$ consisting of measures which are singular with respect to m_G and the closed ideal of $M(G)$ consisting of measures which are absolutely continuous with respect to m_G respectively. We denote by $\mathrm{Trig}(G)$ the set of all trigionometric polynomials on G. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. E^- (or $\bar{E}$) means the closure of E. Let $M^+(G)$ be the subset of $M(G)$ consisting of positive measures. For $\mu \in M(G)$, μ_a and μ_s denote the absolutely continuous part and the singular part of μ respectively. For a subgroup H of G, $H^\perp$ means the annihilator of H.

Helson and Lowdenslager extended the classical F. and M. Riesz theorem as follows:

THEOREM A (*cf.* [**8**], 8.2.3. *Theorem*). *Let G be a compact abelian group with ordered dual, i.e., there exists a semigroup P in $\hat{G}$ such that* (i) $P \cup (-P) = \hat{G}$ *and* (ii) $P \cap (-P) = \{0\}$. *Let μ be a measure in $M(G)$ such that $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$. Then*

(I) $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ *for* $\gamma < 0$;

(II) $\hat{\mu}_s(0) = 0$.

In [**3**] and [**4**], Doss extended Theorem A for a LCA group.

THEOREM B ([4], *Lemma* 1). *Let G be a LCA group such that $\hat{G}$ is algebraically ordered*, i.e., *there exists a semigroup P in $\hat{G}$ such that* (i) $P \cup (-P) = \hat{G}$ *and* (ii) $P \cap (-P) = \{0\}$ (*we do not assume the closedness of P*). *Let μ be a measure in $M(G)$ such that $\hat{\mu}(\gamma) = 0$ for $\gamma < 0$. Then*

(I) $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ *for* $\gamma < 0$;

(II) $\hat{\mu}_s(0) = 0$.

REMARK 1.1. In Theorem B, when G is noncompact, (II) is obtained from (I) and the fact that 0 is an accumulation point of P^c.

On the other hand, deLeeuw and Glicksberg in [2] obtained an analogous result of Theorem A for a compact abelian group G such that there exists a nontrivial homomorphism ψ from $\hat{G}$ into R (the reals). That is,

THEOREM C (*cf.* [2], *Proposition* 5.1, *p.* 189). *Let G be a compact abelian group and ψ a nontrivial homomorphism from $\hat{G}$ into R. Put $M^a(G) = \{\mu \in M(G);\ \hat{\mu}(\gamma) = 0$ for $\gamma \in \hat{G}$ with $\psi(\gamma) < 0\}$. Let μ be a measure in $M^a(G)$. Then μ_a and μ_s belong to $M^a(G)$.*

REMARK 1.2. In general, however, the conclusion of Theorem C can not be strengthened to "$\hat{\mu}_s(0) = 0$".

Our purpose in this paper is to prove that an analogous result of Theorem C is satisfied for a LCA group with partially ordered dual. We state the main theorem of this paper.

MAIN THEOREM. *Let G be a LCA group and P a closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let μ be a measure in $M_{P^c}(G)$. Then μ_a and μ_s belong to $M_{P^c}(G)$.*

COROLLARY. *Let G be a LCA group and P a semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Then the following are satisfied*:

(I) *for $\mu \in M_P(G)$, μ_a and μ_s belong to $M_P(G)$*;

(II) *for $\mu \in M_{P^c}(G)$, μ_a and μ_s belong to $M_{P^c}(G)$.*

Proof of Corollary. Since (II) is easily obtained from the Main Theorem, we only prove (I). We note the following:

$$\begin{aligned}
&\hat{\mu} = 0 && \text{on } (-P \setminus P)\\
\Leftrightarrow\ &\hat{\mu} = 0 && \text{on } \gamma - P \text{ for all } \gamma \in (-P)\setminus P\\
\Leftrightarrow\ &(\gamma\mu)^\wedge = 0 && \text{on } -P \text{ for all } \gamma \in P\setminus(-P)\\
\Leftrightarrow\ &(\gamma\mu)^\wedge = 0 && \text{on } (-P)^- \text{ for all } \gamma \in P\setminus(-P).
\end{aligned}$$

Hence, by the Main Theorem and the fact that $(\gamma\mu)_a = \gamma\mu_a$, we obtain the corollary.

In §2, we prove Main Theorem for a σ-compact metrizable locally compact abelian group by using the theory of disintegration. In §3 we prove the theorem for a general locally compact abelian group by using a certain lemma which is due to Pigno and Saeki ([7], Lemma 4). The author would like to thank the referee for his valuable advice.

2. σ-compact metrizable case. In this section, we prove Main Theorem for a σ-compact metrizable locally compact abelian group. We need the theory of disintegration. The following lemma can be found in ([**1**], Théorème 1, p. 58).

LEMMA 2.1. *Let G be a σ-compact metrizable LCA group and H a closed subgroup of G. Let π be the natural homomorphism from G onto G/H. Let μ be a positive measure in $M(G)$ and put $\eta = \pi(\mu)$ (continuous image under π). Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x}\in G/H}$ consisting of positive measures in $M(G)$ with the following properties*:

(1) *$\dot{x} \mapsto \lambda_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on G,*

(2) $\operatorname{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(3) $\|\lambda_{\dot{x}}\| \leq 1$,

(4) *$\mu(g) = \int_{G/H} \lambda_{\dot{x}}(g)\,d\eta(\dot{x})$ for each bounded Borel measurable function g on G.*

Conversely, let $\{\lambda'_{\dot{x}}\}_{\dot{x}\in G/H}$ be a family of positive measures in $M(G)$ which satisfies (1), (2) *and* (4). *Then we have*

(5) $\lambda_{\dot{x}} = \lambda'_{\dot{x}}$ *a.a.* $\dot{x}(\eta)$.

