THE REGULAR REPRESENTATION OF LOCAL AFFINE MOTION GROUPS

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BRUCE E. BLACKADAR

Let $F$ be a nondiscrete locally compact topological field. Then the regular representation of the group of invertible affine motions of $F^n$, the semidirect product of $F^n$ by $GL_n(F)$, is a type $I_\infty$ factor. An explicit transformation formula is obtained.

1. Introduction. It is of some interest [4] to examine the regular representation of the group of affine motions of $F^n$ for a nondiscrete locally compact field $F$. We show that the regular representation of such a group is a type $I_\infty$ factor, i.e. is a multiple of an irreducible representation on an infinite-dimensional Hilbert space.

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2. Preliminaries. Let $F$ be a nondiscrete locally compact field. It is known (see, for example, [3, Theorem 9.21]) that $F$ is either $\mathbb{R}$, $\mathbb{C}$, a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers, or the field of formal Laurent series in one variable over a finite field. In particular, if $F$ is not $\mathbb{R}$ or $\mathbb{C}$ it has the following properties:

(i) $F$ is the quotient field of a compact open subring $R$.
(ii) $R$ has a unique maximal ideal $M$, which is principal; let $M = (\pi)$.
(iii) $R/M$ is a finite field with (say) $q$ elements.
(iv) There is a character $\chi$ on the additive group of $F$ with $R \subseteq \ker \chi$, $\pi^{-1} \not\in \ker \chi$; any other character on $F$ is of the form $\chi_u(x) = \chi(ux)$ for some $u \in F$.
(v) $R$ has a nonarchimedean absolute value $| \cdot |$ with $|\pi| = 1/q$.
(vi) If $\mu$ (usually denoted $dx$) is additive Haar measure on $F$, normalized so that $\mu(R) = 1$, then $\mu(M) = 1/q$ and $dx/|x|$ is multiplicative Haar measure $\mu^*$ on $F^*$, with the measure of $R^*$ equal to $1 - 1/q$.

If $F$ is $\mathbb{R}$ or $\mathbb{C}$, let $dx$ denote Lebesgue measure normalized to make the Fourier inversion formula valid, $|\cdot|$ the ordinary absolute value (squared if $F = \mathbb{C}$), and $\chi(x) = e^{2\pi i \text{Re} x}$. 

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We now let $G_n$ be the group of invertible affine motions of $F^n$ (the $n$-dimensional “ax + b” group), i.e. $G_n = F^n \cdot GL_n$, the semidirect product of $F^n$ by $GL_n = GL_n(F)$. It will frequently be useful to consider $G_n$ as a subgroup of $GL_{n+1}$ by the identification

$$(b, A) \mapsto \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b & A \end{bmatrix}.$$ 

Using this identification, we will think of $G_2 \subseteq GL_2 \subseteq \cdots \subseteq GL_n \subseteq G_n \subseteq GL_{n+1}$.

3. The results.

**Theorem 3.1.** The right regular representation $\rho_{G_n}$ of $G_n$ is a type $I_\infty$ factor.

**Proof.** By induction on $n$. The case $n = 1$ was done in [2, §3]; we briefly outline the argument for completeness. $G_1 \cong F \times F^*$ topologically, and $\mu \times \mu^*$ is right Haar measure. If $f \in L^2(G_1)$, set $\hat{f}_u(y, x) = \chi(yx)f(z, x)\chi(-uz)\,dz$; then $[\rho_{G_1}(b, a)f]_u(y, x) = \chi(ubx)f_\pi(y, ax)$. If $\rho_u = \text{ind}_{F \times GL_n} \chi_{-u}$, then $\rho_u \cong \rho_v$ for $u, v \neq 0$; since $f(y, x) = \int_F \hat{f}_u(y, x)\,du$, we have $\rho = \int_F \rho_u\,du$.

