THE MAXIMAL ERGODIC HILBERT TRANSFORM WITH WEIGHTS

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This work is concerned with the characterization of those positive functions, \( w \), such that the ergodic maximal Hilbert transform associated to an invertible, measure preserving, ergodic transformation on a probability space, is a bounded operator in \( L^p(wd\mu) \).

1. Introduction. Let \( (X, \mathcal{F}, \mu) \) be a non-atomic probability space, and let \( T: X \to X \) be an ergodic, invertible, measure preserving transformation. We consider the ergodic maximal Hilbert transform associated to \( T \) defined by

\[
Hf(x) = \sup_{s, t \geq 0} \left| \sum_{s < |i| < t} f(T^i(x)) \right| \quad (s, t \in \mathbb{Z})
\]

and acting on measurable functions. Our main result is given by the following theorem.

(1.2) Theorem. Let \( w \) be a positive integrable function. Then \( f \to Hf \) is bounded on \( L^p(wd\mu) \) if and only if \( w \) satisfies condition \( A'_p \), i.e., there exists a constant \( M \) such that for a.e. \( x \in X \) and for all positive integers \( k \)

\[
k^{-1} \sum_{i=0}^{k-1} w(T^i(x)) \cdot \left[ k^{-1} \sum_{i=0}^{k-1} w(T^i(x))^{-1/(p-1)} \right]^{p-1} \leq M.
\]

2. Main results. In this section we will prove the theorem above stated using the concept of ergodic rectangle and some ideas in (3) adapted to our context.

(2.1) Definition. Let \( B \) be a subset of \( X \) with positive measure and \( k \) a positive integer such that

\[
T^iB \cap T^jB = \emptyset, \quad i \neq j, \quad 0 \leq i, j \leq k - 1.
\]

Then the set \( R = \bigcup_{i=0}^{k-1} T^iB \) will be called an "(ergodic) rectangle" with base \( B \) and length \( k \).

Obviously \( \mu(R) = k\mu(B) \).

In the proof of the theorem we will need the following two results which have been proved in (1).
(2.2) **Proposition.** Let $k$ be a positive integer and let $A \subset X$ be a subset with positive measure. Then there exists $B \subset A$ such that $B$ is base of a rectangle of length $k$.

(2.3) **Lemma.** For any positive integer $k$, $X$ can be written as a countable union of bases of rectangles of length $k$.

The boundedness of the operator $f \to Hf$ on $L_p(wd\mu)$, $p > 1$, implies $w$ satisfies $A'_p$. Let $k$ be a positive integer and let's fix a rectangle with base $B$ and length $4k$. We consider, for each integer $n$, the subset of $B$ given by

$$B_n = \left\{ x \in B: 2^n \leq (2k)^{-1} \sum_{l=0}^{k-1} w(T^l x)^{-1/(p-1)} < 2^{n+1} \right\}. \quad (2.4)$$

It's obvious that $B = \bigcup_n B_n$.

Now fix $n$ and let $A \subset B_n$ be an arbitrary measurable subset with positive measure. Consider

$$Q_1 = A \cup TA \cup \ldots \cup T^{k-1}A,$$

$$Q_2 = T^k A \cup T^{k+1} A \cup \ldots \cup T^{2k-1} A.$$ If $f$ is a non-negative function we have

$$Hf(T^j x) \geq (2k)^{-1} \sum_{l=0}^{k-1} f(T^l x) \quad (x \in A, \sup f \subset Q_1, k \leq j \leq 2k - 1), \quad (2.5)$$

$$Hf(T^j x) \geq (2k)^{-1} \sum_{l=k}^{2k-1} f(T^l x) \quad (x \in A, \sup f \subset Q_2, 0 \leq j \leq k - 1). \quad (2.6)$$

Applying (2.6) to $\chi_{Q_2}$ we obtain

$$Hf(T^j x) \geq \frac{1}{2} \quad (x \in A, 0 \leq j \leq k - 1). \quad (2.7)$$

It follows immediately that

$$\left( \frac{1}{2} \right)^p \int_A w(T^j x) \, d\mu \leq \int_A (Hf(T^j x))^p w(T^j x) \, d\mu. \quad (2.8)$$

Summing over $j$, $j = 0, \ldots, k - 1$, and using the boundedness of our operator we have

$$\int_{Q_1} w \, d\mu \leq 2^p C \int_{Q_2} w \, d\mu. \quad (2.9)$$
Throughout this paper \( C \) will denote an universal constant not necessarily the same at each occurrence. Applying now (2.5) to \( f = w^{-1/(p-1)} \chi_{Q_1} \) we find that

\[
Hf(T^j x) \geq (2k)^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \geq 2^n,
\]

since \( k \leq j \leq 2k - 1 \) and \( x \in A \subset B_n \). Thus, for \( f = w^{-1/(p-1)} \chi_{Q_1} \) it follows that

\[
2^n \int_A w(T^j x) \, d\mu \leq \int_A Hf(T^j x)^p w(T^j x) \, d\mu.
\]