LEMMA 2.2. *Let G, H and π be as in Lemma 2.1. Let μ be a positive measure in $M(G)$ and put $\eta = \pi(\mu)$. By* (2) *of Lemma* 2.1, *$\lambda_{\dot{x}}$ can be represented as follows*:

(1) *$\lambda_{\dot{x}} = \nu_{\dot{x}} * \delta_x$ for some $\nu_{\dot{x}} \in M^+(H)$ and $x \in G$ with $\pi(x) = x_{\dot{x}}$.*

If $\nu_{\dot{x}} \in M_s(H)$ a.a. $\dot{x}(\eta)$, we have $\mu \in M_s(G)$.

Proof. It is sufficient to prove the lemma when μ has compact support, so we can assume η supported by K compact. Suppose $\{f_n\} \subset C_0(G)$ is dense. Let ε be a positive real number. Then for each n Lusin's theorem says $\dot{x} \mapsto \lambda_{\dot{x}}(f_n)$ is continuous on a compact subset E_n of K with $\eta(K\setminus E_n) < \varepsilon/2^n$. We put $E = \bigcap_{n=1}^{\infty} E_n$. Then E is compact, $\eta(K\setminus E) < \varepsilon$

and $\dot{x} \mapsto \lambda_{\dot{x}}(f_n)$ is continuous on E for all n. Hence $\dot{x} \mapsto \lambda_{\dot{x}}(h)$ is continuous on E for all $h \in C_0(G)$. By the hypothesis we may assume that $\|\lambda_{\dot{x}}\| = 1$ and $\nu_{\dot{x}} \in M_s(H)$ for all $\dot{x} \in E$. Hence for $\dot{x} \in E$ we can choose $f = f_{\dot{x}} \in C_c(G)$ with $0 \leq f \leq 1$, $1 = \|\lambda_{\dot{x}}\| < \lambda_{\dot{x}}(f) + \varepsilon$ and $\delta_x * m_H(f) < \varepsilon$ $(x \in \pi^{-1}(\{\dot{x}\}))$. Then both inequalities are held on some neighborhood of x in E, say $N_{\dot{x}}$. Since E lies in $N_{\dot{x}_1}, \ldots, N_{\dot{x}_k}$, with $f_1, \ldots, f_k$ the corresponding f's, we set $g = f_1$ on $\pi^{-1}(N_{\dot{x}_1})$, $= f_2$ on $\pi^{-1}(N_{\dot{x}_2} \setminus N_{\dot{x}_1}), \ldots,$ $= f_k$ on $\pi^{-1}(N_{\dot{x}_k} \setminus \bigcup_{j=1}^{k-1} N_{\dot{x}_j})$ and $= 0$ on $\pi^{-1}(E^c)$. Then g is a Borel measurable function on G with $0 \leq g \leq 1$ satisfying $1 - \varepsilon < \lambda_{\dot{x}}(g)$ and $\delta_x * m_H(g) < \varepsilon$ for all $\dot{x} \in E$ $(x \in \pi(\{\dot{x}\}))$. Thus $\mu(g) = \int_{G/H} \lambda_{\dot{x}}(g) d\eta(\dot{x}) > 1 - 2\varepsilon$ and $m_G(g) < \varepsilon m_{G/H}(K)$. Since this holds for each $\varepsilon > 0$, μ is necessarily singular.

LEMMA 2.3. *Let G be a LCA group and H a closed subgroup of G. Let π be the natural homomorphism from G onto G/H. Let μ be a measure in $M^+(G)$. If $\pi(\mu)$ belongs to $M_s(G/H)$, μ is singular with respect to the Haar measure on G.*

Proof. Since $\pi(\mu) \in M_s(G/H)$, there exists a σ-compact subset $\tilde{E}$ of G/H such that $\pi(\mu)(\tilde{E}^c) = 0$ and $m_{G/H}(\tilde{E}) = 0$. Then μ is concentrated on $\pi^{-1}(\tilde{E})$. Therefore it is sufficient to prove that $\pi^{-1}(\tilde{E})$ is a locally null set. For a compact set K in G, we have

$$\begin{aligned} m_G(K \cap \pi^{-1}(\tilde{E})) &= \int_G \chi_K(x) \chi_{\pi^{-1}(\tilde{E})}(x) dm_G(x) \\ &= \int_{G/H} \int_H \chi_K(\dot{x} + y) \chi_{\pi^{-1}(\tilde{E})}(\dot{x} + y) dm_H(y) dm_{G/H}(\dot{x}) \\ &= \int_{G/H} \chi_{\tilde{E}}(\dot{x}) \int_H \chi_K(\dot{x} + y) dm_G(y) dm_{G/H}(\dot{x}) \\ &= 0. \end{aligned}$$

Hence $\pi^{-1}(\tilde{E})$ is a locally null set and the proof is complete.

