Pacific Journal of Mathematics

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Vol. 108, No. 2 April 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 2, 1983

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BRUCE E. BLACKADAR

Let F be a nondiscrete locally compact topological field. Then the regular representation of the group of invertible affine motions of F^n, the semidirect product of F^n by $GL_n(F)$, is a type I_∞ factor. An explicit transformation formula is obtained.

1. Introduction. It is of some interest [4] to examine the regular representation of the group of affine motions of F^n for a nondiscrete locally compact field F. We show that the regular representation of such a group is a type I_∞ factor, i.e. is a multiple of an irreducible representation on an infinite-dimensional Hilbert space.

The results of this paper were part of the author's doctoral dissertation at the University of California, Berkeley, June 1975, under the direction of Calvin C. Moore.

2. Preliminaries. Let F be a nondiscrete locally compact field. It is known (see, for example, [3, Theorem 9.21]) that F is either $\mathbf{R}$, $\mathbf{C}$, a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers, or the field of formal Laurent series in one variable over a finite field. In particular, if F is not $\mathbf{R}$ or $\mathbf{C}$ it has the following properties:

(i) F is the quotient field of a compact open subring R.

(ii) R has a unique maximal ideal M, which is principal; let $M = (\pi)$.

(iii) R/M is a finite field with (say) q elements.

(iv) There is a character χ on the additive group of F with $R \subseteq \ker \chi$, $\pi^{-1} \notin \ker \chi$; any other character on F is of the form $\chi_u(x) = \chi(ux)$ for some $u \in F$.

(v) R has a nonarchimedean absolute value $|\cdot|$ with $|\pi| = 1/q$.

(vi) If μ (usually denoted dx) is additive Haar measure on F, normalized so that $\mu(R) = 1$, then $\mu(M) = 1/q$ and $dx/|x|$ is multiplicative Haar measure μ^* on F^*, with the measure of R^* equal to $1 - 1/q$.

If F is $\mathbf{R}$ or $\mathbf{C}$, let dx denote Lebesgue measure normalized to make the Fourier inversion formula valid, $|\cdot|$ the ordinary absolute value (squared if $F = \mathbf{C}$), and $\chi(x) = e^{2\pi i \operatorname{Re} x}$.

We now let G_n be the group of invertible affine motions of F^n (the n-dimensional "$ax + b$" group), i.e. $G_n = F^n \cdot GL_n$, the semidirect product of F^n by $GL_n = GL_n(F)$. It will frequently be useful to consider G_n as a subgroup of GL_{n+1} by the identification

$$(b, A) \leftrightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b & & A & \end{bmatrix}.$$

Using this identification, we will think of $G_1 \subseteq GL_2 \subseteq \cdots \subseteq GL_n \subseteq G_n \subseteq GL_{n+1}$.

3. The results.

THEOREM 3.1. *The right regular representation ρ_{G_n} of G_n is a type I_∞ factor.*

Proof. By induction on n. The case $n = 1$ was done in [**2**, §3]; we briefly outline the argument for completeness. $G_1 \cong F \times F^*$ topologically, and $\mu \times \mu^*$ is right Haar measure. If $f \in L^2(G_1)$, set $\hat{f}_u(y, x) = \chi(uy)\int_F f(z, x)\chi(-uz)\, dz$; then $[\rho_{G_1}(b, a)f]\hat{}_u(y, x) = \chi(ubx)\hat{f}_u(y, ax)$. If $\rho_u = \mathrm{ind}_{F\uparrow G_1} \chi_{-u}$, then $\rho_u \cong \rho_v$ for $u, v \neq 0$; since $f(y, x) = \int_F \hat{f}_u(y, x)\, du$, we have $\rho = \int_F \rho_u\, du$.

