

# Pacific Journal of Mathematics

## **THE MAXIMAL ERGODIC HILBERT TRANSFORM WITH WEIGHTS**

ENRIQUE ATENCIA AND FRANCISCO JAVIER MARTIN-REYES

# THE MAXIMAL ERGODIC HILBERT TRANSFORM WITH WEIGHTS

E. ATENCIA AND F. J. MARTIN-REYES

**This work is concerned with the characterization of those positive functions,  $w$ , such that the ergodic maximal Hilbert transform associated to an invertible, measure preserving, ergodic transformation on a probability space, is a bounded operator in  $L_p(wd\mu)$ .**

**1. Introduction.** Let  $(X, \mathfrak{F}, \mu)$  be a non-atomic probability space, and let  $T: X \rightarrow X$  be an ergodic, invertible, measure preserving transformation. We consider the ergodic maximal Hilbert transform associated to  $T$  defined by

$$(1.1) \quad Hf(x) = \sup_{s, t \geq 0} \left| \sum_{s < |i| < t} \frac{f(T^i x)}{i} \right| \quad (s, t \in \mathbf{Z})$$

and acting on measurable functions. Our main result is given by the following theorem.

(1.2) **THEOREM.** *Let  $w$  be a positive integrable function. Then  $f \rightarrow Hf$  is bounded on  $L_p(wd\mu)$  if and only if  $w$  satisfies condition  $A'_p$ , i.e., there exists a constant  $M$  such that for a.e.  $x \in X$  and for all positive integers  $k$*

$$(1.3) \quad k^{-1} \sum_{i=0}^{k-1} w(T^i x) \cdot \left[ k^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \right]^{p-1} \leq M.$$

**2. Main results.** In this section we will prove the theorem above stated using the concept of ergodic rectangle and some ideas in (3) adapted to our context.

(2.1) **DEFINITION.** Let  $B$  be a subset of  $X$  with positive measure and  $k$  a positive integer such that

$$T^i B \cap T^j B = \emptyset, \quad i \neq j, 0 \leq i, j \leq k-1.$$

Then the set  $R = \bigcup_{i=0}^{k-1} T^i B$  will be called an “(ergodic) rectangle” with base  $B$  and length  $k$ .

Obviously  $\mu(R) = k\mu(B)$ .

In the proof of the theorem we will need the following two results which have been proved in (1).

(2.2) PROPOSITION. *Let  $k$  be a positive integer and let  $A \subset X$  be a subset with positive measure. Then there exists  $B \subset A$  such that  $B$  is base of a rectangle of length  $k$ .*

(2.3) LEMMA. *For any positive integer  $k$ ,  $X$  can be written as a countable union of bases of rectangles of length  $k$ .*

**The boundedness of the operator  $f \rightarrow Hf$  on  $L_p(wd\mu)$ ,  $p > 1$ , implies  $w$  satisfies  $A'_p$ .** Let  $k$  be a positive integer and let's fix a rectangle with base  $B$  and length  $4k$ . We consider, for each integer  $n$ , the subset of  $B$  given by

$$(2.4) \quad B_n = \left\{ x \in B : 2^n \leq (2k)^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} < 2^{n+1} \right\}.$$

Its obvious that  $B = \bigcup_n B_n$ .

Now fix  $n$  and let  $A \subset B_n$  be an arbitrary measurable subset with positive measure. Consider

$$Q_1 = A \cup TA \cup \dots \cup T^{k-1}A,$$

$$Q_2 = T^kA \cup T^{k+1}A \cup \dots \cup T^{2k-1}A.$$

If  $f$  is a non-negative function we have

$$(2.5) \quad Hf(T^j x) \geq (2k)^{-1} \sum_{l=0}^{k-1} f(T^l x)$$

$(x \in A, \sup f \subset Q_1, k \leq j \leq 2k - 1),$

$$(2.6) \quad Hf(T^j x) \geq (2k)^{-1} \sum_{l=k}^{2k-1} f(T^l x)$$

$(x \in A, \sup f \subset Q_2, 0 \leq j \leq k - 1).$

Applying (2.6) to  $\chi_{Q_2}$  we obtain

$$(2.7) \quad Hf(T^j x) \geq \frac{1}{2} \quad (x \in A, 0 \leq j \leq k - 1).$$

It follows immediately that

$$(2.8) \quad \left(\frac{1}{2}\right)^p \int_A w(T^j x) d\mu \leq \int_A (Hf(T^j x))^p w(T^j x) d\mu.$$