LEMMA 2.4. *Let G be a σ-compact metrizable LCA group and P a closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Put $\Lambda = P \cap (-P)$ and $H = \Lambda^\perp$. Let π be the natural homomorphism from G onto G/H. For a measure $\mu \in M(G)$, we assume that there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in G/H}$ in $M(G)$ with the following properties*:

(1) *$\dot{x} \mapsto \lambda_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel function f on G,*

(2) $\mathrm{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(3) $\|\lambda_{\dot{x}}\| \leq 1$,

(4) $\mu(g) = \int_{G/H} \lambda_{\dot{x}}(g) d\eta(\dot{x})$ *for each bounded Borel measurable function g on G,*

where $\eta = \pi(|\mu|)$. *Then the following is satisfied:*

(5) *If* $\hat{\mu}(\gamma) = 0$ *on* P, $\hat{\lambda}_{\dot{x}}(\gamma) = 0$ *on* P *a.a.* $\dot{x}(\eta)$.

Proof. First we note

$$P + \Lambda \subset P. \tag{6}$$

For $f \in L^1(\hat{G})$ with $\mathrm{supp}(f) \subset P$, we have

$$\begin{aligned} 0 &= \int_{\hat{G}} \hat{\mu}(\gamma) f(\gamma) d\gamma \\ &= \int_G \hat{f}(x) d\mu(x) \\ &= \int_{G/H} \lambda_{\dot{x}}(\hat{f}) d\eta(\dot{x}). \end{aligned} \tag{7}$$

On the other hand, for $\gamma_* \in \Lambda$, by (6), we have $\mathrm{supp}(f_{\gamma_*}) \subset P$, where $f_{\gamma_*}(\gamma) = f(\gamma - \gamma_*)$. Hence, by (7), we have

$$\begin{aligned} 0 &= \int_{G/H} \lambda_{\dot{x}}\left(\hat{f}_{\gamma_*}\right) d\eta(\dot{x}) \\ &= \int_{G/H} \int_G \hat{f}_{\gamma_*}(x) d\lambda_{\dot{x}}(x) d\eta(\dot{x}) \\ &= \int_{G/H} \int_G (-x, \gamma_*) \hat{f}(x) d\lambda_{\dot{x}}(x) d\eta(\dot{x}) \\ &= \int_{G/H} (-\dot{x}, \gamma_*) \int_G \hat{f}(x) d\lambda_{\dot{x}}(x) d\eta(\dot{x}) \qquad \left(\text{by (2) and } \gamma_* \in \Lambda\right) \\ &= \int_{G/H} (-\dot{x}, \gamma_*) \lambda_{\dot{x}}(\hat{f}) d\eta(\dot{x}). \end{aligned}$$

Since γ_* is an arbitrary element in Λ, we have

$$0 = \int_{G/H} p(\dot{x}) \lambda_{\dot{x}}(\hat{f}) d\eta(\dot{x}) \quad \text{for all } p(\dot{x}) \in \mathrm{Trig}(g/H). \tag{8}$$

Since $\mathrm{Trig}(G/H)$ is dense in $L^1(\eta)$ and $\dot{x} \mapsto \lambda_{\dot{x}}(\hat{f})$ is a bounded Borel measurable function, we have

$$\lambda_{\dot{x}}(\hat{f}) = 0 \text{ a.a. } \dot{x}(\eta) \quad \text{for } f \in L^1(\hat{G}) \text{ with } \mathrm{supp}(f) \subset P. \tag{9}$$

Hence, for $f \in L^1(\hat{G})$ with $\operatorname{supp}(f) \subset P$, we have

$$(10) \qquad 0 = \int_G \hat{f}(x)\, d\lambda_{\dot{x}}(x) = \int_{\hat{G}} \hat{\lambda}_{\dot{x}}(\gamma) f(\gamma)\, d\gamma \quad \text{a.a. } \dot{x}(\eta).$$

On the other hand, since $\hat{G}$ is σ-compact and metrizable, there exists a countable subset $\mathcal{Q} = \{f_n\}$ of $L^1(P) = \{f \in L^1(\hat{G});\ \operatorname{supp}(f) \subset P\}$ such that it is dense in $L^1(P)$. By (10), for each $m \in N$ (the natural numbers), there exists a Borel set $\tilde{K}_m$ in G/H such that $\eta(\tilde{K}_m^c) = 0$ and

$$(11) \qquad 0 = \int_{\hat{G}} \hat{\lambda}_{\dot{x}}(\gamma) f_m(\gamma)\, d\gamma \quad \text{for } \dot{x} \in \tilde{K}_m.$$

Put $K = \bigcap_1^\infty \tilde{K}_m$. Then $\eta(\tilde{K}^c) = 0$ and

$$(12) \qquad 0 = \int_{\hat{G}} \hat{\lambda}_{\dot{x}}(\gamma) f_m(\gamma)\, d\gamma \quad \text{for all } \dot{x} \in \tilde{K} \text{ and } f_m \in \mathcal{Q}.$$

Hence,

$$(13) \qquad 0 = \int_{\hat{G}} \hat{\lambda}_{\dot{x}}(\gamma) f(\gamma)\, d\gamma \quad \text{for all } \dot{x} \in \tilde{K} \text{ and } f \in L^1(P),$$

which yields

$$\hat{\lambda}_{\dot{x}}(\gamma) = 0 \quad \text{on } P \text{ a.a. } \dot{x}(\eta).$$

This completes the proof.

