Pacific Journal of Mathematics

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Vol. 109, No. 1 May 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 109, No. 1, 1983

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If F is a local non-Archimedean field, then every irreducible admissible representation π of $\mathrm{GL}(r, F)$ is a quotient of a representation ξ induced by tempered ones. We show that ξ has a Whittaker model, even though it may fail to be irreducible.

1. Introduction and notations.

(1.1) Let F be a local non-Archimedean field and ψ an additive character of F. Let G be the group $\mathrm{GL}(2, F)$ and B the subgroup of triangular matrices in G. If μ_1 and μ_2 are two characters of $F^\times$ we may consider the induced representation $\xi = \mathrm{Ind}(G, B;\ \mu_1, \mu_2)$. There is a nonzero linear form λ on the space V of ξ such that

$$\lambda\left[\xi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right] = \psi(x)\lambda(f), \qquad f \in V.$$

The map which sends f to the function W, defined by

$$W(g) = \lambda[\xi(g)f], \tag{1}$$

is clearly bijective if ξ is irreducible, that is, if $\mu_1 \cdot \mu_2^{-1} \neq \alpha_F^{\pm 1}$ (we denote by α_F or α the module of F). If $\mu_1 \cdot \mu_2^{-1} = \alpha^{-1}$, the kernel of the map is one dimensional. If $\mu_1 \cdot \mu_2^{-1} = \alpha$ the map has trivial kernel. We recall the proof. Suppose more generally that $\mu_1 \cdot \mu_2^{-1} = \chi\alpha^u$ with $\chi\bar{\chi} = 1$ and $0 < u$. Then we may choose λ in such a way that

$$W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \hat{H}(-a)\mu_2(a)\,|a|^{1/2}, \qquad \hat{H}(a) = \int H(x)\psi(xa)\,dx,$$

where H is the element of $L^1(F)$ defined by

$$H(x) = f\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right].$$

From the Fourier inversion formula, $W|B$ implies $H = 0$ and then, by continuity, $f = 0$. Thus we have proved the injectivity of the map $f \mapsto W$ and even the fact that the W's are determined by their restriction to B.

(1.2) In this paper we extend this result (and its proof) to the group $G_r = \mathrm{GL}(r, F)$, $r \geq 2$. In a precise way, let Q be the upper standard parabolic subgroup of type $(r_1, r_2, \ldots, r_m)$, $\sum r_i = r$, in G_r. Then $Q = MU$ where U is the unipotent radical of Q and M isomorphic to $\Pi\mathrm{GL}(r_i)$. Let π_i, $1 \leq i \leq m$, be an irreducible representation of $\mathrm{GL}(r_i, F)$; suppose $\pi_i = \pi_{i,0} \otimes \alpha^{u_i}$, where $\pi_{i,0}$ is irreducible, unitary, *tempered* and $u_1 > u_2 > \cdots > u_m$. We refer to the induced representation

$$(1) \qquad \xi = \mathrm{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m)$$

as an induced representation of "Langlands' type". Let now N_r be the group of upper triangular matrices with unit diagonal and let θ or θ_r be the character of N_r defined by

$$(2) \qquad \theta(n) = \prod_{i=1}^{r-1} \psi(n_{i,i+1}).$$

Then there is a nonzero linear form λ on the space of ξ and, up to a scalar factor, only one such that

$$(3) \qquad \lambda[\xi(n)f] = \theta(n)\lambda(f).$$

Let $\mathcal{W}(\xi; \psi)$ be the space spanned by the functions of the form (1.1.1). Our goal is to prove that the map $f \mapsto W$ is bijective, even though ξ may be reducible. In fact we prove a little more: in the terminology of [**B-Z**] (Theorem 4.9) the representation ξ has a Kirillov model. We remark that when all $\pi_{i,0}$ are supercuspidal, our result is a special case of Theorem 4.11 in [**B-Z**]. In general, one can try to reduce our result to theirs by imbedding each $\pi_{i,0}$ in a representation induced by supercuspidal ones (cf. [**Z**]). For instance, denote by B_r the group of upper-triangular matrices in G_r and by σ_r the (unique) invariant irreducible subspace of

$$\mathrm{Ind}\left(G_r, B_r; \alpha^{(r-1)/2}, \alpha^{(r-1)/2-1}, \ldots, \alpha^{-(r-1)/2}\right).$$