Now assume $\rho_{G_{n-1}}$ is a factor. Regard F^n as a subgroup of G_n by identifying b with $(b, \mathbf{1})$. $\rho_{G_n} = \mathrm{ind}_{F^n\uparrow G_n} \rho_{F^n}$. $\rho_{F^n} = \int_{F^n} \chi_u\, du$, where χ_u ($u \in F^n$) is the character given by $\chi_u(v) = \chi(u \cdot v)$. By moving the direct integral past the induction, we get $\rho_{G_n} = \int_{F^n}(\mathrm{ind}_{F^n\uparrow G_n} \chi_u)\, du$. If u and v are nonzero vectors in F^n, $\mathrm{ind}\, \chi_u \simeq \mathrm{ind}\, \chi_v$, since u and v are conjugate under the action of GL_n on F^n. Set $e_1 = (1, 0, \ldots, 0)$. We then have $\rho_{G_n} \simeq \int_{F^n}(\mathrm{ind}_{F^n\uparrow G_n} \chi_{e_1})\, du$. $G_n = F^n \cdot GL_n$, so, regarding $G_{n-1} \subseteq GL_n$, let $H_n = F^n \cdot G_{n-1}$. Since the action of G_{n-1} on F^n leaves the first coordinate fixed, we have $H_n = F \times (F^{n-1} \cdot G_{n-1})$.

We split the induction into two steps,

$$\rho_{G_n} \simeq \int_{F^n} \mathrm{ind}_{H_n\uparrow G_n}(\mathrm{ind}_{F^n\uparrow H_n} \chi_{e_1})\, du.$$

Let us examine $\pi = \mathrm{ind}_{F^n\uparrow H_n} \chi_{e_1}$. $\chi_{e_1} = \chi \otimes \mathbf{1}$ on $F^n = F \times F^{n-1}$, and $H_n = F \times (F^{n-1} \cdot G_{n-1})$, so $\pi \simeq \chi \otimes (\mathrm{ind}_{F^{n-1}\uparrow(F^{n-1}\cdot G_{n-1})} \mathbf{1}) \simeq \chi \otimes \rho_{G_{n-1}}$ (where $\rho_{G_{n-1}}$ is considered as a representation of $F^{n-1} \cdot G_{n-1}$ with kernel F^{n-1}). By the induction hypothesis, $\rho_{G_{n-1}}$ is a I_∞ factor representation of G_{n-1}, so π is a I_∞ factor representation of H_n. We now use Mackey's theorem ([**1**], Theorem 6, p. 58) to show that $\mathrm{ind}_{H_n\uparrow G_n} \pi$ is a I_∞ factor representation of G_n, since H_n is precisely the stability group of χ_{e_1} under the action of G_n on F^n. □

We now get an explicit formula for this transformation. Throughout, we will always consider $GL_k \subseteq G_k \subseteq GL_{k+1} \subseteq G_{k+1}$, so that all groups will be thought of as being embedded in GL_{n+1}. Let $f \in L^2(G_n)$. We first take the Fourier transform along F^n: define

$$\hat{f}_u(y, X) = \chi(u \cdot y) \int_{F^n} f(z, X) \chi(-u \cdot z)\, dz.$$

Then

$$\hat{f}_u \in \mathcal{H}_u^n = \left\{ f: G_n \to \mathbf{C}: f(y, X) = \chi(u \cdot y) f(0, X), \right.$$
$$\left. \int_{GL_n} |f(0, X)|^2\, dX < \infty \right\}$$

where dX is Haar measure on GL_n.

By the Fourier inversion formula, $f(y, X) = \int_{F^n} \hat{f}_u(y, X)\, du$.

$$[\rho(b, A)f]_u^{\wedge}(y, X) = \chi(u \cdot y) \int_{F^n} [\rho(b, A)f](z, X) \chi(-u \cdot z)\, dz$$
$$= \chi(u \cdot y) \int_{F^n} f(z + Xb, XA) \chi(-u \cdot z)\, dz$$

Set $t = z - Xb$.

$$= \chi(u \cdot y) \int_{F^n} f(t, XA) \chi(-u \cdot t) \chi(u \cdot Xb)\, dt$$
$$= \chi(u \cdot Xb) \hat{f}_u(y, XA).$$