LEMMA 2.5. *Let G be a σ-compact metrizable LCA group and H a closed subgroup of G. Let π be the natural homomorphism from G onto G/H. Let $\{\lambda_{\dot{x}}\}_{\dot{x} \in G/H}$ be a family in $M^+(G)$ with the following properties*:

(1) *$\dot{x} \mapsto \lambda_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on G,*

(2) $\operatorname{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(3) $\|\lambda_{\dot{x}}\| \leq 1$.

By (2), $\lambda_{\dot{x}} = \nu_{\dot{x}} * \delta_x$ for some $\nu_{\dot{x}} \in M^+(H)$ and $x \in G$ with $\pi(x) = \dot{x}$. We define measures $\lambda^a_{\dot{x}}, \lambda^s_{\dot{x}} \in M^+(G)$ as follows:

(4) $\lambda^a_{\dot{x}} = \nu^a_{\dot{x}} * \delta_x$, $\lambda^s_{\dot{x}} = \nu^s_{\dot{x}} * \delta_x$,

where $\nu^a_{\dot{x}}$ and $\nu^s_{\dot{x}}$ are the absolutely continuous part and the singular part of $\nu_{\dot{x}}$ with respect to m_H respectively. Then the following is satisfied:

(5) *$\dot{x} \mapsto \lambda^a_{\dot{x}}(f)$ and $\dot{x} \mapsto \lambda^s_{\dot{x}}(f)$ are Borel measurable functions for each bounded Borel function f on G.*

Proof. For $\dot{x} \in G/H$, let $L^1(\pi^{-1}(\{\dot{x}\}))$ be the space of functions on $\pi^{-1}(\{\dot{x}\})$ which are integrable with respect to $m_{\dot{x}}$, where $m_{\dot{x}}$ is the measure on the coset $\pi^{-1}(\{\dot{x}\})$ which is given by translating m_H on $\pi^{-1}(\{\dot{x}\})$.

Step 1. There exists a countable dense subset $\mathcal{C}$ of $L^1(G)$ such that $\mathcal{C}|_{\pi^{-1}(\{\dot{x}\})}$ is dense in $L^1(\pi^{-1}(\{\dot{x}\}))$ for each $\dot{x} \in G/H$.

Since G is σ-compact and metrizable, there exist open sets U_n in G with compact closures such that $\overline{U}_n \subset U_{n+1}$ and $\bigcup_1^\infty U_n = G$. Then, for each $n \in N$, there exists a countable set $\mathcal{C}_n$ in $C_c(G)$ such that

(6) $\operatorname{supp}(f) \subset U_n$ for $f \in \mathcal{C}_n$, $\mathcal{C}_n|_{U_n}$ is dense in $C_c(U_n)$ with respect to the supremum norm.

Now we put $\mathcal{C} = \bigcup_1^\infty \mathcal{C}_n$. Then, by (6), $\mathcal{C}$ is a countable dense subset of $L^1(G)$. Put $S_{n,\dot{x}} = \pi^{-1}(\{\dot{x}\}) \cap U_n$ and $B_{n,\dot{x}} = \{u \in C_c(\pi^{-1}(\{\dot{x}\}));\ \operatorname{supp}(u) \subset S_{n,\dot{x}}\}$.

Claim 1. $\mathcal{C}_n|_{S_{n,\dot{x}}}$ is dense in $B_{n,\dot{x}}$.

In fact, let u be a function in $B_{n,\dot{x}}$ and ε a positive real number. By Tietze's extension theorem, there exists a bounded continuous function k_n on G such that $k_n|_{\overline{S}_{n,\dot{x}}} = u|_{\overline{S}_{n,\dot{x}}}$, where $\overline{S}_{n,\dot{x}}$ is the closure of $S_{n,\dot{x}}$ in $\pi^{-1}(\{\dot{x}\})$. We choose an open set V_n in G and a nonnegative continuous function p_n on G with the compact support such that

$$\overline{V}_n \subset U_n \quad \text{and} \quad \operatorname{supp}(u) \subset V_n,$$

$$p_n = \begin{cases} 1 & \text{for } x \in \overline{V}_n, \\ 0 & \text{for } x \notin U_n \end{cases} \tag{7}$$

and $\|p_n\|_\infty \leq 1$. Put $u_n(x) = k_n(x)p_n(x)$. Then u_n is a continuous function on G such that $\operatorname{supp}(u_n) \subset U_n$. Moreover, by the construction of u_n, we have $u_n|_{S_{n,\dot{x}}} = u|_{S_{n,\dot{x}}}$. Since $\mathcal{C}_n|_{U_n}$ is dense in $C_c(U_n)$, there exists a function f_n in $\mathcal{C}_n$ such that $\|f_n|_{U_n} - u_n|_{U_n}\|_\infty < \varepsilon$. Hence we have

$$\begin{aligned} \left\|f_n|_{S_{n,\dot{x}}} - u|_{S_{n,\dot{x}}}\right\|_\infty &= \left\|f_n|_{S_{n,\dot{x}}} - u_n|_{S_{n,\dot{x}}}\right\|_\infty \\ &\leq \left\|f_n|_{U_n} - u_n|_{U_n}\right\|_\infty \\ &< \varepsilon. \end{aligned}$$

Thus Claim is proved.