Then σ_r is a square-integrable representation (ordinary special representation). Consider now the induced representation

$$\xi = \mathrm{Ind}\left(G_5, Q; \sigma_3 \otimes \alpha^{1/2}, \sigma_2\right),$$

where Q has type (3, 2). Then ξ is a subrepresentation of

$$\eta = \mathrm{Ind}(G_5, B_5; \rho_1, \rho_2, \ldots, \rho_5)$$

where $\rho_3 = \alpha^{-1/2}$, $\rho_4 = \alpha^{1/2}$. Since $\rho_4 = \rho_3 \otimes \alpha$, Theorem 4.11 of [**B-Z**] does not apply to η. Thus our result does not follow directly from Theorem 4.11 of [**B-Z**]; some extra work is needed.

At any rate, our approach is more direct and we use the results of Bernstein-Zelevinski only in an auxiliary way. In more detail, let P_r be the subgroup of matrices p in G_r of the form

$$p = \begin{pmatrix} g & * \\ 0 & 1 \end{pmatrix}, \qquad g \in G_{r-1}.$$

Call τ_r the unitary representation of P_r induced (in Mackey's sense) by θ_r. Then τ_r is irreducible and the right regular representation of P_r is a multiple of τ_r; the right regular representation of G_r has the same property, when restricted to P_r. Thus, if π is an irreducible (preunitary) square-integrable representation, then denoting by $\bar{\pi}$ the corresponding unitary representation, we see that $\bar{\pi} \mid P_r$ is a multiple of τ_r. (Cf., for instance, [**J**]). Thus π is generic, that is, there is a linear form $\lambda \neq 0$ on the space V of π satisfying (1.2.3). Since λ is unique, within a scalar factor, we see that in fact $\bar{\pi} \mid P_r \simeq \tau_r$. Finally if η is an induced representation of the form

$$\eta = \operatorname{Ind}(G_r Q; \pi_1, \pi_2, \ldots, \pi_m),$$

where the π_i are irreducible square-integrable, then η is pre-unitary and $\bar{\eta} \mid P_r \simeq \tau_r$ (loc. cit.). In particular η is irreducible. This shows that if π is any irreducible pre-unitary tempered representation of G_r then $\bar{\pi} \mid P_r \simeq \tau_r$. This is, *essentially*, all we need to know about tempered representations (cf. §2 below).

We also remark that the problem of finding all irreducible square-integrable representations of G_r is equivalent to the problem of finding all irreducible generic ones. Indeed, if π is a square-integrable representation, then π is generic by the above remarks, thus by Theorem 9.7 of [**B-Z**] (classification of all generic representations) π is equivalent to an induced representation of the form

$$\xi = \operatorname{Ind}(G_r, Q; \sigma_1, \sigma_2, \ldots, \sigma_m)$$

where the σ_i are "generalized special representations". But then Casselman's criterion for square-integrability shows that, in fact, ξ is itself a generalized special representation: this is a sketch of the proof of Theorem 9.3 stated in [**Z**] and due to I. N. Bernstein. Conversely if ξ is a representation of the form (1.2.1) then ξ has a unique irreducible quotient $J(\pi_1, \pi_2, \ldots, \pi_m)$ ("Langlands' quotient": cf. [**B-W**] XI, §2). If ξ is irreducible then our result implies that $J(\pi_1, \pi_2, \ldots, \pi_m)$ is degenerate (not generic). Since any irreducible representation π of G_r has the form $J(\pi_1, \pi_2, \ldots, \pi_r)$ for appropriate π_i, we see that if π is generic then π must be equivalent to a representation of the form (1.2.1); that is, we have another proof of Theorem 9.7 of [**B-Z**].