Adding up in \( j \) for \( j = k, \ldots, 2k - 1 \) and applying again our assumption of boundedness we can write

\[
2^n \int_{Q_2} w \, d\mu \leq C \int_{Q_1} w^{-1/(p-1)} \, d\mu
\]

which, because of (2.9) yields

\[
2^n \int_{Q_1} w \, d\mu \cdot \left( \int_{Q_1} w^{-1/(p-1)} \, d\mu \right)^{-1} \leq 2^n C^2.
\]

On the other hand we also have:

\[
\mu(A)^{-1} \int_A (2k)^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \leq 2^{n+1},
\]

raising to the power \( p \) and applying (2.12) it follows that

\[
\left( \left( k \mu(A) \right)^{-1} \int_A \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \right)^p
\]

\[
\cdot \int_{Q_1} w \, d\mu \left( \int_{Q_1} w^{-1/(p-1)} \, d\mu \right)^{-1} \leq 2^{3p} C^2
\]

or equivalently

\[
\left( \mu(A)^{-1} \int_A \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \, d\mu \right)^{p-1}
\]

\[
\cdot \left( \mu(A)^{-1} \int_A \sum_{i=0}^{k-1} w(T^i x) \, d\mu \right) \leq 2^{3p} C.
\]
This, immediately, gives
\[
k^{-1} \sum_{i=0}^{k-1} w(T^i x) \circ \left( k^{-1} \sum_{i=0}^{k-1} w(T^i x)^{-1/(p-1)} \right)^{p-1} \leq 2^3 p C^2 \quad \text{(a.e. in } B_n \text{)}.
\]

Now a straightforward application of Lemma (2.3) gives us that \( w \) satisfies condition \( A'_{\infty} \).

In order to prove the converse we first assume that \( w \) satisfies condition \( A'_{\infty} \) and for that we mean that there are positive constants \( C, \delta > 0 \) so that given any finite set \( I \) consisting of consecutive integers and any subset \( E \subseteq I \)
\[
\frac{\sum_{i \in E} w(T^i x)}{\sum_{i \in I} w(T^i x)} \leq C \left( \frac{\# E}{\# I} \right)^{\delta} \quad \text{(a.e. in } X\text{)}
\]
where \( \# E \) is the number of elements of \( E \).

In the following the subsets \( I \) above described will be called intervals in the integers. Theorem (1.2) will, then, be a consequence of the following results:

(2.13). Theorem. If \( w \) satisfies \( A'_{\infty} \) then
\[
\int_X (Hf)^p w d\mu \leq C \int_X (f*)^p w d\mu
\]
where \( f^* \) is the ergodic no centered maximal function associated to the transformation \( T \).

(2.15). Lemma. Condition \( A'_p \) implies condition \( A'_{\infty} \).

(2.16). Theorem.
\[
\int_X (f*)^p w d\mu \leq C \int_X |f|^p w d\mu, \quad \text{if } w \text{ satisfies } A'_p.
\]

Theorem (2.16) has been proved in (1).

The proof of Lemma (2.15) runs as follows:

Let's call \( I \) to the interval \( \{0, 1, \ldots, k-1\} \) and let \( E \) be an arbitrary subset of \( I \).

It was shown in (1) that if \( w \) satisfies \( A'_p \) then the following "reverse Hölder" inequality holds:
\[
k^{-1} \sum_{j=0}^{k-1} w(T^j x)^v \leq C k^{-v} \left( \sum_{j=0}^{k-1} w(T^j x) \right)^v,
\]
with constants \( C, v > 1 \) independent of \( k \).
Applying Hölder's inequality we obtain
\[
\sum_{j \in E} w(T^j k) \leq \left( \sum_{j \in E} w(T^j x)^\frac{1}{v} \right)^{1/v} (\# E)^{1 - 1/v} \\
\leq \left( \sum_{j=0}^{k-1} w(T^j x)^\frac{1}{v} \right)^{1/v} (\# E)^{1 - 1/v}.
\]

The result now holds using inequality (2.17).

In the proof of Theorem (2.13) we will use the fact (4) that there exists a constant \( C \) such that for any sequence \( \{b_k\}_{k=-\infty}^{\infty} \) and any \( \lambda > 0 \) holds
\[
(2.18) \quad \sum_{k : Hb_k > \lambda} \leq \frac{C}{\lambda} \cdot \sum_{k=-\infty}^{+\infty} |b_k|
\]
where
\[
Hb_k = \sup_{s,t \geq 0} \left| \sum_{s < |k-j| < t} \frac{b_j}{k-j} \right| \quad (s, t \in \mathbb{Z}).
\]

Combining this result with condition \( A'_\infty \) we will prove, for any \( f \in L^1(d\mu) \), the following fundamental inequality
\[
(2.19) \quad \int_{\{x : Hf(x) > \lambda, f^*(x) \leq \gamma \lambda\}} w \, d\mu \leq C \left( \frac{\gamma}{\beta'} \right)^{\delta} \int_{\{x : Hf(x) > \lambda\}} \, d\mu.
\]
where \( \beta' \) depends on \( \beta \) and \( \gamma \).