We return to the proof of Step 1. Let f be a function in $L^1(\pi^{-1}(\{\dot{x}\}))$ and ε a positive real number. Since $\bigcup_1^\infty S_{n,\dot{x}} = \pi^{-1}(\{\dot{x}\})$, there exists a positive integer n such that $\int_{(S_{n,\dot{x}})^c} |f(y)|\, dm_{\dot{x}}(y) < \varepsilon/3$. We can also choose a function $f_n \in B_{n,\dot{x}}$ such that $\int_{S_{n,\dot{x}}} |f(y) - f_n(y)| \, dm_{\dot{x}}(y) < \varepsilon/3$. By Claim 1, there exists a function $g_n \in \mathcal{A}_n$ such that $\| g_n|_{S_{n,\dot{x}}} - f_n|_{S_{n,\dot{x}}} \|_\infty < \varepsilon/3(m_{\dot{x}}(S_{n,\dot{x}}) + 1)$. Noting $g_n|_{\pi^{-1}(\{\dot{x}\})}(y) = 0$ if $y \in \pi^{-1}(\{\dot{x}\}) \setminus S_{n,\dot{x}}$, we have

$$
\begin{aligned}
\int_{\pi^{-1}(\{\dot{x}\})} & |f(y) - g_n(y)| dm_{\dot{x}}(y) \\
&= \int_{\pi^{-1}(\{\dot{x}\}) \setminus S_{n,\dot{x}}} |f(y)| dm_{\dot{x}}(y) + \int_{S_{n,\dot{x}}} |f(y) - g_n(y)| dm_{\dot{x}}(y) \\
&< \varepsilon/3 + \int_{S_{n,\dot{x}}} |f(y) - f_n(y)| dm_{\dot{x}}(y) \\
&\qquad + \int_{S_{n,\dot{x}}} |f_n(y) - g_n(y)| dm_{\dot{x}}(y) \\
&< \varepsilon.
\end{aligned}
$$

Thus Step 1 is proved. In order to prove the lemma, it is sufficient to show that $\dot{x} \mapsto \lambda^s_{\dot{x}}(f)$ is a Borel measurable function for each $f \in C_0(G)$.

Step 2. For a nonnegative function $f \in C_0(G)$, $\dot{x} \mapsto \lambda^s_{\dot{x}}(f)$ is a Borel measurable function.

Let $\mathcal{A}$ be the countable subset of $L^1(G)$ obtained in Step 1 and $\mathcal{B}$ a countable dense subset of $C_0(G)$. Then we have

$$
\begin{aligned}
(8) \qquad \lambda^s_{\dot{x}}(f) &= \|f\lambda^s_{\dot{x}}\| \\
&= \inf_{g \in \mathcal{A}} \|f\lambda_{\dot{x}} - \chi_{\pi^{-1}(\{\dot{x}\})} g\| \\
&= \inf_{g \in \mathcal{A}} \sup_{\substack{h \in \mathcal{B}, \\ \|h\|_\infty \leq 1}} |\lambda_{\dot{x}}(fh) - (\chi_{\pi^{-1}(\{\dot{x}\})} g)(h)| \\
&= \inf_{g \in \mathcal{A}} \sup_{\substack{h \in \mathcal{B}, \\ \|h\|_\infty \leq 1}} \left| \lambda_{\dot{x}}(fh) - \int_{\pi^{-1}(\{\dot{x}\})} g(z)h(z) dm_{\dot{x}}(z) \right|.
\end{aligned}
$$

We note that $\int_{\pi^{-1}(\{\dot{x}\})} g(z)h(z)dm_{\dot{x}}(z) = \int_H g(\dot{x} + y)h(\dot{x} + y)dm_H(y)$. Hence, $\dot{x} \mapsto \int_{\pi^{-1}(\{\dot{x}\})} g(z)h(z)dm_{\dot{x}}(z)$ is a continuous function on G/H. Therefore, by (1)and (8), Step 2 is proved.

By Step 2, $\dot{x} \mapsto \lambda^s_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on G. This completes the proof.

LEMMA 2.6. *Let G be a σ-compact metrizable LCA group and P a closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let μ be a measure in $M_{P^c}(G)$. Then μ_a and μ_s belong to $M_{P^c}(G)$.*

Proof. Put $\Lambda = P \cap (-P)$ and $H = \Lambda^{\perp}$. Let π be the natural homomorphism from G onto G/H, and put $\eta = \pi(|\mu|)$. Then, by Lemma 2.1, there exists a family $\{\xi_{\dot{x}}\}_{\dot{x} \in G/H}$ in $M^+(G)$ with the following properties:

(1) $\dot{x} \mapsto \xi_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on G,

(2) $\mathrm{supp}(\xi_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(3) $\|\xi_{\dot{x}}\| \leq 1$,

(4) $|\mu|(g) = \int_{G/H} \xi_{\dot{x}}(g) d\eta(\dot{x})$ for each bounded Borel measurable function g on G.