Finally we also remark that our result and its proof apply to the case $F = \mathbf{R}$ or $\mathbf{C}$ as well. Naturally λ in (1.1.3) and (1.1.1) is then a linear form defined and continuous on an appropriate space of smooth vectors to which f belongs. One needs to duplicate the estimates of §2 and check that in (3.1.2), the linear form $f \mapsto W(e)$ can be taken to be λ, that is, is continuous. Furthermore in (3.2.15) the right-hand side does not have support in the set (3.2.16) but is "of rapid decrease for $|a_i|$ large". Rather than dealing with these minor changes now we prefer to wait for another occasion. We also remark that, taking again into account Langlands' classification and Theorem D of [**K**], we get, for $\mathrm{GL}(r, F)$, another easy proof of the difficult Theorem 6.2 of [**V**].

However, on the whole, our motivations are global. In [**J-P-S**] Theorem (13.6) and [**G-J**], §4 we used this result for GL(3). Similar applications are expected for higher r's.

(1.3) In addition to the notations already introduced we will use the following ones: q will be the cardinality of the residual field of F, $\mathfrak{R}$ the ring of integers in F; K_r will be the subgroup $\mathrm{GL}(r, \mathfrak{R})$. We will denote by Z_r the center of G_r, by A_r the subgroup of diagonal matrices in G_r, by $B_r = A_r N_r$ the group of upper triangular matrices and, finally, by P_r the subgroup of matrices of the form

$$(1) \qquad p = \begin{pmatrix} g & * \\ 0 & 1 \end{pmatrix}, \qquad g \in G_{r-1}.$$

2. Estimate of tempered Whittaker functions.

(2.1) Let π be an irreducible pre-unitary tempered representation of G_r. Then there is a linear form $\lambda \neq 0$ on the space V of π satisfying (1.2.3) and, within a scalar factor, only one. We denote by $\mathcal{W}(\pi; \psi)$ the space spanned by functions of the form (1.1.1) with f in V. We recall some known facts on the elements of $\mathcal{W}(\pi; \psi)$.

(2.2) If W is in $\mathcal{W}(\pi; \psi)$ then the integral

$$\Psi(s, W, \overline{W}, \Phi) = \int_{N_r \backslash G_r} W(g)\overline{W}(g)\Phi[(0,0,\ldots,0,1)g]\,|\det g|^s\,dg,$$

where Φ is in the space $\mathcal{S}(F^r)$ of Schwartz-Bruhat functions on F^r, converges for $\mathrm{Re}\, s \gg 0$ and represents a rational fraction in q^{-s} without pole for $\mathrm{Re}\, s > 0$ ([**J-P-S**] Prop. (8.4)); in passing we note that this result is independent of the classification of all square-integrable representations.

(2.3) The unitary representation of G_r corresponding to π has the property that its restriction to the subgroup P_r is equivalent to the representation τ_r of P_r induced (in Mackey's sense) by θ_r. It amounts to the same to say that

(1) $$B(W, W') = \int_{N_{r-1}\backslash G_{r-1}} W\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] \overline{W'}\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh$$

defines a G_r-invariant form on $\mathcal{W}(\pi; \psi)$ (cf. [**J**]). From this or Theorem (4.9) of [**B-Z**] it follows that any W is determined by its restriction to P_r.

(2.4) Finally, the space of these restrictions contains the space $\mathcal{K}_0(\pi; \psi)$ of functions f on G_r, transforming on the left under θ_r, right smooth and of compact support mod N_r ([**G-K**] (5.2)).

(2.5) We need an estimate for the elements of $\mathcal{W}(\pi; \psi)$. The quickest proof uses (2.2). Let δ_r denote the module of the Borel subgroup B_r in G_r. We will extend δ_r to a function on G_r which is K_r-invariant on the right. We remark that

(1) $$\delta_r\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] = \delta_{r-1}(g)\,|\det g|$$

if g is in G_{r-1}. We also define a function Λ_r on G_r by setting

(2) $$\Lambda_r\left(zn\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}k\right) = |\det g|$$

for $z \in Z_r$, $n \in N_r$, $k \in K_r$, $g \in G_{r-1}$.

