NORMS ON $F(X)$

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It is well known that if $\| \cdot \|$ is a norm on the field $F(X)$ of rational functions over a field $F$ for which $F$ is bounded, then $\| \cdot \|$ is equivalent to the supremum of a finite family of absolute values on $F(X)$, each of which is improper on $F$. Moreover, $\| \cdot \|$ is equivalent to an absolute value if and only if the completion of $F(X)$ for $\| \cdot \|$ is a field. We show that the analogous characterization of norms on $F(X)$ for which $F$ is discrete is impossible by constructing for each infinite field $F$, a norm $\| \cdot \|$ on $F(X)$ such that $F$ is discrete, $\| X \| < 1$, the completion of $F(X)$ for $\| \cdot \|$ is a field, but $\| \cdot \|$ is not equivalent to the supremum of finitely many absolute values.

1. Introduction and basic definitions. Let $R$ be a ring and let $\mathcal{T}$ be a ring topology on $R$, that is, $\mathcal{T}$ is a topology on $R$ making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to $R$. A subset $A$ of $R$ is bounded for $\mathcal{T}$ if given any neighborhood $U$ of zero, there exists a neighborhood $V$ of zero such that $AV \subseteq U$ and $VA \subseteq U$. $\mathcal{T}$ is a locally bounded topology on $R$ if there exists a fundamental system of neighborhoods of zero for $\mathcal{T}$ consisting of bounded sets.

We recall that a norm $\| \cdot \|$ on a ring $R$ is a function from $R$ to the nonnegative reals satisfying $\| x \| = 0$ if and only if $x = 0$, $\| x - y \| \leq \| x \| + \| y \|$ and $\| xy \| \leq \| x \| \| y \|$ for all $x$ and $y$ in $R$. If $\| \cdot \|$ is a norm on $R$, for each $\varepsilon > 0$ define $B_\varepsilon$ by, $B_\varepsilon = \{ r \in R : \| r \| < \varepsilon \}$. Then $\{ B_\varepsilon : \varepsilon > 0 \}$ is a fundamental system of neighborhoods of zero for a Hausdorff locally bounded topology $\mathcal{T}_{\| \cdot \|}$ on $R$. Two norms on $R$ are equivalent if they define the same topology. We note further that if $\| \cdot \|$ is a nontrivial norm on a field $K$ (that is, $\mathcal{T}_{\| \cdot \|}$ is nondiscrete), then a subset $A$ of $K$ is bounded for the topology defined by $\| \cdot \|$ if and only if $A$ is bounded in norm.

It is classic that, to within equivalence, the only valuations on the field $F(X)$ of rational functions over a field $F$ that are improper on $F$ are the valuations $v_p$, where $p$ is a prime polynomial of $F[X]$, and the valuation $v_\infty$ defined by the prime polynomial $X^{-1}$ of $F[X^{-1}]$([1, Corollary 2, p. 94]). For each valuation $v$, the function $| \cdot |_v$ defined by $| y |_v = 2^{-v(y)}$ for all $y$ in $F(X)$ is an absolute value on $F(X)$ for which $F$ is discrete. In [2, Theorem 2] we showed that if $\| \cdot \|$ is a nontrivial norm on $F(X)$ for which $F$ is bounded, then $\| \cdot \|$ is equivalent to the supremum of finitely
many of these absolute values. (This result was also obtained by Kiyek [5, Satz 2.11].) The analogous question of characterizing those norms \( \| \cdot \| \)
on \( F(X) \) for which \( F \) is discrete has been considered in several papers. (See for example [4, Theorem 4] and [10, Lemma 3]. We note that in each case the author has actually assumed that \( F \) is bounded.) In this paper we modify a technique of Mutylin [6] to show that such a characterization is impossible by constructing for each infinite field \( F \), a norm \( \| \cdot \| \) on \( F(X) \) for which \( F \) is discrete, \( \| X \| < 1 \), the completion of \( F(X) \) is a field but \( \| \cdot \| \) is not equivalent to the supremum of any finite family of absolute values on \( F(X) \). In the process, we also obtain a norm \( \| \cdot \| \) on the polynomial ring \( F[X] \) such that \( F \) is discrete and \( \| X \| < 1 \) but \( \| \cdot \| \) is not equivalent to the supremum of finitely many absolute values on \( F[X] \). (For a characterization of all norms on \( F[X] \) for which \( F \) is a bounded set, see [3, Theorem 2].)