Let h be a unimodular Borel measurable function on G such that $\mu = h|\mu|$. By (2), there exists a measure $\nu_{\dot{x}} \in M^+(H)$ and $x \in G$ such that $\pi(x) = \dot{x}$ and $\xi_{\dot{x}} = \nu_{\dot{x}} * \delta_x$. Let $\nu_{\dot{x}}^a$ and $\nu_{\dot{x}}^s$ be the absolutely continuous part and the singular part of $\nu_{\dot{x}}$ with respect to m_H respectively. We define measures $\xi_{\dot{x}}^a$ and $\xi_{\dot{x}}^s$ in $M^+(G)$ by $\xi_{\dot{x}}^a = \nu_{\dot{x}}^a * \delta_x$ and $\xi_{\dot{x}}^s = \nu_{\dot{x}}^s * \delta_x$. Put $\eta = \eta_a + \eta_s$, where $\eta_a \in L^1(G/H) \cap M^+(G/H)$ and $\eta_s \in M_s(G/H) \cap M^+(G/H)$. Then, by Lemma 2.5, we can define $\Phi_{aa}, \Phi_{sa}, \Phi_s \in M^+(G)$ as follows:

(5)
$$\Phi_{aa}(f) = \int_{G/H} \xi_{\dot{x}}^a(f) d\eta_a(\dot{x}),$$

$$\Phi_{sa}(f) = \int_{G/H} \xi_{\dot{x}}^s(f) d\eta_a(\dot{x}),$$

$$\Phi_s(f) = \int_{G/H} \xi_{\dot{x}}(f) d\eta_s(\dot{x}) \quad \text{for } f \in C_0(G).$$

Claim 1. $\Phi_{sa} \in M_s(G) \cap M^+(G)$.

We define a measure $\zeta_{\dot{x}}^s \in M_s(G) \cap M^+(G)$ as follows:

$$\zeta_{\dot{x}}^s = \begin{cases} (1/\|\xi_{\dot{x}}^s\|)\xi_{\dot{x}}^s & \text{if } \|\xi_{\dot{x}}^s\| \neq 0, \\ 0 & \text{if } \|\xi_{\dot{x}}^s\| = 0. \end{cases}$$

Then we have $\Phi_{sa}(f) = \int_{G/H} \zeta_{\dot{x}}^s(f) \|\xi_{\dot{x}}^s\| d\eta_a(\dot{x})$ for $f \in C_0(G)$. By Lemma 2.5, we can define a measure $\eta_a' \in L^1(G/H) \cap M^+(G/H)$ by $\eta_a'(\tilde{E}) = \int_{\tilde{E}} \|\xi_{\dot{x}}^s\| d\eta_a(\dot{x})$ for a Borel set $\tilde{E}$ in G/H. Then we have $\pi(\Phi_{sa}) = \eta_a'$. In fact, for $g \in C_0(G/H)$, we get

$$\begin{aligned}\pi(\Phi_{sa})(g) &= \int_G g \circ \pi(x)\, d\Phi_{sa}(x) \\ &= \int_{G/H} \zeta_{\dot{x}}^s(g \circ \pi)\|\xi_{\dot{x}}^s\|\, d\eta_a(\dot{x}) \\ &= \int_{G/H} g(\dot{x})\|\xi_{\dot{x}}^s\|\, d\eta_a(\dot{x}) \\ &= \int_{G/H} g(\dot{x})\, d\eta_a'(\dot{x}).\end{aligned}$$

Hence, for $\{\zeta_{\dot{x}}^s\}_{\dot{x} \in G/H}$ and η_a', we have

(6) $\pi(\Phi_{sa}) = \eta_a'$,

(7) $\dot{x} \mapsto \zeta_{\dot{x}}^s(f)$ is a Borel measurable function for each bounded Borel function f on G,

(8) $\operatorname{supp}(\zeta_{\dot{x}}^s) \subset \pi^{-1}(\{\dot{x}\})$,

(9) $\|\zeta_{\dot{x}}^s\| \leq 1$,

(10) $\Phi_{sa}(g) = \int_{G/H} \zeta_{\dot{x}}^s(g)\, d\eta_a'(\dot{x})$ for each bounded Borel measurable function g on G

and

(11) $\zeta_{\dot{x}}^s * \delta_{-x} \in M_s(H)$, where x is an element in G such that $\pi(x) = \dot{x}$.

Hence, by (6)–(11) and Lemma 2.2, Claim 1 is proved.

Claim 2. $\Phi_s \in M_s(G) \cap M^+(G)$.

This is obtained from Lemma 2.3.

Claim 3. $\Phi_{aa} \in L^1(G)$.

Let E be a Borel measurable set in G such that $m_G(E) = 0$. Then, since

$$0 = m_G(E) = \int_{G/H} \int_H \chi_E(\dot{x} + y)\, dm_H(y)\, dm_{G/H}(\dot{x}),$$

there exists a Borel set $\tilde{F}$ in G/H with $m_{G/H}(\tilde{F}) = 0$ such that $m_{\dot{x}}(E \cap \pi^{-1}(\{\dot{x}\})) = 0$ if $\dot{x} \notin \tilde{F}$, where $m_{\dot{x}}$ is the measure on the coset $\pi^{-1}(\{\dot{x}\})$ translated m_H on $\pi^{-1}(\{\dot{x}\})$. Then we have

$$\begin{aligned}\Phi_{aa}(E) &= \int_{G/H} \xi_{\dot{x}}^a(\chi_E)\, d\eta_a(\dot{x}) \\ &= \int_{\tilde{F}} \xi_{\dot{x}}^a(\chi_E)\, d\eta_a(\dot{x}) + \int_{\tilde{F}^c} \xi_{\dot{x}}^a(\chi_E)\, d\eta_a(\dot{x}). \\ &= 0.\end{aligned}$$

Thus Claim 3 is proved.