PROPOSITION. *Suppose π is a tempered representation of G_r and W is in $\mathcal{W}(\pi; \psi)$. Then, for any $s > 0$, there is a constant $c_s > 0$ such that $|W|^2 \leq c_s \delta_r \Lambda_r^{-s}$.*

Proof. Let $\Phi \geq 0$ be an element of $\mathcal{S}(F^r)$ which is K_r invariant on the right. Then, for $s \gg 0$, setting $\eta_r = (0, 0, \ldots, 0, 1)$, we have:

(1) $$\Psi(s, W, \overline{W}, \Phi)$$
$$= \int_{K_r} dk \int_{A_{r-1}} |W|^2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}k\right] \delta_r^{-1}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] |\det a|^s \, d^\times a$$
$$\times \int_{F^\times} \Phi[\eta_r bk]\, |b|^{rs}\, d^\times b.$$

The convergence of the integral for Res $\gg 0$ amounts to the convergence of a power series in $x = q^{-s}$,

(2) $$\Psi(s, W, \overline{W}, \Phi) = \sum_{m \geq m_0} a_m x^m,$$

say for $0 <|x|< \varepsilon$ (cf. (4.1) and (4.2) in [J-P-S]). By (2.2), the series in (2) actually converges for $0 <|x|< 1$. But then since the integrand in (1) is ≥ 0, the integral for Ψ must actually converge for $s > 0$. In particular

$$(3) \qquad \int |W|^2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] \delta_r^{-1}\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right] |\det a|^s \, d^\times a < +\infty$$

(for $s > 0$) for all $k \in K_r$. Fix k then and let us denote by $f(a)$, $a \in (F^\times)^{r-1} \simeq A_{r-1}$, the integrand in (3). Clearly there is an open compact subgroup, U say, of $(F^\times)^{r-1}$ such that $f(a\varepsilon) = f(a)$ for all a in $(F^\times)^{r-1}$, ε in U. We deduce at once that, for all $b \in (F^\times)^{r-1}$,

$$|f(b)| \leq c \int |f(a)| \, d^\times a,$$

c a positive constant. In other words the integrand in (3) is bounded. This is precisely what we wanted to prove. □

3. Induced representations of Langlands' type.

(3.1) Consider a representation

$$(1) \qquad \xi = \operatorname{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m)$$

(notations as in (1.2)). A vector f in the space of ξ may be regarded as a function on G_r with values in $\otimes_{i=1}^m \mathcal{W}(\pi_i; \psi)$; it may also be regarded as a scalar function on $G_r \times G_{r_1} \times \cdots \times G_{r_m}$ whose value at $(g, h_1, h_2, \ldots, h_m)$ we denote by $f(g; h_1, h_2, \ldots, h_m)$. The integral

$$(2) \qquad W(g) = \int_U f(wug; e, e, \ldots, e)\bar{\theta}(u) \, du,$$

where

$$(3) \qquad w = \begin{pmatrix} 0 & & 1_{r_1} \\ & 1_{r_2} & \\ & \cdot^{\cdot^{\cdot}} & \\ 1_{r_m} & & 0 \end{pmatrix},$$

and du is a Haar-measure on the unipotent radical U of the parabolic subgroup of type $(r_m, r_{m-1}, \ldots, r_2, r_1)$, defines an element of $\mathcal{W}(\xi; \psi)$ provided it converges. We are going to show that it converges for all f; in fact, we are going to obtain a majorization of the function

$$(4) \qquad h \mapsto \int f(wug; e, e, \ldots, e, h) \, du.$$

It will be sufficient to obtain an upper bound for the integral

$$\int |f|(wug;\, e, e, \dots, e, h)\, du. \tag{5}$$

This integral, finite or infinite, is equal to

$$|\det h|^{-(r-r_m)/2} \int |f| \left[wu \begin{pmatrix} h & 0 \\ 0 & 1_{r-r_m} \end{pmatrix} g;\, e, e, \dots, e \right] du. \tag{6}$$