2. Norms on \( F(X) \).

**Lemma 1.** Let \( F \) be an infinite field and let \( E \) be its prime subfield.

1. If \( F \) is finitely generated over \( E \), then there exists a nested sequence \( F_0, F_1, F_2, \ldots \) of subrings of \( F \) such that \( F_n \) is properly contained in \( F_{n+1} \) for all \( n \geq 0 \), \( 1 \in F_0 \) and \( F = \bigcup_{n=0}^{\infty} F_n \).

2. If \( F \) is not finitely generated over \( E \), then there exists a nested sequence \( F_0, F_1, F_2, \ldots \) of subfields of \( F \) such that \( F_n \) is properly contained in \( F_{n+1} \) for all \( n \geq 0 \) and \( F = \bigcup_{n=0}^{\infty} F_n \).

**Proof.** (1) \( F \) is either a finite algebraic extension of \( Q \) or there exists a subfield \( K \) of \( F \) and an element \( z \) in \( F \) which is transcendental over \( K \) such that \( F \) is a finite algebraic extension of \( K(z) \). If \( F \) is a finite algebraic extension of \( Q \), let \( p_0, p_1, \ldots \) be a sequence of distinct positive primes in \( Z \) and for each \( n \), let \( \delta_n \) be an extension of the \( p_n \)-adic valuation from \( Q \) to \( F \). Define \( F_n \) by,

\[
F_n = O(\{\delta_{n+1}, \delta_{n+2}, \ldots\}) = \{a \in F: v_i(a) \geq 0 \text{ for } i \geq n + 1\}.
\]

Then \( 1 \in F_0 \), each \( F_n \) is clearly a subring of \( F \) and \( F_n \subseteq F_{n+1} \) for all \( n \geq 0 \). As \( p_{n+2}/p_{n+1} \in F_{n+1} \setminus F_n \), \( F_n \) is properly contained in \( F_{n+1} \) for all \( n \geq 0 \). Finally, if \( a \in F \setminus \{0\} \), then \( \delta_p(a) = 0 \) for all but finitely many primes \( p \). Hence \( F = \bigcup_{n=0}^{\infty} F_n \).

If \( F \) is a finite algebraic extension of \( K(z) \), let \( p_0, p_1, \ldots \) be a sequence of distinct prime polynomials in \( K[z] \) and proceed as before.
(2) Suppose $F \setminus E$ is a countably infinite set $\{s_0, s_1, \ldots \}$. By induction on $n$, we define integers $k_0, k_1, \ldots$ and subfields $F_0, F_1, \ldots$ of $F$ satisfying:

(i) $k_0 < k_1 < \cdots$;

(ii) $F_n = E(s_0, s_1, \ldots, s_{k_n})$;

(iii) $F_n$ is properly contained in $F_{n+1}$.

Let $k_0 = 0$ and let $F_0 = E(s_0)$. Assume $k_0, k_1, \ldots, k_n$ and $F_0, F_1, \ldots, F_n$ have been defined satisfying (i)–(iii). As $F$ is not finitely generated over $E$, there exists an integer $t$ such that $s_t \notin F_n$. Let $k_{n+1}$ be the smallest integer $t$ satisfying this property and let $F_{n+1} = E(s_0, s_1, \ldots, s_{k_{n+1}})$. Properties (i)–(iii) obviously hold for $k_{n+1}$ and $F_{n+1}$ thus defined. By (i) and (ii), $F = \bigcup_{n=0}^{\infty} F_n$ and hence $F_0, F_1, \ldots$ is the desired sequence of subfields of $F$.

Suppose $F \setminus E$ is uncountable. Then the transcendence degree of $F$ over $E$ is infinite. Hence there exists a subfield $E_0$ of $F$ and distinct elements $x_0, x_1, \ldots$ of $F$ such that $\{x_i; i \geq 0\}$ is a transcendence base for $F$ over $E_0$. For each $n \geq 0$, let $F_n = \{a \in F: a$ is algebraic over $E_0(x_0, x_1, \ldots, x_n)\}$. $F_0, F_1, \ldots$ is then a sequence of subfields of $F$ satisfying the desired properties.

(The author is grateful to the referee for suggesting the above lemma as well as its proof.)

Henceforth, let $F$ be an infinite field and let $F_0, F_1, F_2, \ldots$ be a nested sequence of subrings of $F$ such that $F_n$ is properly contained in $F_{n+1}$ for all $n \geq 0$, $1 \in F_0$ and $F = \bigcup_{n=0}^{\infty} F_n$. For each $a \in F$, let $\phi(a)$ denote the smallest nonnegative integer $n$ such that $a \in F_n$. Clearly:

(1) $\phi(a \pm b) \leq \max\{\phi(a), \phi(b)\}$ for all $a, b$ in $F$.

