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We introduce spaces $\mathcal{C}^\gamma(G, K)$, $0 < \gamma \leq 2$, of rapidly decreasing K-bi-invariant functions on a reductive $\mathfrak{p}$-adic group G where K is a maximal compact subgroup of G, and we study spherical transformations on these spaces. The image of $\mathcal{C}^\gamma(G, K)$ under spherical transformation is completely described when K is a standard maximal compact subgroup of G.

Introduction. Let G be the group of rational points of a connected reductive algebraic group defined over a locally compact totally disconnected nondiscrete field k, with anisotropic center.

The group G is a totally disconnected locally compact group. A maximal compact subgroup K of G is fixed. The set of all (zonal) spherical functions on G with respect to K is parametrized naturally by orbits of the Weyl group W in a commutative complex Lie group $\hat{T}_{\mathbf{C}}$.

For $0 < \gamma \leq 2$ a family of Schwartz spaces $\mathcal{C}^\gamma(G, K)$ of rapidly decreasing K-bi-invariant functions on G is defined. It is shown that they are Fréchet algebras under convolution. For $f \in \mathcal{C}^\gamma(G, K)$ its spherical transform is defined by

$$\hat{f}(s) = \int_G f(g)\Gamma_s(g^{-1})\, dg$$

for $s \in \hat{T}$, where $\hat{T}$ is the greatest compact subgroup of $\hat{T}_{\mathbf{C}}$ and Γ_s is the spherical function corresponding to s. $\hat{T}$ is a compact real Lie group.

Let $C_W^\infty(\hat{T})$ denote the algebra of all W-invariant infinitely differentiable functions on $\hat{T}$ under pointwise multiplication. Then the spherical transformation $f \mapsto \hat{f}$ is a continuous epimorphism of $\mathcal{C}^2(G, K)$ onto $C_W^\infty(\hat{T})$. With certain additional conditions on K, we show that the spherical transformation is an isomorphism of topological algebras. The situation when these additional conditions are fulfilled is called the standard case and the contrary case is called the exceptional [**5**]. A standard maximal compact subgroup of G may always be found. If G is a split group, these additional conditions are always fulfilled.

If $\gamma < 2$, every spherical transform $\hat{f}$ of $f \in \mathcal{C}^\gamma(G, K)$ may be uniquely extended to a holomorphic function defined on a W-invariant domain D^γ

containing $\hat{T}$. If the algebra of all W-invariant analytic functions on D^{γ} with bounded derivatives is denoted by $\mathcal{H}_W(D^{\gamma})$, then the spherical transformation is a morphism of topological algebras $\mathcal{C}^{\gamma}(G, K)$ and $\mathcal{H}_W(D^{\gamma})$. If K is standard, then it is an isomorphism.

In §6 the spherical transformation of elements of $\mathcal{C}^2(G, K)$ with K exceptional is discussed.

We conclude the introduction with a few basic notations.

If X is a locally compact space, then the vector space of all complex-valued continuous functions on X is denoted by $C(X)$ and the subspace of all compactly supported functions is denoted by $C_0(X)$. For G a totally disconnected locally compact unimodular group and K its open compact subgroup, the vector space of all K-bi-invariant functions on G is denoted by $C(G, K)$. Let $C_0(G, K) = C_0(G) \cap C(G, K)$. The space of all γ-integrable functions in $C(G, K)$ is denoted by $L^{\gamma}(G, K)$, $0 < \gamma \leq 2$. We regard $L^{\gamma}(G, K)$ as a topological vector space in a standard way.

Z will denote the ring of rational integers, **R** and **C** will denote the fields of real and complex numbers, respectively. The set of all non-negative rational integers is denoted by $\mathbf{Z}_+$. $\sqrt{-1}$ is denoted by i.

The present paper is an account of the results of the author's thesis. It is a pleasure to acknowledge the help given to me by my advisor, Professor D. Miličić. The results presented in this paper are analogous to the results of P. C. Trombi and V. S. Varadarajan for real semisimple Lie groups.

1. Spherical functions. Let k be a locally compact totally disconnected non-discrete field. For an algebraic group **X** defined over k, X will denote the group of its k-rational points.