Now assume $\rho_{G_{n-1}}$ is a factor. Regard $F^n$ as a subgroup of $G_n$ by identifying $b$ with $(b, 1)$. $\rho_{G_n} = \text{ind}_{F^n \times G_n} \rho_{F^n}$. $\rho_{F^n} = \int_{F^n} \chi_u\,du$, where $\chi_u (u \in F^n)$ is the character given by $\chi_u(v) = \chi(u \cdot v)$. By moving the direct integral past the induction, we get $\rho_{G_n} = \int_{F^n} (\text{ind}_{F^n \times G_n} \chi_u)\,du$. If $u$ and $v$ are nonzero vectors in $F^n$, $\text{ind} \chi_u \cong \text{ind} \chi_v$, since $u$ and $v$ are conjugate under the action of $GL_n$ on $F^n$. Set $e_1 = (1, 0, \ldots, 0)$. We then have $\rho_{G_n} \cong \int_{F^n} (\text{ind}_{F^n \times G_n} \chi_{e_1})\,du$. $G_n = F^n \cdot GL_n$, so, regarding $G_{n-1} \subseteq GL_n$, let $H_n = F^n \cdot G_{n-1}$. Since the action of $G_{n-1}$ on $F^n$ leaves the first coordinate fixed, we have $H_n = F \times (F^{n-1} \cdot G_{n-1})$.

We split the induction into two steps,

$$\rho_{G_n} \cong \int_{F^n} \text{ind}_{H_n \times G_n} (\text{ind}_{F^n \times H_n} \chi_{e_1})\,du.$$ 

Let us examine $\pi = \text{ind}_{F^n \times H_n} \chi_{e_1}$. $\chi_{e_1} = \chi \otimes 1$ on $F^n = F \times F^{n-1}$, and $H_n = F \times (F^{n-1} \cdot G_{n-1})$, so $\pi \cong \chi \otimes (\text{ind}_{F^{n-1} \times (F^{n-1} \cdot G_{n-1})} 1) \cong \chi \otimes \rho_{G_{n-1}}$ (where $\rho_{G_{n-1}}$ is considered as a representation of $F^{n-1} \cdot G_{n-1}$ with kernel $F^{n-1}$). By the induction hypothesis, $\rho_{G_{n-1}}$ is a $I_\infty$ factor representation of $G_{n-1}$, so $\pi$ is a $I_\infty$ factor representation of $H_n$. We now use Mackey’s theorem ([1], Theorem 6, p. 58) to show that $\text{ind}_{H_n \times G_n} \pi$ is a $I_\infty$ factor representation of $G_n$, since $H_n$ is precisely the stability group of $\chi_{e_1}$ under the action of $G_n$ on $F^n$. \qed
We now get an explicit formula for this transformation. Throughout, we will always consider $GL_k \subseteq G_k \subseteq GL_{k+1} \subseteq G_{k+1}$, so that all groups will be thought of as being embedded in $GL_{n+1}$. Let $f \in L^2(G_n)$. We first take the Fourier transform along $F^n$: define

$$\hat{f}_u(y, X) = \chi(u \cdot y) \int_{F^n} f(z, X) \chi(-u \cdot z) \, dz.$$ 

Then

$$\hat{f}_u \in \mathcal{H}_u^n = \left\{ f: G_n \to \mathbb{C}: f(y, X) = \chi(u \cdot y)f(0, X), \right.$$ 

$$\left. \int_{GL_n} |f(0, X)|^2 \, dX < \infty \right\}$$

where $dX$ is Haar measure on $GL_n$.