This is precisely the representation $\operatorname{ind}_{F^n \uparrow G_n} \chi_u$ on $\mathcal{H}_u^n[\chi_u(v) = \chi(u \cdot v)]$. So we have written

$$L^2(G_n) \simeq \int_{F^n} \mathcal{H}_u^n\, du, \qquad \rho_{G_n} \simeq \int_{F^n} \Big(\operatorname*{ind}_{F^n \uparrow G_n} \chi_u \Big)\, du.$$

Let $e_1^n = (1, 0, \ldots, 0) \in F^n$. We now take an equivalence in each piece, $\mathcal{H}_u^n \to \mathcal{H}_{e_1^n}^n$, ind $\chi_u \to$ ind $\chi_{e_1^n}$ by setting $\tilde{f}_u(y, X) = \hat{f}_u(B_u(y, X))$ where

$$B_u = \begin{bmatrix} 1/u_1 & -u_2/u_1 & \cdots & -u_n/u_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{for } u = (u_1, \ldots, u_n),\ u_1 \neq 0.$$

We interchangeably think of B_u as an element of GL_n, G_n, and GL_{n+1} to simplify notation. The reason for choosing this B_u is that $u \cdot B_u v = B_u^t u \cdot v = e_1^n \cdot v$ for all v.

$\hat{f}_u \to \tilde{f}_u$ is an isometry of $\mathcal{H}_u^n$ onto $\mathcal{H}_{e_1^n}^n$: this can be seen most easily by identifying $\mathcal{H}_u^n$ with $L^2(GL_n)$ by $\hat{f}_n \leftrightarrow \hat{f}_u(0, \cdot)$ and noting that GL_n is unimodular (we have assumed right Haar measure). By associating f with $\int_{F^n} \tilde{f}_u \, du$, we get

$$L^2(G_n) \simeq \int_{F^n} \mathcal{H}_{e_1^n}^n \, du, \qquad \rho_{G_n} \simeq \int_{F^n} \Big(\operatorname*{ind}_{F^n \uparrow G_n} \chi_{e_1^n} \Big) \, du.$$

$\tilde{f}_u(y, X) = \chi(e_1^n \cdot y) \int_{F^n} f(v, B_u X) \chi(-u \cdot v) \, dv$. We now change variables, setting $v = B_u t$, $dv = 1/|u_1| \, dt$.

$$\tilde{f}_u(y, X) = \chi(e_1^n \cdot y) \int_{F^n} f(B_u(t, X)) \chi(-u \cdot B_u t) \frac{1}{|u_1|} \, dt$$

$$= \chi(e_1^n \cdot y) \int_{F^n} f(B_u(t, X)) \chi(e_1^n \cdot t) \, dt.$$

Now we split the induction into two steps,

$$\operatorname*{ind}_{F^n \to G_n} \chi_{e_1^n} = \operatorname*{ind}_{H_n \uparrow G_n} \Big(\operatorname*{ind}_{F^n \uparrow H_n} \chi_{e_1^n} \Big).$$

Set

$$\bar{f}_u(y, X)(Z) = \tilde{f}_u(y, ZX) \quad \text{for } y \in F^n, X \in GL_n, Z \in G_{n-1} \subseteq GL_n.$$

$$\bar{f}_u \in \Big\{ f\colon G_n \to L^2(G_{n-1}) \colon f([(b, C)(y, X)])(Z) = \chi(e_1^n \cdot b) f(y, X)(ZC)$$
$$\text{for } X \in GL_n, Z, C \in G_{n-1}, b, y \in F^n; \int_{GL_n} |f(X)(\mathbf{1})|^2 \, dX < \infty \Big\}.$$

If we look at the representation σ^n of H_n on $L^2(G_{n-1})$ given by $[\sigma^n(b, C)g](Z) = \chi(e_1^n \cdot b) g(ZC)$ for $b \in F^n$, $C \in G_{n-1}$, we see that

$$\sigma^n \simeq \operatorname*{ind}_{F^n \uparrow H_n} \chi_{e_1^n}, \quad \text{and} \quad \operatorname*{ind}_{F^n \uparrow G_n} \chi_{e_1^n} \simeq \operatorname*{ind}_{H_n \uparrow G_n} \sigma^n.$$

Also, $\sigma^n \simeq \chi_{e_1^n} \otimes \rho_{G_{n-1}}$ as an inner tensor product.