We define a measure $\lambda_{\dot{x}} \in M(G)$ by $\lambda_{\dot{x}}(f) = \xi_{\dot{x}}(hf)$ for $f \in C_0(G)$, where h is the unimodular Borel function on G such that $\mu = h \mid \mu \mid$. Then the following are satisfied:

(12) $\dot{x} \mapsto \lambda_{\dot{x}}(f)$ is a Borel measurable function for each bounded Borel measurable function f on G,

(13) $\operatorname{supp}(\lambda_{\dot{x}}) = \operatorname{supp}(\xi_{\dot{x}}) \subset \pi^{-1}(\{\dot{x}\})$,

(14) $\|\lambda_{\dot{x}}\| \leq 1$,

(15) $\mu(g) = \int_{G/H} \lambda_{\dot{x}}(g) d\eta(\dot{x})$ for each bounded Borel measurable function g on G.

We define measures $\lambda^a_{\dot{x}}$, $\lambda^s_{\dot{x}} \in M(G)$ by $\lambda^a_{\dot{x}} = h\xi^a_{\dot{x}}$ and $\lambda^s_{\dot{x}} = h\xi^s_{\dot{x}}$ respectively. Then we have

$$\lambda_{\dot{x}} = \lambda^a_{\dot{x}} + \lambda^s_{\dot{x}} \quad \text{for } \dot{x} \in G/H, \text{ and}$$

(16) $\lambda^a_{\dot{x}}$ and $\lambda^s_{\dot{x}}$ are absolutely continuous and singular with respect to $m_{\dot{x}}$ respectively.

By (13), there exist an element x in G with $\pi(x) = \dot{x}$ and a measure $\omega_{\dot{x}} \in M(H)$ such that $\lambda_{\dot{x}} = \omega_{\dot{x}} * \delta_x$, $\lambda^a_{\dot{x}} = \omega^a_{\dot{x}} * \delta_x$ and $\lambda^s_{\dot{x}} = \omega^s_{\dot{x}} * \delta_x$, where $\omega^a_{\dot{x}}$ and $\omega^s_{\dot{x}}$ are the absolutely continuous part and the singular part of $\omega_{\dot{x}}$ with respect to m_H respectively. Since $\hat{\mu}(\gamma) = 0$ on P, by Lemma 2.4, we have

$$\hat{\lambda}_{\dot{x}}(\gamma) = 0 \quad \text{on } P \text{ a.a. } \dot{x}(\eta), \tag{17}$$

hence

$$\hat{\omega}_{\dot{x}}(\gamma) = 0 \quad \text{on } P \text{ a.a. } \dot{x}(\eta). \tag{18}$$

Let β be the natural homomorphism from $\hat{G}$ onto $\hat{G}/\Lambda$. Then $\beta(P)$ is a closed semigroup in $\hat{G}/\Lambda$. We note that $\beta(P)$ induces a totally order on $\hat{G}/\Lambda$, and moreover, $\beta(P) = \{\beta(\gamma) \in \hat{G}/\Lambda;\ \beta(\gamma) \geq 0\}$. Hence, by (18) and Theorem B, we have

$$\widehat{\omega^a_{\dot{x}}}(\gamma) = \widehat{\omega^s_{\dot{x}}}(\gamma) = 0 \quad \text{on } P \text{ a.a. } \dot{x}(\eta), \tag{19}$$

hence

$$\widehat{\lambda^a_{\dot{x}}}(\gamma) = \widehat{\lambda^s_{\dot{x}}}(\gamma) = 0 \quad \text{on } P \text{ a.a. } \dot{x}(\eta). \tag{20}$$

On the other hand, by Lemma 2.5 and the construction of $\lambda^a_{\dot{x}}$ and $\lambda^s_{\dot{x}}$, $\dot{x} \mapsto \lambda^a_{\dot{x}}(f)$ and $\dot{x} \mapsto \lambda^s_{\dot{x}}(f)$ are Borel measurable functions for each bounded Borel measurable function f on G. Hence we can define measures

$\mu_i \in M(G)$ $(i = 1, 2, 3)$ as follows:

$$(21) \qquad \mu_1(f) = \int_{G/H} \lambda^a_{\dot{x}}(f)\, d\eta_a(\dot{x}),$$

$$\mu_2(f) = \int_{G/H} \lambda^s_{\dot{x}}(f)\, d\eta_a(\dot{x}),$$

$$\mu_3(f) = \int_{G/H} \lambda_{\dot{x}}(f)\, d\eta_s(\dot{x}) \quad \text{for } f \in C_0(G).$$

Then $\mu = \mu_1 + \mu_2 + \mu_3$, and, by the construction of $\lambda_{\dot{x}}$, $\lambda^a_{\dot{x}}$ and $\lambda^s_{\dot{x}}$, we have

$$\mu_1 \ll \Phi_{aa}, \quad \mu_2 \ll \Phi_{sa} \quad \text{and} \quad \mu_3 \ll \Phi_s.$$

Therefore, by Claims 1–3, we have $\mu_a = \mu_1$ and $\mu_s = \mu_2 + \mu_3$. By (20) and (21), we can easily verify that $\mu_i \in M_{P^c}(G)$ $(i = 1, 2, 3)$. Hence we have μ_a, $\mu_s \in M_{P^c}(G)$ and the proof is complete.