In this paper a connected reductive algebraic group, whose center is anisotropic, is denoted by **G**. Following [3], in this section we introduce necessary notations and we present Macdonald's explicit formula for spherical functions on G.

Let $\mathbf{G}^{\mathrm{der}}$ be the derived group of **G**, $\tilde{\mathbf{G}}$ the simply connected covering of $\mathbf{G}^{\mathrm{der}}$, and $\psi: \tilde{\mathbf{G}} \to \mathbf{G}^{\mathrm{der}}$ the canonical homomorphism. If **X** is a subgroup of **G**, $\psi^{-1}(\mathbf{X})$ is denoted by $\tilde{\mathbf{X}}$.

Fix a minimal parabolic subgroup **P** of **G**. Let **A** be a maximal split torus contained in **P**, **M** its centralizer, **U** the unipotent radical of **P**, $\mathbf{U}^-$ the unipotent radical of the parabolic opposite to **P**.

Let $\mathfrak{B}$ be the Bruhat-Tits building of $\tilde{G}$ [5]. The group $\tilde{G}$ acts on $\mathfrak{B}$. Let $\mathcal{Q}$ be the unique apartment in $\mathfrak{B}$ stabilized by $\tilde{A}$. If $\tilde{N}$ is the normalizer of $\tilde{A}$ in $\tilde{G}$, then $\tilde{N}$ stabilizes $\mathcal{Q}$. Let $\nu: \tilde{N} \to \mathrm{Aut}(\mathcal{Q})$ be the corresponding homomorphism.

There exists a canonical affine root system Σ_{aff} on an affine space $\mathcal{Q}$. Let W_{aff} be the Weyl group of Σ_{aff}. We have $\operatorname{Im} \nu = W_{\text{aff}}$. With a special point $x_0 \in \mathcal{Q}$ fixed, we will consider only the vector space structure with the origin x_0 on $\mathcal{Q}$. Set

$$\Sigma = \{a \in \Sigma_{\text{aff}};\, a(0) = 0\}, \quad W = \{w \in W_{\text{aff}};\, w(0) = 0\};$$

Σ is a reduced root system and W is its Weyl group. With a W-invariant scalar product $(\ ,\ )$ on $\mathcal{Q}$ fixed, a canonical isomorphism $\Lambda\colon \mathcal{Q} \to \mathcal{Q}^*$ is fixed, so we also have the scalar product on $\mathcal{Q}^*$, the space of linear forms on $\mathcal{Q}$.

For $a \in \Sigma_{\text{aff}}$ let $\tilde{U}(a)$ be the group of all $u \in \tilde{U} \cup \tilde{U}^-$ such that $ux = x$ for all $x \in \mathcal{Q}$ with $a(x) \geq 0$. Let Σ^+ be the set of all $\alpha \in \Sigma$ such that $\tilde{U}(\alpha) \subseteq \tilde{U}$. Σ^+ is a system of positive roots in Σ. Let Δ be the set of simple roots in Σ^+. The rank of Σ is denoted by l. Set $r_0 = \operatorname{card} \Sigma^+$.

Set

$$C_0 = \{x \in \mathcal{Q};\, \alpha(x) > 0 \text{ for all } \alpha \in \Sigma^+\},$$
$$C_0^{\perp} = \{x \in \mathcal{Q};\, (y, x) \geq 0 \text{ for all } y \in C_0\}.$$

The closure of C_0 is denoted by $\overline{C}_0$. Let C be the affine chamber of $\mathcal{Q}$ contained in C_0 having 0 as vertex. Let $\tilde{B}$ be the Iwahori subgroup fixing the chamber C. For $w \in W_{\text{aff}}$, $q(w)$ denotes $[\tilde{B}w\tilde{B}\colon \tilde{B}]$.

If $x \in \mathcal{Q}$, $x \neq 0$, we write $\check{x} = 2\Lambda^{-1}x/(x, x)$. For $x \in \mathcal{Q}$, $\tau(x)$ denotes the translation of $\mathcal{Q}$ by vector x. If $a \in \Sigma_{\text{aff}}$, the orthogonal reflection of $\mathcal{Q}$ with respect to the hyperplane $\{x \in \mathcal{Q};\, a(x) = 0\}$ is denoted by w_a. For $\alpha \in \Sigma$ set $a_\alpha = w_\alpha w_{\alpha-1}$. Then $a_\alpha = \tau(-\check{\alpha})$.