By the Fourier inversion formula, $f(y, X) = \int_{F^n} \hat{f}_u(y, X) \, du$.

$$[\rho(b, A)f]_u(y, X) = \chi(u \cdot y) \int_{F^n} [\rho(b, A)f](z, X) \chi(-u \cdot z) \, dz$$

$$= \chi(u \cdot y) \int_{F^n} f(z + Xb, XA) \chi(-u \cdot z) \, dz$$

Set $t = z - Xb$.

$$= \chi(u \cdot y) \int_{F^n} f(t, XA) \chi(-u \cdot t) \chi(u \cdot Xb) \, dt$$

$$= \chi(u \cdot Xb) \hat{f}_u(y, XA).$$

This is precisely the representation $\text{ind}_{F^n \cdot G_n} \chi_u$ on $\mathcal{H}_u^n[\chi_u(v) = \chi(u \cdot v)]$. So we have written

$$L^2(G_n) \cong \int_{F^n} \mathcal{H}_u^n \, du, \quad \rho_{G_n} \cong \int_{F^n} \left( \text{ind} \chi_u \right) \, du.$$ 

Let $e^n_1 = (1, 0, \ldots, 0) \in F^n$. We now take an equivalence in each piece, $\mathcal{H}_u^n \to \mathcal{H}_{e^n_1}$, $\text{ind} \chi_u \to \text{ind} \chi_{e^n_1}$ by setting $\hat{f}_u(y, X) = \hat{f}_u(B_u(y, X))$ where

$$B_u = \begin{bmatrix} 1/u_1 & -u_2/u_1 & \cdots & -u_n/u_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{for } u = (u_1, \ldots, u_n), u_1 \neq 0.$$ 

We interchangeably think of $B_u$ as an element of $GL_n$, $G_n$, and $GL_{n+1}$ to simplify notation. The reason for choosing this $B_u$ is that $u \cdot B_u v = B_u^t u \cdot v = e^n_1 \cdot v$ for all $v$. 


\[ \hat{f}_u \to \hat{f}_u \] is an isometry of \( \mathcal{K}_u \) onto \( \mathcal{K}_{e_r} \); this can be seen most easily by identifying \( \mathcal{K}_u \) with \( L^2(\text{GL}_n) \) by \( \hat{f}_u \mapsto \hat{f}_u(0, \cdot) \) and noting that \( \text{GL}_n \) is unimodular (we have assumed right Haar measure). By associating \( f \) with \( \int_{F^n} \hat{f}_u \, du \), we get

\[ L^2(G_n) \simeq \int_{F^n} \mathcal{K}_{e_r} \, du, \quad \rho_{G_n} \simeq \int_{F^n} \left( \text{ind } \chi_{e_r} \right) \, du. \]

\[ \hat{f}_u(y, X) = \chi(e^n_1 \cdot y) \int_{F^n} f(v, B_u X) \chi(-u \cdot v) \, dv. \]

We now change variables, setting \( v = B_u t \), \( dv = 1/|u_1| \, dt \).

\[ \hat{f}_u(y, X) = \chi(e^n_1 \cdot y) \int_{F^n} f(B_u(t, X)) \chi(-u \cdot B_u t) \frac{1}{|u_1|} \, dt \]

\[ = \chi(e^n_1 \cdot y) \int_{F^n} f(B_u(t, X)) \chi(e^n_1 \cdot t) \, dt. \]

Now we split the induction into two steps,

\[ \text{ind } F^n \to G_n \chi_{e_r} = \text{ind } H_n \uparrow G_n \left( \text{ind } F^n \uparrow H_n \chi_{e_r} \right). \]

Set

\[ \hat{f}_u(y, X)(Z) = \hat{f}_u(y, ZX) \quad \text{for } y \in F^n, X \in \text{GL}_n, Z \in G_{n-1} \subseteq \text{GL}_n. \]

\[ \hat{f}_u \in \left\{ f : G_n \to L^2(G_{n-1}) : f([((b, C)(y, X)])(Z) = \chi(e^n_1 \cdot b) f(y, X)(ZC) \right\}. \]

for \( X \in \text{GL}_n, Z, C \in G_{n-1}, b, y \in F^n; \int_{\text{GL}_n} |f(X)(1)|^2 \, dX < \infty \).