3. Proof of Main Theorem.

LEMMA 3.1. *Let G be a metrizable LCA group and P a proper closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let μ be a measure in $M(G)$. Then there exists a σ-compact open subgroup G_1 of G such that* (1) $\operatorname{supp}(\mu) \subset G_1$ *and* (2) $G_1^\perp \subset P \cap (-P)$.

Proof. Put $\Lambda = P \cap (-P)$, and let β be the natural homomorphism from $\hat{G}$ onto $\hat{G}/\Lambda$. Then $\beta(P)$ is a closed semigroup in $\hat{G}/\Lambda$ such that (i) $\beta(P) \cup (-\beta(P)) = \hat{G}/\Lambda$ and (ii) $\beta(P) \cap (-\beta(P)) = \{0\}$. Hence, by ([**8**], 8.1.5. Theorem), we have

$$(3) \qquad \hat{G}/\Lambda = D, \quad \text{or} \quad \hat{G}/\Lambda = R \oplus D,$$

where D is a discrete abelian group which is torsion-free. Put $H = \Lambda^\perp$. Then, by (3), H is a σ-compact closed subgroup of G. Since μ is regular, there exists a σ-compact open subgroup G_0 of G such that $\operatorname{supp}(\mu) \subset G_0$. We put $G_1 = G_0 + H$. Then G_1 is a σ-compact open subgroup of G which satisfies (1) and (2). This completes the proof.

LEMMA 3.2. *Let G be a metrizable LCA group and P a closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let μ be a measure in $M_{P^c}(G)$. Then μ_a and μ_s belong to $M_{P^c}(G)$.*

Proof. We may assume that $P \subsetneqq \hat{G}$. Let G_1 be the σ-compact open subgroup of G obtained in Lemma 3.1. Let π be the natural homomorphism from $\hat{G}$ onto $\hat{G}/G_1^{\perp}$. Then, by (2) in Lemma 3.1, $\pi(P)$ is a closed semigroup in $\hat{G}/G_1^{\perp}$ such that $\pi(P) \cup (-\pi(P)) = \hat{G}/G_1^{\perp}$. We can regard μ as a measure in $M_{\pi(P)^c}(G_1)$. Since G_1 is σ-compact and metrizable, by Lemma 2.6, we have $\mu_a, \mu_s \in M_{\pi(P)^c}(G_1)$, which yields $\mu_a, \mu_s \in M_{P^c}(G)$. This completes the proof.

Now we prove the main theorem of this paper.

THEOREM 3.3 (*Main Theorem*). *Let G be a LCA group and P a closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let μ be a measure in $M_{P^c}(G)$. Then μ_a and μ_s belong to $M_{P^c}(G)$.*

Proof. It is sufficient to show that $\mu_s \in M_{P^c}(G)$. Let γ_0 be an element in P. Since μ_s is a singular measure, there exists a σ-compact subset E of G such that $m_G(E) = 0$ and $|\mu_s|(E^c) = 0$. Then, by ([7], Lemma 4), there exists a σ-compact open subgroup Γ of $\hat{G}$ containing γ_0 such that

(1) $$m_G(E + \Gamma^{\perp}) = 0.$$

Let π be the natural homomorphism from G onto $G/\Gamma^{\perp}$. Then, by (1), we have

(2) $$\pi(\mu)_s = \pi(\mu_s).$$

Put $P_\Gamma = P \cap \Gamma$. Then P_Γ is a closed semigroup in Γ such that $P_\Gamma \cup (-P_\Gamma) = \Gamma$, and $\pi(\mu)$ belongs to $M_{P_\Gamma^c}(G/\Gamma^{\perp})$. Since $G/\Gamma^{\perp}$ is metrizable, by (2) and Lemma 3.2, we have $\pi(\mu_s) = \pi(\mu)_s \in M_{P_\Gamma^c}(G/\Gamma^{\perp})$, so that $\hat{\mu}_s(\gamma_0) = \pi(\mu_s)\hat{\ }(\gamma_0) = 0$. Since γ_0 is an arbitrary element in P, we have $\mu_s \in M_{P^c}(G)$. This completes the proof.

REMARK 3.4. In the proof of Lemma 2.6, when $\hat{G}/\Lambda$ is not discrete, we needed Theorem B. However, in this case, we have $\hat{G}/\Lambda \cong R \oplus D$ and $\beta(P) \cong \{(x, d) \in R \oplus D;\ d > 0, \text{ or } d = 0 \text{ and } x \geq 0\}$, where D is a discrete ordered group (cf. [8], 8.1.5. Theorem). Using Theorem A and our method, we can prove Theorem B if P is closed. Hence the Main Theorem can be obtained by employing only Theorem A.

Appendix. The author has recently extended Theorem A(II) as follows (cf. [**10**], Lemma 1.2):

THEOREM 3.5. *Let G be a LCA group and P a semigroup in $\hat{G}$ such that $P \cup (-P) = G$. Put $\Lambda = P \cap (-P)$ and $H = \Lambda^{\perp}$. If P is open, then we have*

$$(*) \qquad m_H * \{M_P(G) \cap M_s(G)\} \subset M_P(G) \cap M_s(G).$$

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