We now decompose $\rho_{G_{n-1}}$ in the same manner as before. Let

$$\hat{f}_{u,r}(y, X)(t, S) = \chi(r \cdot t) \int_{F^{n-1}} \bar{f}_u(y, X)(w, S) \chi(-r \cdot w) \, dw$$
$$\big(t \in F^{n-1}, S \in GL_{n-1} \big).$$

Then

$$\bar{f}_u(y, X)(t, S) = \int_{F^{n-1}} \hat{f}_{u,r}(y, X)(t, S) \, dr; \qquad \hat{f}_{u,r}(y, X) \in \mathcal{H}_r^{n-1}.$$

Let

$$B_r = \begin{bmatrix} 1/r_1 & -r_2/r_1 & \cdots & -r_{n-1}/r_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in GL_{n-1}$$

$$\left(\text{for } r \in F^{n-1}, r_1 \neq 0\right).$$

Set $\tilde{f}_{u,r}(y, X)(t, S) = \hat{f}_{u,r}(y, X)(B_r(t, S))$.

$$[\sigma^n(b, (d, C))f]^{\hat{}}_{u,r}(y, X)(t, S)$$

$$= \chi(r \cdot t)\int_{F^{n-1}} [\sigma^n(b, (d, C))f]^{\tilde{}}_u(y, X)(w, S)\chi(-r \cdot w)dw$$

$$= \chi(r \cdot t)\int_{F^{n-1}} \chi(e_1 \cdot b)\tilde{f}_u(y, X)(w + Sd, SC)\chi(-r \cdot w)dw.$$

Set $v = w + Sd$.

$$= \chi(e_1 \cdot b)\chi(r \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(v, SC)\chi(-r \cdot v)\chi(r \cdot Sd)dv$$

$$= \chi(e_1 \cdot b)\chi(r \cdot Sd)\hat{f}_{u,r}(y, X)(t, SC).$$

$$[\sigma^n(b, (d, C))f]^{\tilde{}}_{u,r}(y, X)(t, S)$$

$$= \chi(e_1 \cdot b)\chi(r \cdot B_r Sd)\hat{f}_{u,r}(y, X)(B_r(t, SC))$$

$$= \chi(e_1 \cdot b)\chi(e_1 \cdot Sd)\tilde{f}_{u,r}(y, X)(t, SC).$$

Thus by associating $\tilde{f}_u$ with

$$\int_{F^{n-1}} \tilde{f}_{u,r}\, dr, \qquad \sigma^n \simeq \int_{F^{n-1}} \chi_{e_1^n} \otimes \left(\operatorname*{ind}_{F^{n-1} \uparrow G_{n-1}} \chi_{e_1^{n-1}} \right).$$

$$\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(w, B_r S)\chi(-r \cdot w)dw.$$

We want to pull the B_r past the w, so we change variables as before. Set $w = B_r v$, $dw = 1/|r_1|\, dv$. Then

$$\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(B_r(v, S))\chi(-r \cdot B_r v)\frac{1}{|r_1|}dv$$

$$= \chi(e_1 \cdot t)\int_{F^{n-1}} \tilde{f}_u(y, X)(B_r(v, S))\chi(e_1 \cdot v)\frac{1}{|r_1|}dv$$

$$= \chi(e_1^n \cdot y)\chi(e_1^{n-1} \cdot t)$$

$$\cdot \int_{F^{n-1}} \left[\int_{F^n} f(B_u(w, B_r(v, S)X))\chi(-w_1)\frac{1}{|u_1|}dw \right] \chi(-v_1)\frac{1}{|r_1|}dv.$$

We now pull the B_r past the w, by letting $w = B_r z$, $dw = 1/|r_1|\, dz$. Note that $z_1 = w_1$ since B_r does not affect the first column.