With notation as in (2.5), let f_0 be the function defined by

$$f_0(g) = \delta_Q^{1/2}(q) \prod_{j=1}^{m} \delta_{r_j}^{1/2}(g_j) \Lambda_{r_j}(g_j)^{-s_j} |\det g_j|^{u_j}, \tag{7}$$

for g of the form $g = qk$, $q \in Q$, $k \in K_r$ and q of the form

$$q = \begin{pmatrix} g_1 & & & * \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_m \end{pmatrix}, \qquad g_i \in G_{r_i}. \tag{8}$$

Here $(s_1, s_2, \dots, s_m)$ is an m-tuple of positive numbers to be chosen below. Next we apply Proposition (2.5) to the (quasi-) tempered representations π_i $(1 \le i \le m)$ to conclude that given $g_0 \in G_r$, there is a constant $c > 0$ such that

$$|f|(gg_0;\, e, e, \dots, e) \le cf_0(g). \tag{9}$$

Thus all we need to do is to obtain an upper bound for the function

$$|\det h|^{-(r-r_m)/2} \int f_0 \left[wu \begin{pmatrix} h & 0 \\ 0 & 1_{r-r_m} \end{pmatrix} \right] du. \tag{10}$$

This is actually equal to

$$\int f_0(wu)\, du\, \delta_{r_m}^{1/2}(h) |\det h|^{u_m} \Lambda_{r_m}(h)^{-s_m}. \tag{11}$$

We are thus reduced to proving that

$$\int f_0(wu)\, du < +\infty. \tag{12}$$

For that let V denote the unipotent radical of the lower parabolic subgroup of G_r of type $(r_1, \dots, r_m)$. Then the integral (11) is the same as the integral

$$\int_V f_0(v)\, dv. \tag{13}$$

Next for q a diagonal matrix of the form (8), we have

$$\delta_B(q) = \delta_Q(q) \prod_{1 \le j \le m} \delta_{r_j}(g_j),$$

from which we see that for $q = \mathrm{diag}(a_1, a_2, \dots, a_r)$

(14) $$f_0(q) = \delta_B^{1/2}(a)\,|a_1 a_2 \cdots a_{r_1-1}|^{u_1 - s_1}\,|a_{r_1}|^{(r_1-1)s_1+u_1}$$
$$\cdot |a_{r_1+1} \cdots a_{r_1+r_2-1}|^{u_2-s_2}\,|a_{r_1+r_2}|^{(r_2-1)s_2+u_2} \cdots .$$

We have seen then that to insure the convergence of (13) it suffices to choose the $s_i > 0$ so that

(15) $$u_1 + (r_1 - 1)s_1 > u_1 - s_1 > u_2 + (r_2 - 1)s_2 > u_2 - s_2 > \cdots .$$

Each inequality in (15) is either true or can be made true by making the s_i positive and sufficiently small. We have now proved that the integral in (2) is indeed convergent and, moreover, obtained the inequality

(16) $$\int_U |f|(wug; e, e, \dots, e, h)\,du \le c_v \delta_{r_m}^{1/2}(h) \Lambda_{r_m}(h)^{-v} |\det h|^{u_m},$$

where v is any sufficiently small positive number and w is given by (3).

(3.2) PROPOSITION. *Let ξ be the representation* (3.1.1). *Then the map $f \mapsto W$ from the space of ξ to $\mathcal{W}(\xi;\,\psi)$ defined by* (3.1.2) *is bijective. Moreover, if $W \in \mathcal{W}(\xi;\,\psi)$ then the relation $W | P_r = 0$ implies $W = 0$.*

Proof. Our assertion is trivial for $m = 1$. Thus we may assume $m > 1$ and our assertion proved for $m - 1$. Consider then the induced representation