If \( \mu\{x : Hf(x) > \lambda\} = 1 \) the weak type \( (1 - 1) \) of \( H \) with respect to the measure \( \mu \) tells us
\[
1 \leq \frac{C}{\lambda} \int_X |f| \, d\mu
\]
and choosing \( \gamma < C^{-1} \) we have
\[
\gamma \lambda < \int_X |f| \, d\mu.
\]

By the individual ergodic theorem:
\[
\gamma \lambda < f^*(x) \quad \text{a.e. in } X
\]
and that implies (2.19)

Therefore we may assume that \( \mu\{x : Hf(x) > \lambda\} < 1 \). In particular, if
\[
D = \{x : T^i x \in O_\lambda : i = 0, -1, -2, \ldots\}
\]
where \( O_\lambda = \{x : Hf(x) > \lambda\} \), then \( \mu(D) = 0 \), since \( T \) is ergodic.
From this fact is clear that if we call
\[ B_t = \{ x: x, Tx, \ldots, T^{-1}x \in O_\lambda, T^{-1}x, T^i x \notin O_\lambda \} \]
and \( R_i = B_i \cup \cdots \cup T^{-1}B_i \) then \( O_\lambda = \bigcup_{i=1}^{\infty} R_i \) (a.e.).

The former decomposition of \( O_\lambda \) and the study of distribution function inequalities in the integers (2), that we now proceed to develop, will be used in the proof of (2.19). So we consider a function \( F \) defined in the integers and the associated maximal Hilbert transform

\[
(2.20) \quad HF(k) = \sup_{s,t \geq 0} \left| \sum_{s < |k-j| < t} \frac{F(j)}{k-j} \right| \quad (s, t \in \mathbb{Z})
\]

and the maximal function

\[
(2.21) \quad F^*(k) = \sup_{n,m \geq 0} \frac{1}{n+m+1} \sum_{j=-n}^{m} |F(k+j)|.
\]

Let \( \lambda \) be a positive number. The set
\[
\{ k: HF(k) > \lambda \}
\]

can be written as a countable union of disjoint intervals \( I_i \) in the integers and of maximum length. In this situation we can state the following lemma.

\[
(2.22). \text{Lemma. There exists positive constants } C \text{ and } C' \text{ such that}
\]

\[
\# \{ j \in I_i: HF(j) > \beta \lambda, F^*(j) \leq \gamma \lambda \} \leq C \frac{\gamma}{\beta - 1 - \gamma C'} \# I_i
\]

for any \( I_i \) and where \( \beta \) is bigger than 1.

For the proof just look at the proof of inequality (4) in (2) and write it in the integers.

\[
\text{Proof of inequality (2.19). For } n \text{ fixed we call } E_{n,l} \text{ the nonempty subsets of } \{0, 1, \ldots, n-1\} \text{ (} l = 1, 2, \ldots, 2^n - 1 \text{).}
\]

For each \( x \) of \( B_n \) we write
\[
E_n^x = \{ i: 0 \leq i \leq n - 1: HF(T^i x) > \beta \lambda, f^*(T^i x) \leq \gamma \lambda \}
\]

and
\[
B_{n,l} = \{ x \in B_n: E_n^x = E_{n,l} \}.
\]

By Lemma (2.22) if \( x \in B_n \) we have
\[
\# E_n^x \leq \frac{C \gamma}{\beta'} \# \{0, 1, \ldots, n - 1\}
\]
which implies
\[
\sum_{j \in E_n^x} w(T^j x) \leq C \cdot \left( \frac{\gamma}{\beta'} \right)^{\delta n - 1} \sum_{j=0}^{\delta n - 1} w(T^j x) \quad (x \in B_n)
\]
since \( w \) satisfies \( A'_\infty \). Integrating over \( B_n \) we obtain
\[
\int_{\bigcup_{j \in E_n, T^j B_n,j}} w \, d\mu \leq C \cdot \left( \frac{\gamma}{\beta'} \right)^{\delta} \int_{\bigcup_{j=0}^{\delta n - 1} T^j B_n,j} w \, d\mu.
\]
Summing first over \( l \) and then over \( n \) and keeping in mind that \( O_{\lambda} = \bigcup_{n=1}^{\infty} R_n \) (a.e.) we get inequality (2.19).

As is well known a standard argument shows that the "good-\( \lambda \) inequality" (2.19) implies (2.14) (see for example (2)). Therefore we have Theorem (2.13) for \( f \) in \( L^1(d\mu) \).

Theorem (1.2) now follows combining Theorem (2.16) with standard density arguments.

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