If we look at the representation \( \sigma^n \) of \( H_n \) on \( L^2(G_{n-1}) \) given by \( [\sigma^n(b, C)g](Z) = \chi(e^n_1 \cdot b) g(ZC) \) for \( b \in F^n, C \in G_{n-1} \), we see that

\[ \sigma^n \simeq \text{ind } H_n \uparrow G_n \chi_{e_r}, \quad \text{and } \text{ind } H_n \uparrow G_n \chi_{e_r} \simeq \text{ind } \sigma^n. \]

Also, \( \sigma^n \simeq \chi_{e_r} \otimes \rho_{G_{n-1}} \) as an inner tensor product.

We now decompose \( \rho_{G_{n-1}} \) in the same manner as before. Let

\[ \hat{f}_{u,r}(y, X)(t, S) = \chi(r \cdot t) \int_{F^n} \hat{f}_u(y, X)(w, S) \chi(-r \cdot w) \, dw \]

\[ (t \in F^{n-1}, S \in \text{GL}_{n-1}). \]

Then

\[ \hat{f}_u(y, X)(t, S) = \int_{F^{n-1}} \hat{f}_{u,r}(y, X)(t, S) \, dr; \quad \hat{f}_{u,r}(y, X) \in \mathcal{K}_{e_r} \].
Let

\[
B_r = \begin{bmatrix}
1/r_1 & -r_2/r_1 & \cdots & -r_{n-1}/r_1 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix} \in GL_{n-1}
\]

(for \( r \in F^{n-1}, r_1 \neq 0 \)).

Set \( \tilde{f}_{u,r}(y, X)(t, S) = \hat{f}_{u,r}(y, X)(B_r(t, S)) \).

\[
[\sigma^n(b, (d, C)) \tilde{f}_{u,r}(y, X)(t, S)]
\]

\[
= \chi(r \cdot t) \int_{F_{n-1}} [\sigma^n(b, (d, C))\tilde{f}_u(y, X)(w, S)] \chi(-r \cdot w) dw
\]

\[
= \chi(r \cdot t) \int_{F_{n-1}} \chi(e_1 \cdot b) \tilde{f}_u(y, X)(w + Sd, SC) \chi(-r \cdot w) dw.
\]

Set \( v = w + Sd \).

\[
= \chi(e_1 \cdot b) \chi(r \cdot t) \int_{F_{n-1}} \tilde{f}_u(y, X)(v, SC) \chi(-r \cdot v) \chi(r \cdot Sd) dv
\]

\[
= \chi(e_1 \cdot b) \chi(r \cdot Sd) \hat{f}_{u,r}(y, X)(t, SC).
\]

Thus by associating \( \tilde{f}_u \) with

\[
\int_{F_{n-1}} \tilde{f}_{u,r} dr, \quad \sigma^n \simeq \int_{F_{n-1}} \chi_{e_1^n} \otimes \left( \text{ind }_{F_{n-1} G_{n-1}} \chi_{e_1^{n-1}} \right).
\]

\[
\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t) \int_{F_{n-1}} \tilde{f}_u(y, X)(w, B_r S) \chi(-r \cdot w) dw.
\]

We want to pull the \( B_r \) past the \( w \), so we change variables as before. Set \( w = B_r v, dw = 1/|r_1| dv \). Then

\[
\tilde{f}_{u,r}(y, X)(t, S) = \chi(e_1 \cdot t) \int_{F_{n-1}} \tilde{f}_u(y, X)(B_r(v, S)) \chi(-r \cdot B_r v) \frac{1}{|r_1|} dv
\]

\[
= \chi(e_1 \cdot t) \int_{F_{n-1}} \tilde{f}_u(y, X)(B_r(v, S)) \chi(e_1 \cdot v) \frac{1}{|r_1|} dv
\]

\[
= \chi(e_1^n \cdot y) \chi(e_1^{n-1} \cdot t)
\]

\[
\cdot \int_{F_{n-1}} \left[ \int_{F_{n-1}} f(B_u(w, B_r(v, S) X)) \chi(-w_1) \frac{1}{|u_1|} dw \right] \chi(-v_1) \frac{1}{|r_1|} dv.
\]
We now pull the $B_r$ past the $w$, by letting $w = B_r z$, $dw = 1/|r_1| dz$. Note that $z_1 = w_1$ since $B_r$ does not affect the first column.