Set $q_a = [\tilde{U}(a-1)\colon \tilde{U}(a)]$ for $a \in \Sigma_{\text{aff}}$. Then q_{a+2} is always the same as q_a. For $\alpha \in \Sigma$ we set $q_{\alpha/2} = q_{\alpha+1}q_\alpha^{-1}$. Let

$$\Sigma_1 = \{\alpha \in (\Sigma \cup \tfrac{1}{2}\Sigma);\, q_\alpha \neq 1\}.$$

Then Σ_1 is a root system.

The group G acts on $\tilde{G}$ and B. If the normalizer of A in G is denoted by N, then $\mathcal{Q}$ is stabilized by N. Let $\nu\colon N \to \operatorname{Aut}(\mathcal{Q})$ be the corresponding homomorphism. The inverse image of the set of translations is M. Set $T = \nu(M)$. The kernel of ν is denoted by M_0. We identify M/M_0 and T.

The groups N/M, $\tilde{N}/\tilde{M}$, and W are isomorphic. The Weyl group W acts on $\tilde{M}$ and M by inner automorphisms and W acts on T as well.

Let

$$B = \{g \in G;\, gx = x \text{ for all } x \in C\}, \quad K = \{g \in G;\, g0 = 0\}.$$

The groups B and K are open compact subgroups and K is a special good maximal compact subgroup of G [**1**].

The case when $q_{\alpha/2} \geq 1$ for all $\alpha \in \Sigma$ will be called standard. The situation when $q_{\alpha/2} < 1$ for some $\alpha \in \Sigma$ is called the exceptional case [**5**]. We will also say K is a standard maximal compact subgroup of G or K is exceptional. There is always a special point $x_0 \in \mathcal{C}$ such that K is standard.

Set

$$\begin{aligned} M^- &= \{m \in M;\ -\nu(m)(0) \in \overline{C_0}\}, \\ M_- &= \{m \in M;\ -\nu(m)(0) \in C_0^{\perp}\}, \\ T^- &= \nu(M^-), \qquad T_- = \nu(M_-). \end{aligned}$$

Owing to the Iwasawa decomposition $G = UMK$, the mapping $m \mapsto UmK$ induces a bijection from $M/M_0 = T$ onto $U\backslash G/K$ ([**1**], Proposition (4.4.3)). The composition $G \to U\backslash G/K \to T$ is denoted by h.

There is the Cartan decomposition $G = KM^-K$. The mapping $m \mapsto KmK$ induces a bijection from $T^- = M^-/M_0$ onto $K\backslash G/K$ ([**1**], Proposition (4.4.3)). Let σ be the composition $G \to K\backslash G/K \to T^-$.

From the Cartan decomposition we deduce that mapping $f \mapsto f \circ \sigma$ is a bijection from the space of all complex-valued functions on T^- onto $C(G, K)$. Sometimes we shall identify these two spaces.

If $m_1^{-1} \in M^-$, $m_2 \in M$ and $Km_1K \cap Um_2K \neq \varnothing$, then $m_2m_1^{-1} \in M_-$, or, equivalently,

$$\nu(m_1)(0) - \nu(m_2)(0) \in C_0^{\perp} \tag{1.1}$$

([**1**], Proposition (4.4.4), (i)).

If $m_1, m_2, m_3 \in M^-$ and $Km_1Km_2K \cap Km_3K \neq \varnothing$, then $m_1m_2m_3^{-1} \in M_-$, or, equivalently,

$$\nu(m_1)(0) + \nu(m_2)(0) - \nu(m_3)(0) \in (-C_0^{\perp}) \tag{1.2}$$

([**1**], Proposition (4.4.4), (iii)).

For $m^{-1} \in M^-$ we have

$$KmK \cap UmK = mK \tag{1.3}$$

([**1**], Proposition (4.4.4), (ii)).

Let δ be the modular character of P. It is easy to see that $p \mapsto \delta(\psi(p))$ is the modular character of $\tilde{P}$.