(1) $$\xi^* = \mathrm{Ind}(G_r, Q^*; \xi', \pi_m),$$

where

(2) $$\xi' = \mathrm{Ind}\big(G_{r-r_m}, Q'; \pi_1, \pi_2, \dots, \pi_{m-1}\big),$$

where Q^* has type $(r - r_m, r_m)$ and Q' has type $(r_1, r_2, \dots, r_{m-1})$. Furthermore, by the induction hypothesis, we may regard ξ' as acting on $\mathcal{W}(\xi';\,\psi)$. Thus we may regard an element f^* of ξ^* as a function on G_r with values in $\mathcal{W}(\xi';\,\psi) \otimes \mathcal{W}(\pi_m;\,\psi)$, or as a scalar function on $G_r \times G_{r-r_m} \times G_{r_m}$. We denote its value at (g, h_1, h_2) by $f^*(g;\, h_1, h_2)$. Of course the representations ξ and ξ^* are equivalent. If f, as in (3.1), is in the space of ξ then the exact relation between f and f^* is given by

(3) $$f^*[g; e, e] = \int_{V'} f\left[\begin{pmatrix} w' & 0 \\ 0 & 1_{r_m} \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & 1_{r_m} \end{pmatrix} g; e, e, \dots, e\right] \bar{\theta}_{r-r_m}(v)\,dv,$$

where

$$(4)\qquad w' = \begin{pmatrix} 0 & & & 1_{r_1} \\ & & 1_{r_2} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1_{r_{m-1}} & & & 0 \end{pmatrix},$$

and V' is the unipotent radical of the (upper) parabolic in G_{r-r_m} of type $(r_{m-1}, r_{m-2}, \ldots, r_1)$. Writing (11.2) as an iterated integral, we readily find that in terms of f^*,

$$(5)\qquad W(g) = \int_{V^*} f^*[w^* vg; e, e]\bar{\theta}_r(v)\, dv,$$

where now

$$(6)\qquad w^* = \begin{pmatrix} 0 & 1_{r-r_m} \\ 1_{r_m} & 0 \end{pmatrix},$$

and V^* is the unipotent radical of the parabolic in G_r of type $(r_m, r - r_m)$. Of course the convergence of the integral (3.1.2) implies that of both integrals (3) and (5) (for *all* $g \in G_r$). Since the map $f \mapsto f^*$ is bijective, all of our assertions will be proved if we show

$$(7)\qquad W | P_r = 0 \text{ implies that } f^* = 0.$$

Assume then that $W | P_r = 0$. Explicitly this reads

$$(8)\qquad \int f^*\left[w^* \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, e \right] \psi(\mathrm{tr}(\varepsilon x))\, dx = 0$$

for all $p \in P_r$. Here

$$(9)\qquad \varepsilon = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (r - r_m \text{ rows, } r_m \text{ columns}).$$

Replacing p by

$$\begin{pmatrix} g_1 & 0 \\ 0 & 1_{r-r_m} \end{pmatrix} p,$$

where $g_1 \in G_{r_m}$, and changing variables, we can write this condition in the form

$$(10) \qquad \int f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m}\end{pmatrix} p;\, e,\, g_1\right]\psi(\mathrm{tr}(\varepsilon g_1 x))\, dx = 0,$$

for all $p \in P_r$, $g_1 \in G_{r_m}$. We can also replace g_1 by hg_1 where $h \in P_{r_m}$. Note that $\varepsilon h = \varepsilon$. Thus if we set, for $h \in G_{r_m}$,

$$(11) \qquad F(h) = \int f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m}\end{pmatrix} p;\, e,\, hg_1\right]\psi(\mathrm{tr}(\varepsilon g_1 x))\, dx,$$

then we see that the function F defined on G_{r_m} has a zero restriction to P_{r_m}. At this point we may assume $u_m = 0$. We are going to show that F is actually zero. To see that we first need a majorization of F. Using (3) to express f^* in terms of f we obtain at once from (3.1.16):

$$(12) \qquad |F(h)| \leq c_v \delta_{r_m}^{1/2}(h)\Lambda_{r_m}(h)^{-v},$$

again for $v > 0$ sufficiently small, and all $h \in G_{r_m}$.