Summing over  $j$ ,  $j = 0, \dots, k - 1$ , and using the boundedness of our operator we have

$$(2.9) \quad \int_{Q_1} w d\mu \leq 2^p C \int_{Q_2} w d\mu.$$

Throughout this paper  $C$  will denote an universal constant not necessarily the same at each occurrence. Applying now (2.5) to  $f = w^{-1/(p-1)}\chi_{Q_1}$  we find that

$$(2.10) \quad Hf(T^jx) \geq (2k)^{-1} \sum_{l=0}^{k-1} w(T^l x)^{-1/(p-1)} \geq 2^n,$$

since  $k \leq j \leq 2k - 1$  and  $x \in A \subset B_n$ . Thus, for  $f = w^{-1/(p-1)}\chi_{Q_1}$  it follows that

$$(2.11) \quad 2^{np} \int_A w(T^jx) \, d\mu \leq \int_A Hf(T^jx)^p w(T^jx) \, d\mu.$$

Adding up in  $j$  for  $j = k, \dots, 2k - 1$  and applying again our assumption of boundedness we can write

$$2^{np} \int_{Q_2} w \, d\mu \leq C \int_{Q_1} w^{-1/(p-1)} \, d\mu$$

which, because of (2.9) yields

$$(2.12) \quad 2^{np} \int_{Q_1} w \, d\mu \cdot \left( \int_{Q_1} w^{-1/(p-1)} \, d\mu \right)^{-1} \leq 2^p C^2.$$

On the other hand we also have:

$$\mu(A)^{-1} \int_A (2k)^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \leq 2^{n+1},$$

raising to the power  $p$  and applying (2.12) it follows that

$$\begin{aligned} & \left( (k\mu(A))^{-1} \int_A \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \right)^p \\ & \circ \int_{Q_1} w \, d\mu \left( \int_{Q_1} w^{-1/(p-1)} \, d\mu \right)^{-1} \leq 2^{3p} C^2 \end{aligned}$$

or equivalently

$$\begin{aligned} & \left( \mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \right)^{p-1} \\ & \cdot \left( \mu(A)^{-1} \int_A k^{-1} \sum_{i=0}^{k-1} w(T^i x) \, d\mu \right) \leq 2^{3p} C. \end{aligned}$$

This, immediately, gives

$$k^{-1} \sum_{i=0}^{k-1} w(T^i x) \circ \left( k^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \right)^{p-1} \leq 2^{3p} C^2 \quad (\text{a.e. in } B_n).$$

Now a straightforward application of Lemma (2.3) gives us that  $w$  satisfies condition  $A'_p$ .

In order to prove the converse we first assume that  $w$  satisfies condition  $A'_\infty$  and for that we mean that there are positive constants  $C, \delta > 0$  so that given any finite set  $I$  consisting of consecutive integers and any subset  $E \subset I$

$$\frac{\sum_{i \in E} w(T^i x)}{\sum_{i \in I} w(T^i x)} \leq C \left( \frac{\#E}{\#I} \right)^\delta \quad (\text{a.e. in } X)$$

where  $\#E$  is the number of elements of  $E$ .

In the following the subsets  $I$  above described will be called intervals in the integers. Theorem (1.2) will, then, be a consequence of the following results:

(2.13). THEOREM. *If  $w$  satisfies  $A'_\infty$  then*

$$(2.14) \quad \int_X (Hf)^p w \, d\mu \leq C \int_X (f^*)^p w \, d\mu$$

where  $f^*$  is the ergodic no centered maximal function associated to the transformation  $T$ .

(2.15). LEMMA. *Condition  $A'_p$  implies condition  $A'_\infty$ .*

(2.16). THEOREM.

$$\int_X (f^*)^p w \, d\mu \leq C \int_X |f|^p w \, d\mu, \quad \text{if } w \text{ satisfies } A'_p.$$

Theorem (2.16) has been proved in (1).

The proof of Lemma (2.15) runs as follows:

Let's call  $I$  to the interval  $\{0, 1, \dots, k - 1\}$  and let  $E$  be an arbitrary subset of  $I$ .

It was shown in (1) that if  $w$  satisfies  $A'_p$  then the following “reverse Hölder” inequality holds:

$$(2.17) \quad k^{-1} \sum_{j=0}^{k-1} w(T^j x)^v \leq C k^{-v} \left( \sum_{j=0}^{k-1} w(T^j x) \right)^v,$$

with constants  $C, v > 1$  independent of  $k$ .

Applying Hölder’s inequality we obtain

$$\begin{aligned} \sum_{j \in E} w(T^j k) &\leq \left( \sum_{j \in E} w(T^j x)^v \right)^{1/v} (\#E)^{1-1/v} \\ &\leq \left( \sum_{j=0}^{k-1} w(T^j x)^v \right)^{1/v} (\#E)^{1-1/v}. \end{aligned}$$

The result now holds using inequality (2.17).