\[
\tilde{f}_{u, r}(y, X)(t, S)
= \int_{F^n} \left[ \int_{F^n} f(B_u B_r(z, (v, S) X)) \chi(-z_1) \frac{1}{|u_1 r_1|} dz \right] \chi(-v_1) \frac{1}{|r_1|} dv.
\]

We now repeat the process until we get down to $F^1$. We end up with

\[
\tilde{f}_{u, r, \ldots, s}(y, X)(t, S) \cdot \cdots (q, T)
= \chi(e_1^n \cdot y) \chi(e_1^{n-1} \cdot t) \cdots \chi(q)
\]

\[
\cdot \int_{F^1} \cdots \int_{F^n} f(B_u B_r \cdots B_s(w, (v, \ldots (z, T) \ldots), S) X))
\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{u_1 r_1^2 \cdots s_1^n} dw \, dv \cdots dz.
\]

\[
\tilde{f}_{u, r, \ldots, s} \in \mathcal{S}^n = \left\{ f: G_n \to \mathcal{S}^{n-1}: f([(b, C)(y, X)])(Z)
= \chi(e_1^n \cdot b) f(y, X)(ZC) \right\}.
\]

\[
\tilde{f}_{u, r, \ldots, s} \in \mathcal{S} = \left\{ f: G_n \to \mathcal{S}: f(C(y, X)) = \phi(C) f(y, X) \right\}
\]

where

\[
\Gamma_n = \begin{bmatrix}
1 & 1 & 0 \\
\ast & \ddots & \ast \\
* & & 1
\end{bmatrix},
\phi \begin{bmatrix}
1 & 1 & 0 \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix} = \Sigma a_{ii}.
\]
\[
\bar{f}_{u,r,\ldots,s}(y, X) = \int_{F^s} \cdots \int_{F^s} f \left( B_u B_r \cdots B_s \begin{bmatrix}
1 \\
w_1 & 1 & 0 \\
\vdots & & \vdots \\
w_n & 0 & \cdots & 1
\end{bmatrix} \right) \\
\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{u_1 r_1^2 \cdots s_1^n} \, dw \, dv \, \cdots \, dz.
\]

\[
\int_{F^s} \cdots \int_{F^s} f \left( \begin{bmatrix}
1 & 0 & \cdots & 0 \\
u_1 & & & u_n \\
0 & 0 & r_1 & \cdots & r_{n-1} \\
\vdots & & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_1
\end{bmatrix} \right)^{-1} \\
\cdot \chi(-w_1 - v_1 - \cdots - z_1) \frac{1}{u_1 r_1^2 \cdots s_1^n} \, dw \, dv \, \cdots \, dz
\]

\[
= \int_{\Gamma_n} f \left( \begin{bmatrix}
1 & 0 & \cdots & 0 \\
u_1 & & & u_n \\
0 & 0 & r_1 & \cdots & r_{n-1} \\
\vdots & & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_1
\end{bmatrix} \right)^{-1} \gamma(y, X) \phi(-\gamma) \frac{1}{u_1 r_1^2 \cdots s_1^n} \, d\gamma
\]

since Haar measure on \( \Gamma_n \) is \( dw \, dv \, \cdots \, dz \).
\[\left[\rho(b, A)f\right]_{u, r \ldots s}(y, X)\]

\[= \int_{\Gamma_n} \left[\rho(b, A)f\right] \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & u_1 & \cdots & u_n \\
0 & 0 & r_1 & \cdots & r_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_1
\end{bmatrix}^{-1} \gamma(y, X)\]