(2) $\phi(ab) \leq \max\{\phi(a), \phi(b)\}$ for all $a, b$ in $F$.

Define $| \cdot |$ from $F$ to $N \cup \{0\}$ by,

$$|a| = \begin{cases} 2^{\phi(a)} & \text{if } a \in F \setminus \{0\}, \\ 0 & \text{if } a = 0. \end{cases}$$

Obviously, $|a| = 0$ if and only if $a = 0$. Furthermore from (1) and (2) we obtain

$$|a \pm b| \leq \max\{|a|, |b|\} \quad \text{and} \quad |ab| \leq \max\{|a|, |b|\}$$

for all $a$ and $b$ in $F$. As $|a| \geq 1$ for all $a \in F \setminus \{0\}$, $|ab| \leq |a| |b|$ for all $a$ and $b$ in $F$. Thus $| \cdot |$ is a norm on $F$.

Let $x$ be any transcendental element over $F$ in some field extension, let $F(x)$ be the field of rational functions over $F$ and let $F((x))$ denote the
field of formal power series over $F$, that is, $F((x)) = \{ \sum_{i=m}^{\infty} a_i x^i : m \in \mathbb{Z}, a_i \in F \text{ for all } i \geq m \}$. As $F((x))$ is the completion of $F(x)$ for the $x$-adic valuation $v_x$ defined on $F(x)$ [8, p. 243], we may identify $F(x)$ with a subfield of $F((x))$.

Define $N$ from $F((x))$ to $[0, \infty]$ by,

$$N(y) = \sup_i |a_i| 2^{-i} \text{ for } y = \sum a_i x^i \in F((x)).$$

**Lemma 2.** (1) $N(y) = 0$ if and only if $y = 0$.

(2) $N(y_1 + y_2) \leq \max\{N(y_1), N(y_2)\}$ for all $y_1, y_2$ in $F((x))$.

(3) $N(y_1 y_2) \leq N(y_1)N(y_2)$ for all $y_1, y_2$ in $F((x))$.

**Proof.** As (1) and (2) follow easily from the corresponding properties of $| \cdots |$, it suffices to prove (3). Let $y_1 = \sum a_i x^i$ and $y_2 = \sum b_i x^i$ be elements of $F((x))$. Then $y_1 y_2 = \sum c_i x^i$ where $c_n = \sum_{i+j=n} a_i b_j$ for all $n \in \mathbb{Z}$. Hence

$$N(c_n x^n) = N \left( \sum_{i+j=n} a_i x^i b_j x^j \right) \leq \max_{i+j=n} N(a_i x^i b_j x^j)$$

$$\leq \max_{i+j=n} N(a_i x^i) N(b_j x^j) \leq N(y_1)N(y_2).$$

Therefore

$$N(y_1 y_2) = \sup_n N(c_n x^n) \leq N(y_1)N(y_2) \text{ for } y_1, y_2 \in F((x)).$$

By the above lemma, the set $R$ defined by, $R = \{ y \in F((x)) : N(y) < \infty \}$, is a subring of $F((x))$ and $N$ is a norm on $R$. Let $D$ be the subset of $R$ defined by,

$$D = \left\{ \sum_{i=m}^{\infty} a_i x^i : m \in \mathbb{Z}, a_i \in F \text{ for all } i \geq m \text{ and } \lim_{i \to \infty} |a_i| 2^{-i} = 0 \right\}.$$

**Lemma 3.** $D$ is a subfield of $R$, $D$ is complete with respect to the $N$-topology and $F(x)$ is a dense subfield of $D$.

**Proof.** Clearly, for any $a \in F$ and any $m \in \mathbb{Z}$, $aD \subseteq D$ and $x^m D \subseteq D$. We first show that for any $y \in D \setminus \{0\}$, $y^{-1} \in D$. By the preceding observation, we may assume that $y = \sum_{i=0}^{\infty} a_i x^i$ where $a_0 = 1$. Then $y^{-1} = \sum_{i=0}^{\infty} b_i x^i$ where $b_0 = 1$ and for all $n \geq 1$, $b_n = -\sum_{i+j=n; 0 \leq j < n} a_i b_j$. For
each $n \geq 0$, let $\gamma_n = \max\{|a_i| : 0 \leq i \leq n\}$. An inductive argument establishes that $|b_n| \leq \gamma_n$ for all $n \geq 0$. As $|a_n|2^{-n} \to 0$, it follows that $\gamma_n2^{-n} \to 0$ and so $|b_n|2^{-n} \to 0$, that is, $y^{-1} \in D$.