Thus, for $W' \in \mathcal{W}(\pi_m; \psi)$, we have the inequality

$$(13) \quad \int_{N_{r_m-1}\backslash G_{r_m-1}} |FW'|\left[\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\right] dh$$

$$\leq c_v \int_{N_{r_m-1}\backslash G_{r_m-1}} |W'|\left[\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\right]\delta_{r_m}^{1/2}\left[\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\right]\Lambda_{r_m}\left[\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix}\right]^{-v} dh.$$

We claim now that both integrals are finite. It suffices to check that the integral

$$(14) \qquad \int_{A_{r_m-1}} |W'|\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]\delta_{r_m-1}^{-1}(a)\delta_{r_m}^{1/2}\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]|\det a|^{-v}\, d^\times a$$

is finite for any $v > 0$. Now by (2.5) we have

$$(15) \qquad \left|W'\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]\right| \leq c'_v \delta_{r_m}^{1/2}\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right]|\det a|^{-v}.$$

Moreover the support of $W'[(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix})]$ is contained in the set C defined by the conditions

$$(16) \qquad a = \mathrm{diag}(a_1 a_2 \cdots a_{r-1}, a_2 \cdots a_{r-1}, \ldots, a_{r-1}), \quad |a_i| \leq c_i,$$

for suitable c_i. Since

$$\delta_{r_m}\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right] = \delta_{r_m-1}(a)\,|\det a|,$$

we are reduced to considering the integral $\int_c |\det a|^{1-v}\, d^\times a$. This is indeed finite, provided $0 < v < 1$. Our argument shows in fact that, if in

$$\int_{N_{r_{m-1}} \backslash G_{r_{m-1}}} FW'\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh, \tag{17}$$

we replace F by its expression (11), then the resulting double integral converges. Thus (17) can be written as

$$\int \psi(\operatorname{tr}(\varepsilon g x))\, dx \int f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g\right] \cdot \overline{W'}\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh. \tag{18}$$

Next, since we have taken $u_m = 0$, the representation π_m of G_r is pre-unitary. Thus (2.1.2) defines an *invariant* Hermitian form on $\mathcal{W}(\pi_m; \psi)$. Hence the inner integral in (18) can also be written as

$$\int f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] \overline{W'}\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} g^{-1}\right] dh.$$

Since W' is arbitrary we can replace W' by any of its right translates. We get that

$$\int \overline{W'}\left[\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] dh \int \psi(\operatorname{tr}(\varepsilon g x))\, dx \cdot \int f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right] = 0 \tag{19}$$

for all $g \in G_{r_m}$ and all $p \in P_r$. Here W' can be taken arbitrary in $\mathcal{K}_0(\psi)$ (cf. (2.2.1)). Thus we finally get

$$\int \psi(\operatorname{tr}(\varepsilon g x))\, dx\, f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, e\right] = 0, \tag{20}$$

again for all $g \in G_{r_m}$ and $p \in P_r$. But $\operatorname{tr}(\varepsilon g x) = y x_1$, where y is the last row of $g \in G_{r_m}$ and x_1 is the first column of x. Thus we get at first for all $y \in F^{r_m}$ nonzero, and then for all y, the relation

$$\int \psi(y x_1) f^*\left[w^*\begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p;\, e, e\right] dx = 0. \tag{21}$$

Since the integral (21) is absolutely convergent, we may apply Fourier inversion to conclude that

$$\int f^*\left[w^*\begin{pmatrix} 1_{r_m} & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1_{r-r_m-1} \end{pmatrix} p; e, e\right] dx = 0, \tag{22}$$

for all $p \in P_r$.