In the proof of Theorem (2.13) we will use the fact (4) that there exists a constant  $C$  such that for any sequence  $\{b_k\}_{k=-\infty}^{\infty}$  and any  $\lambda > 0$  holds

$$(2.18) \quad \sum_{k: Hb_k > \lambda} \leq \frac{C}{\lambda} \cdot \sum_{k=-\infty}^{+\infty} |b_k|$$

where

$$Hb_k = \sup_{s, t \geq 0} \left| \sum_{s < |k-j| < t} \frac{b_j}{k-j} \right| \quad (s, t \in \mathbf{Z}).$$

Combining this result with condition  $A'_{\infty}$  we will prove, for any  $f \in L^1(d\mu)$ , the following fundamental inequality

$$(2.19) \quad \int_{\{x: Hf(x) > \beta\lambda, f^*(x) \leq \gamma\lambda\}} \leq C \left( \frac{\gamma}{\beta'} \right)^{\delta} \int_{\{x: Hf(x) > \lambda\}} w \, d\mu.$$

where  $\beta'$  depends on  $\beta$  and  $\gamma$ .

If  $\mu\{x: Hf(x) > \lambda\} = 1$  the weak type  $(1-1)$  of  $H$  with respect to the measure  $\mu$  tells us

$$1 \leq \frac{C}{\lambda} \int_X |f| \, d\mu$$

and choosing  $\gamma < C^{-1}$  we have

$$\gamma\lambda < \int_X |f| \, d\mu.$$

By the individual ergodic theorem:

$$\gamma\lambda < f^*(x) \quad \text{a.e. in } X$$

and that implies (2.19)

Therefore we may assume that  $\mu\{x: Hf(x) > \lambda\} < 1$ . In particular, if

$$D = \{x: T^i x \in O_{\lambda}: i = 0, -1, -2, \dots\}$$

where  $O_{\lambda} = \{x: Hf(x) > \lambda\}$ , then  $\mu(D) = 0$ , since  $T$  is ergodic.

From this fact is clear that if we call

$$B_i = \{x: x, Tx, \dots, T^{i-1}x \in O_\lambda, T^{-1}x, T^i x \notin O_\lambda\}$$

and  $R_i = B_i \cup \dots \cup T^{i-1}B_i$  then  $O_\lambda = \bigcup_{i=1}^\infty R_i$  (a.e.).

The former decomposition of  $O_\lambda$  and the study of distribution function inequalities in the integers (2), that we now proceed to develop, will be used in the proof of (2.19). So we consider a function  $F$  defined in the integers and the associated maximal Hilbert transform

$$(2.20) \quad HF(k) = \sup_{s, t \geq 0} \left| \sum_{s < |k-j| < t} \frac{F(j)}{k-j} \right| \quad (s, t \in \mathbf{Z})$$

and the maximal function

$$(2.21) \quad F^*(k) = \sup_{n, m \geq 0} \frac{1}{n+m+1} \sum_{j=-n}^m |F(k+j)|.$$

Let  $\lambda$  be a positive number. The set

$$\{k: HF(k) > \lambda\}$$

can be written as a countable union of disjoint intervals  $I_i$  in the integers and of maximum length. In this situation we can state the following lemma.

(2.22). LEMMA. *There exists positive constants  $C$  and  $C'$  such that*

$$\#\{j \in I_i: HF(j) > \beta\lambda, F^*(j) \leq \gamma\lambda\} \leq C \frac{\gamma}{\beta - 1 - \gamma C'} \#I_i$$

for any  $I_i$  and where  $\beta$  is bigger than 1.

For the proof just look at the proof of inequality (4) in (2) and write it in the integers.

*Proof of inequality (2.19).* For  $n$  fixed we call  $E_{n,l}$  the nonempty subsets of  $\{0, 1, \dots, n-1\}$  ( $l = 1, 2, \dots, 2^n - 1$ ).

For each  $x$  of  $B_n$  we write

$$E_n^x = \{i: 0 \leq i \leq n-1: HF(T^i x) > \beta\lambda, f^*(T^i x) \leq \gamma\lambda\}$$

and

$$B_{n,l} = \{x \in B_n: E_n^x = E_{n,l}\}.$$

By Lemma (2.22) if  $x \in B_n$  we have

$$\#E_n^x \leq \frac{C\gamma}{\beta'} \#\{0, 1, \dots, n-1\}$$

which implies

$$\sum_{j \in E_n^x} w(T^j x) \leq C \cdot \left(\frac{\gamma}{\beta'}\right)^{\delta n-1} \sum_{j=0}^{\delta n-1} w(T^j x) \quad (x \in B_n)$$

since  $w$  satisfies  $A'_\infty$ . Integrating over  $B_n$  we obtain

$$\int_{\cup_{j \in E_{n,l}} T^j B_{n,l}} w d\mu \leq C \cdot \left(\frac{\gamma}{\beta'}\right)^{\delta} \int_{\cup_{j=0}^{n-1} T^j B_{n,l}} w d\mu.$$

Summing first over  $l$  and then over  $n$  and keeping in mind that  $O_\lambda = \cup_{n=1}^\infty R_n$  (a.e.) we get inequality (2.19).

As is well known a standard argument shows that the “good- $\lambda$  inequality” (2.19) implies (2.14) (see for example (2)). Therefore we have Theorem (2.13) for  $f$  in  $L^1(d\mu)$ .

Theorem (1.2) now follows combining Theorem (2.16) with standard density arguments.

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