To complete the proof of the lemma we shall make use of the following alternate construction of $R$, $D$ and $N$. Let $Z$ be given the discrete topology and let $v: Z \to (0, \infty)$ be defined by $v(n) = 2^{-n}$ for all $n \in Z$. Denote the set of all continuous functions $f$ from $Z$ to $F$ for which $\|f\|_v = \sup_{i \in Z} v(i)|f(i)| < \infty$ by $C^v(Z, F)$, the set of all $f$ in $C^v(Z, F)$ such that $f$ vanishes at $\infty$ (that is, for each $\varepsilon > 0$, there exists a compact subset $K$ of $Z$ such that $\|f \cdot \chi_{Z \setminus K}\|_v < \varepsilon$) by $C^\infty_v(Z, F)$, and the set of all $f$ in $C^\infty_v(Z, F)$ with compact support by $C^\circ_v(Z, F)$. Then $Z$ is a locally compact space, $v$ is continuous, $C^\infty_v(Z, F)$ is a closed subgroup (under $+$) of the complete, normable group $C^v(Z, F)$ and $C^\circ_v(Z, F)$ is a dense subset of $C^\infty_v(Z, F)$. (The proof of this assertion is similar to the proof in the classical case where $F$ is $\mathbb{R}$ or $\mathbb{C}$. For a discussion of this case see, for example, [7]). For each $y = \sum a_i x^i \in F((x))$, we may identify $y$ with the function $f$ defined from $Z$ to $F$ by $f(i) = a_i$ for all $i \in Z$. With this identification,

$$R = C^v(Z, F), \quad D = C^\infty_v(Z, F), \quad F[x] \subseteq C^\circ_v(Z, F) \subseteq F(x),$$

$$C^\circ_v(Z, F) \subseteq D \quad \text{and} \quad N(y) = \|y\|_v \quad \text{for all} \quad y \in R.$$ 

Moreover, $C^v(Z, F)$ and $C^\circ_v(Z, F)$ are topological rings under the multiplication $(f \cdot g)(i) = \sum_{m+n=i} f(m)g(n)$. As $(C^\circ_v(Z, F), \|\cdot\|_v)$ is complete, $(D, N)$ is complete as well. Further, as $C^\circ_v(Z, F) = C^\infty_v(Z, F)$, $D$ is a subring of $R$ and hence a subfield of $R$ by the previous observation. Thus $F(x) \subseteq D$ and so $D = \overline{C^\circ_v(Z, F)} \subseteq \overline{F(x)} \subseteq D$, that is, $F(x)$ is a dense subfield of $D$.

**Theorem 1.** Let $F$ be an infinite field, let $F_0, F_1, F_2, \ldots$ be a nested sequence of subrings of $F$ such that $F_n$ is properly contained in $F_{n+1}$ for all $n \geq 0$, $1 \in F_0$ and $F = \bigcup_{n=0}^{\infty} F_n$, and let $x$ be any transcendental element over $F$ in some field extension. Then there exists a norm $\|\cdot\|$ on $F(x)$ such that $F$ is discrete, $\|x\| < 1$, the completion of $F(x)$ for $\|\cdot\|$ is a field but $\|\cdot\|$ is not equivalent to the supremum of a finite family of absolute values on $F(x)$. Moreover for each $n \geq 0$, the topology induced on $F_n(x)$ by $\|\cdot\|$ is the same as that induced on $F_n(x)$ by the $x$-adic valuation $v_x$ defined on $F(x)$.

**Proof.** Let $\|\cdot\|$ denote the restriction of $N$ to $F(x)$. By Lemmas 2 and 3, $\|\cdot\|$ is a norm on $F(x)$ and the completion of $F(x)$ for $\|\cdot\|$ is a
field. By definition, \( \| x \| = 2^{-1} < 1 \) and for each nonzero \( a \) in \( F \), \( \| a \| = | a | \geq 1 \). Hence \( F \) is discrete for \( \| \cdot \| \).