We shall now prove that, for any j with $1 \leq j \leq r - r_m - 1$, the relation

$$\int f^*\left[w^*\begin{pmatrix} 1_{r_m} & 0 & x \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{r-r_m-j} \end{pmatrix} p; e, e\right] dx = 0, \tag{23}$$

for all $p \in P_r$, implies the same relation with j replaced by $j + 1$. For this let

$$p' = \begin{pmatrix} g & 0 \\ 0 & 1_{r-r_m-j} \end{pmatrix},$$

where g is an element of G_{r_m+j} of the form

$$g = \begin{pmatrix} 1_{r_m} & 0 & 0 \\ 0 & 1_{j-1} & 0 \\ z & 0 & 1 \end{pmatrix},$$

z being a row of length r_m. Our hypothesis on j implies $p' \in P_{r_m}$. Thus we can replace p by $p'p$ in (23). Then, after a simple computation, we get

$$\int f^*\left[\begin{pmatrix} 1_j & vx & 0 \\ 0 & 1_{r-r_m-j} & 0 \\ 0 & 0 & 1_{r_m} \end{pmatrix} w^*\begin{pmatrix} 1_{r_m} & 0 & x \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{r-r_m-j} \end{pmatrix} p; e, e\right] dx = 0. \tag{24}$$

Here v is the $r_m \times j$ matrix given by

$$v = \begin{bmatrix} 0 \\ -z \end{bmatrix}.$$

Since f^* belongs to the space of ξ^*, this reduces to the relation

$$\int f^*\left[w^*\begin{pmatrix} 1_{r_m} & 0 & x \\ 0 & 1_j & 0 \\ 0 & 0 & 1_{r-r_m-j} \end{pmatrix} p; e, e\right] \psi(-zx_1)\, dx = 0; \tag{25}$$

as before x_1 is the first column of x. If we again use Fourier inversion, we arrive at (23) with j replaced by $j+1$.

Thus we have now proved that $f^*[w^*p;\ e, e] = 0$ for all $p \in P_r$. Replacing p by

$$\begin{pmatrix} 1_{r_m} & 0 \\ 0 & g \end{pmatrix} p, \qquad g \in P_{r-r_m},$$

we get

$$f^*[w^*p;\ g, e] = 0 \tag{26}$$

for all $g \in P_{r-r_m}$, $p \in P_{r_m}$. Since the function $g \mapsto f^*[w^*p;\ g, e]$ belongs to $\mathcal{W}(\xi';\ \psi)$, at this point we can apply the second part of our induction hypothesis to the representation ξ' to conclude that

$$f^*[w^*p;\ g, e] = 0 \tag{27}$$

for all $p \in P_{r_m}$ and now for all $g \in G_{r-r_m}$. But then (27) implies that $f^*[uw^*q;\ e, e] = 0$ for all q in the parabolic subgroup of type $(r_m, r - r_m)$ and all u in the unipotent radical of Q^*. By continuity we get $f^*[g;\ e, e] = 0$ for all g, that is, $f = 0$. □

References

[B-Z] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups*, Ann. Scient. Éc. Norm. Sup., 4e série, t. **10** (1977), 441–472.

[B-W] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Annals of Math. Studies, No. 94, Princeton University Press, 1980.

[G-J] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Scient. Éc. Norm. Sup., 4e série, t. **11** (1978), 471–542.

[G-K] J. M. Galfand and D. A. Kazdan, *Representations of* GL(n, K) *where* K *is a Local Field*, in Lie Groups and Their Representations, John Wiley and Sons, 1975, pp. 95–118.

[J] H. Jacquet, *Generic Representations*, in Non-commutative harmonic analysis, Marseille-Luminy, 1976, Springer-Verlag, Lecture Notes No. 587, (1976), 376–378.

[J-P-S] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms in* GL(3) I, II, Annals of Math., **109** (1979), 169–212 and 213–518.

[J-P-S] ______, *On Rankin Selberg convolution*, to appear.

[K] B. Kostant, *On Whittaker vectors and representation theory*, Inventiones Math., **48** (1978), 101–184.

[V] D. A. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Inventiones Math., **48** (1978), 75–98.

[Z] A. V. Zelevinsky, *Induced representations of reductive p-adic groups II*, Ann. Scient. Éc. Norm. Sup., 4^e série, t. **13** (1980), 165–210.

Received October 10, 1981 and in revised form July 19, 1982. This research was partially supported by National Science Foundation Grants: MCS 79-01712 and MCS 79-13799.

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Pacific Journal of Mathematics

Vol. 109, No. 1 May, 1983