$$\tilde{f}_{u,r}(y, X)(t, S) = \int_{F^{n-1}} \left[\int_{F^n} f(B_u B_r(z, (v, S)X))\chi(-z_1)\frac{1}{|u_1 r_1|}\, dz \right] \chi(-v_1)\frac{1}{|r_1|}\, dv.$$

We now repeat the process until we get down to F^1. We end up with

$$\begin{aligned} &\tilde{f}_{u,r,\ldots,s}(y, X)(t, S) \cdots (q, T) \\ &\qquad ((y, X) \in G_n, (t, S) \in G_{n-1}, \ldots, (q, T) \in G_1) \\ &= \chi(e_1^n \cdot y)\chi(e_1^{n-1} \cdot t) \cdots \chi(q) \\ &\quad \cdot \int_F \int_{F^2} \cdots \int_{F^n} f(B_u B_r \cdots B_s(w, (v, \ldots (z, T) \ldots, S)X)) \\ &\quad \cdot \chi(-w_1 - v_1 - \cdots - z_1)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\, dw\, dv \cdots dz. \end{aligned}$$

$$\begin{aligned} \tilde{f}_{u,r,\ldots,s} \in \mathcal{H}^n = \Big\{ f\colon G_n \to \mathcal{H}^{n-1} \colon & f([(b, C)(y, X)])(Z) \\ &= \chi(e_1^n \cdot b) f(y, X)(ZC) \quad \text{for } X \in GL_n, Z, C \in G_n, \\ &\qquad b, y \in F^n;\ \int_{G_{n-1}\backslash G_n} |f(y, X)|^2 < \infty \Big\}. \end{aligned}$$

$[\mathcal{H}^0 = \mathbf{C}]$.

Set $\bar{f}_{u,r,\ldots,s}(y, X) = \tilde{f}_{u,r,\ldots,s}(y, X)(0, \mathbf{1}) \cdots (0, \mathbf{1})$.

$$\bar{f}_{u,r,\ldots,s} \in \mathcal{H} = \Big\{ f\colon G_n \to \mathbf{C}\colon f(C(y, X)) = \phi(C) f(y, X) \quad \text{for } C \in \Gamma_n,\ \int_{\Gamma_n\backslash G_n} |f(y, X)|^2 < \infty \Big\}$$

where

$$\Gamma_n = \left\{ \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{bmatrix} \right\}, \qquad \phi\left(\begin{bmatrix} 1 & & & & 0 \\ a_{11} & 1 & & & \\ \vdots & & \ddots & & \\ a_{n1} & \cdots & & a_{nn} & 1 \end{bmatrix} \right) = \Sigma a_{ii}.$$

$$\bar{f}_{u,r,\ldots,s}(y, X)$$

$$= \int_F \cdots \int_{F^n} f\left(B_u B_r \cdots B_s \begin{bmatrix} 1 & & & \\ w_1 & 1 & & 0 \\ \vdots & & \ddots & \\ w_n & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & v_1 & \ddots & \\ 0 & v_{n-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & \cdots & z_1 & 1 \end{bmatrix} (y, X) \right)$$

$$\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} \, dw \, dv \cdots dz.$$

$$\int_F \cdots \int_{F^n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & & & & \\ w_1 & 1 & & & \\ w_2 & v_1 & 1 & & \\ \vdots & \vdots & & \ddots & \\ w_n & v_{n-1} & \cdots & z_1 & 1 \end{bmatrix} (y, X) \right)$$

$$\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} \, dw \, dv \cdots dz$$

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(y, X) \right) \phi(-\gamma) \frac{1}{|u_1 r_1^2 \cdots s_1^n|} \, d\gamma$$

since Haar measure on Γ_n is $dw \, dv \cdots dz$.

$$[\rho(b, A)f]^{-}_{u,r,\ldots,s}(y, X)$$

$$= \int_{\Gamma_n} [\rho(b, A)f]\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(y, X)\right)$$

$$\phi(-\gamma)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\,d\gamma$$

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \gamma(Xb, \mathbf{1})(y, XA)\right)$$

$$\cdot\phi(-\gamma)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\,d\gamma$$