Suppose \( \| \cdot \| \) is equivalent to the supremum of a finite family \( \{ | \cdot |_i : 1 \leq i \leq n \} \) of absolute values on \( F(x) \). As the completion of \( F(x) \) for \( \| \cdot \| \) is a field, \( n = 1 \) by the Approximation Theorem for Absolute Values [1, Theorem 2, p. 136]. As \( F \) is discrete for \( \| \cdot \| \), \( F \) is discrete for \( | \cdot |_1 \) as well, that is, \( | a |_1 = 1 \) for all \( a \) in \( F \{0\} \). Thus \( F \) is a bounded set for the topology induced on \( F(x) \) by \( | \cdot |_1 \). However, if \( n \) is any positive integer and \( a_n \) is any element of \( F_n \{F_{n-1}\} \), then \( \| a_n \| = | a_n | = 2^n \). Therefore \( F \) is not bounded for the topology defined on \( F(x) \) by \( \| \cdot \| \), a contradiction.

To prove the last assertion of the theorem, we note that for any \( n \geq 0 \) and for any \( y \) in \( F_n(x) \),
\[ 2^{-v_p(y)} \leq \| y \| \leq 2^n 2^{-v_p(y)}. \]

In [9] Weber showed that if \( F \) is a field and \( x \) is any transcendental element over \( F \), then \( F \) is finite if and only if for each Hausdorff, nondiscrete locally bounded topology \( \mathcal{T} \) on \( F(x) \), there exists a nonempty proper subset \( S \) of \( \mathcal{P}' = \{ p: \text{p is a prime polynomial of } F[x] \} \cup \{ \infty \} \) such that the set \( O(S) \) defined by, \( O(S) = \{ y \in F(x): v_p(y) \geq 0 \text{ for all } p \in S \} \), is a bounded neighborhood of zero for \( \mathcal{T} \) (Satz 3.3). The following is a generalization of this result.

**Corollary.** Let \( F \) be a field and let \( x \) be any transcendental element over \( F \). The following are equivalent.

1. \( F \) is a finite field.
2. If \( \mathcal{T} \) is a Hausdorff, nondiscrete locally bounded topology on \( F(x) \), then there exists a nonempty, proper subset \( S \) of \( \mathcal{P}' \) such that \( O(S) \) is a bounded neighborhood of zero for \( \mathcal{T} \).
3. If \( \| \cdot \| \) is a nontrivial norm on \( F(x) \) such that \( F \) is discrete and the completion of \( F(x) \) for \( \| \cdot \| \) is a field, then \( \| \cdot \| \) is equivalent to an absolute value which is improper on \( F \).

**Proof.** By the above remarks, (1) and (2) are equivalent. By Theorem 1, (3) implies (1). So it suffices to show that (1) implies (3). Suppose \( F \) is a finite field and \( \| \cdot \| \) is a nontrivial norm on \( F(x) \) such that the completion of \( F(x) \) for \( \| \cdot \| \) is a field. Then \( F \) is bounded in norm and so by the corollary to Theorem 2 of [2], \( \| \cdot \| \) is equivalent to an absolute value on \( F(x) \) which is improper on \( F \).

In [3] we characterized all norms on the polynomial ring \( F[x] \) for which \( F \) is bounded (Theorem 2). We conclude this paper by showing that
if $F$ is any infinite field, the analogous characterization of the norms on $F[x]$ for which $F$ is discrete is impossible.

**Theorem 2.** Let $F$ be an infinite field and let $x$ be any transcendental element over $F$ in some field extension. Then there exists a norm $\| \cdot \|$ on $F[x]$ such that $F$ is discrete and $\| x \| < 1$ but $\| \cdot \|$ is not equivalent to the supremum of a finite family of absolute values on $F[x]$.

**Proof.** Let $\| \cdot \|$ be the norm on $F(x)$ constructed in the proof of Theorem 1 and let $\| \cdot \|'$ denote its restriction to $F[x]$. Clearly, $F$ is discrete for $\| \cdot \|'$ and $\| x \|' < 1$. Suppose $\| \cdot \|'$ is equivalent to the supremum of a finite family $\{ | \cdot |_i; 1 \leq i \leq n \}$ of absolute values on $F[x]$. Then each $| \cdot |_i$ is improper on $F$. Indeed, suppose there exist $i$, $1 \leq i \leq n$, and $a \in F$ such that $| a |_i > 1$. Let $m$ be such that $| a^m x |_i > 1$. The sequence $\langle (a^m x)^r \rangle_{r=1}^{\infty}$ converges to 0 for $\| \cdot \|'$ but not for $| \cdot |_i$, a contradiction. Hence each $| \cdot |_i$ is improper on $F$. It then follows that $F$ is bounded for the supremum topology but not for the topology defined on $F[x]$ by $\| \cdot \|'$, a contradiction.

**References**


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