\[\phi(-\gamma) \frac{1}{\left| u_1 r_1^2 \cdots s_1^n \right|} d\gamma\]

\[= \int_{\Gamma_n} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & u_1 & \cdots & u_n \\
0 & 0 & r_1 & \cdots & r_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_1
\end{bmatrix}^{-1} \gamma(Xb, 1)(y, XA)\]

\[\cdot \phi(-\gamma) \frac{1}{\left| u_1 r_1^2 \cdots s_1^n \right|} d\gamma\]

\[\text{[Set } \beta = \gamma(Xb, 1).]\]

\[= \int_{\Gamma_n} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & u_1 & \cdots & u_n \\
0 & 0 & r_1 & \cdots & r_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s_1
\end{bmatrix}^{-1} \beta(y, XA)\]

\[\cdot \chi(e_1 \cdot Xb) \phi(-\beta) \frac{1}{\left| u_1 r_1^2 \cdots s_1^n \right|} d\beta\]

\[= \chi(e_1 \cdot Xb) \hat{f}_{u, r \ldots s}(y, XA).\]

This is precisely \(\text{ind}_{\Gamma_n} \phi\) on \(\mathcal{K}\). So we have

\[L^2(G_n) \simeq \int_{F} \cdots \int_{F^n} \mathcal{K} du \, dr \cdots ds,\]

\[\rho_{c_n} \simeq \int_{F} \cdots \int_{F^n} \left(\text{ind} \phi\right) du \, dr \cdots ds.\]
Let

\[ \Delta_n = \begin{pmatrix}
  u_1 & \cdots & u_n \\
  0 & r_1 & \cdots & r_{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & s_1
\end{pmatrix} : u_1 \neq 0, \ldots, s_1 \neq 0 \]

= group of upper triangular invertible \( n \times n \) matrices.

Right Haar measure on \( \Delta_n \) is

\[
\frac{du_1 \cdots du_n dr_1 \cdots dr_{n-1} \cdots ds_1}{|u_1 r_1^2 \cdots s_1^n|}.
\]

We may identify \( \Delta_n \) with \( \Gamma_n \backslash G_n \) as a measure space, and hence we may regard \( \text{ind}_{\Gamma_n \backslash G_n} \phi \) as a representation \( \sigma \) on \( L^2(\Delta_n) \).

We now renormalize \( f_{u,r,\ldots,s} \) so that we can recapture \( f \) as an integral over \( \Delta_n \).

We have

\[
f = \int_F \cdots \int_{F^n} f_{u,r,\ldots,s} du \, dr \cdots ds.
\]

Set \( f_{u,r,\ldots,s} = \sqrt{|u_1 r_1^2 \cdots s_1^n|} \tilde{f}_{u,r,\ldots,s} \); then

\[
f = \int_F \cdots \int_{F^n} f_{u,r,\ldots,s} \frac{du \, dr \cdots ds}{|u_1 r_1^2 \cdots s_1^n|} = \int f_\alpha d\alpha;
\]

\[
f_\alpha(y, X) = (|u_1 r_1^2 \cdots s_1^n|)^{-1/2} \int_{\Gamma_n} f(\alpha^{-1} \gamma(y, X)) \phi(-\gamma) \, d\gamma,
\]

where

\[
\alpha = \begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & u_1 & \cdots & u_n \\
  0 & 0 & r_1 & \cdots & r_{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & s_1
\end{pmatrix}.
\]

We thus have \( L^2(G_n) \simeq \int_{\Delta_n} L^2(\Delta_n) \, d\alpha, \ \rho_{G_n} \simeq \int_{\Delta_n} \sigma \, d\alpha. \) We may identify \( \int_{\Delta_n} L^2(\Delta_n) \, d\alpha \) with \( L^2(\Delta_n) \otimes L^2(\Delta_n), \ \rho_{G_n} \simeq \sigma \otimes 1. \)
REFERENCES


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UNIVERSITY OF NEVADA, RENO
RENO, NV 89557
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