[Set $\beta = \gamma(Xb, \mathbf{1})$.]

$$= \int_{\Gamma_n} f\left(\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}^{-1} \beta(y, XA)\right)$$

$$\cdot\chi(e_1 \cdot Xb)\phi(-\beta)\frac{1}{|u_1 r_1^2 \cdots s_1^n|}\,d\beta$$

$$= \chi(e_1 \cdot Xb)\bar{f}_{u,r,\ldots,s}(y, XA).$$

This is precisely $\operatorname{ind}_{\Gamma_n \uparrow G_n} \phi$ on $\mathcal{H}$. So we have

$$L^2(G_n) \simeq \int_F \cdots \int_{F^n} \mathcal{H}\, du\, dr \cdots ds,$$

$$\rho_{C_n} \simeq \int_F \cdots \int_{F^n} \Big(\operatorname*{ind}_{\Gamma_n \uparrow G_n} \phi\Big)\, du\, dr \cdots ds.$$

Let

$$\Delta_n = \left\{ \begin{bmatrix} u_1 & & \cdots & u_n \\ 0 & r_1 & \cdots & r_{n-1} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & s_1 \end{bmatrix} : u_1 \neq 0, \ldots, s_1 \neq 0 \right\}$$

$$= \text{group of upper triangular invertible } n \times n \text{ matrices.}$$

Right Haar measure on Δ_n is

$$\frac{du_1 \cdots du_n dr_1 \cdots dr_{n-1} \cdots ds_1}{|u_1 r_1^2 \cdots s_1^n|}.$$

We may identify Δ_n with $\Gamma_n \backslash G_n$ as a measure space, and hence we may regard $\operatorname{ind}_{\Gamma_n \uparrow G_n} \phi$ as a representation σ on $L^2(\Delta_n)$.

We now renormalize $\bar{f}_{u,r,\ldots,s}$ so that we can recapture f as an integral over Δ_n.

We have

$$f = \int_F \cdots \int_{F^n} \bar{f}_{u,r,\ldots,s}\, du\, dr \cdots ds.$$

Set $f_{u,r,\ldots,s} = \sqrt{|u_1 r_1^2 \cdots s_1^n|}\, \bar{f}_{u,r,\ldots,s}$; then

$$f = \int_F \cdots \int_{F^n} f_{u,r,\ldots,s} \frac{du\, dr \cdots ds}{|u_1 r_1^2 \cdots s_1^n|} = \int_{\Delta_n} f_\alpha\, d\alpha;$$

$$f_\alpha(y, X) = \left(|u_1 r_1^2 \cdots s_1^n|\right)^{-1/2} \int_{\Gamma_n} f(\alpha^{-1}\gamma(y, X))\phi(-\gamma)\, d\gamma,$$

where

$$\alpha = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & u_1 & & \cdots & u_n \\ 0 & 0 & r_1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \cdots & s_1 \end{bmatrix}.$$

We thus have $L^2(G_n) \simeq \int_{\Delta_n} L^2(\Delta_n)\, d\alpha$, $\rho_{G_n} \simeq \int_{\Delta_n} \sigma\, d\alpha$. We may identify $\int_{\Delta_n} L^2(\Delta_n)\, d\alpha$ with $L^2(\Delta_n) \otimes L^2(\Delta_n)$, $\rho_{G_n} \simeq \sigma \otimes \mathbf{1}$.

REFERENCES

[1] L. Auslander and C. Moore, *Unitary representations of solvable Lie groups*, Amer. Math. Soc. Memoir no. 62, 1966.

[2] B. Blackadar, *The regular representation of restricted direct product groups*, J. Funct. Anal., **25** (1977), 267–274.

[3] N. Jacobson, *Basic Algebra II*, W. H. Freeman, San Francisco, 1980.

[4] H. Jacquet, *Generic Representations*, Lecture Notes in Mathematics v. 587, Springer-Verlag, 1977, 91–101.

Received March 21, 1981.

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The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

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