For $\alpha \in \Delta$ choose $n \in \tilde{N}$ such that $\nu(n) = a_\alpha$. Then $\delta(n)^{-1} = q_{\alpha/2}q_\alpha^2$ ([**5**], Corollary (3.2.7)). Since $q_{\alpha/2}q_\alpha^2 = q_{\alpha+1}q_\alpha$, it follows that

$$\delta(n)^{-1} > 1. \tag{1.4}$$

Let μ be the Haar measure on G such that $\mu(K) = 1$. For $g \in G$ we define $v_g = \mu(KgK)$. If $t \in T$ and $m \in M$ such that $\nu(m) = t$, then we write $v_t = \mu(KmK)$.

The next lemma is proved in [**8**].

1.1. LEMMA. *If K_0 is an open compact subgroup of G, then there exist c_1, $c_2 > 0$ such that*

$$c_1\delta(m)^{-1} \leq \mu(K_0 m K_0) \leq c_2\delta(m)^{-1},$$

for all $m \in M^-$.

A complex character of M is called unramified if it is trivial on M_0. The mapping $s \mapsto s \circ \nu$ is an isomorphism from the group of all characters of T onto the group of all unramified characters of M, so we identify these two groups and denote them by $\hat{T}_{\mathbf{C}}$. The character $\delta \mid M$ is unramified and this character is denoted by δ as well.

T is a free abelian group of rank l. Fix $b_1, b_2, \ldots, b_l$ a basis of T. The mapping Θ: $s \mapsto (s(b_1), \ldots, s(b_l))$ is an isomorphism of $\hat{T}_{\mathbf{C}}$ onto $(\mathbf{C}^*)^l$. As $(\mathbf{C}^*)^l$ is a complex Lie group, there is the unique complex Lie group structure such that Θ is a complex Lie group isomorphism. This structure on $\hat{T}_{\mathbf{C}}$ does not depend on the choice of basis $b_1, \ldots, b_l$.

The image of the invariant analytic vector field is ${}_k\partial/\partial s_k$ on $(\mathbf{C}^*)^l$, $k = 1, 2, \ldots, l$, under the mapping induced on vector fields by Θ^{-1}, is denoted by S_k. The set $S = \{S_k;\ k = 1, \ldots, l\}$ is a basis of the Lie algebra $\mathfrak{t}$ of $\hat{T}_{\mathbf{C}}$. The universal enveloping algebra of $\mathfrak{t}$ is denoted by $\mathfrak{U}(\mathfrak{t})$. Let $\mathfrak{S}$ be the basis of $\mathfrak{U}(\mathfrak{t})$ which is obtained from S by the Poincaré-Birkhoff-Witt Theorem. There is the canonical filtration $(\mathfrak{U}_n(\mathfrak{t});\ n \in \mathbf{Z}_+)$ of $\mathfrak{U}(\mathfrak{t})$. The sets $\mathfrak{S}_n = \mathfrak{S} \cap \mathfrak{U}_n(\mathfrak{t})$ are finite.

Let $\hat{T}$ be the set of all unitary characters in $\hat{T}_{\mathbf{C}}$. Then $\hat{T}$ is a real Lie subgroup of $\hat{T}_{\mathbf{C}}$ and the greatest compact subgroup of $\hat{T}_{\mathbf{C}}$. Clearly, $\Theta(\hat{T}) = \{z \in \mathbf{C}^*;\ |z| = 1\}^l$. Denoting by (r_k, φ_k) the polar coordinates of z_k, we obtain an invariant vector field $\partial/\partial\varphi_k$ on $\Theta(\hat{T})$ and the corresponding vector field $\tilde{S}_k$ on $\hat{T}$. The set $\{\tilde{S}_k;\ k = 1, \ldots, l\}$ is a basis of the Lie algebra $\tilde{\mathfrak{t}}_0$ of $\hat{T}$. Let $\tilde{\mathfrak{t}}$ be the complexification of $\tilde{\mathfrak{t}}_0$. The elements of $\tilde{\mathfrak{t}}$ act on complex-valued smooth functions on $\hat{T}$.

We extend the mapping $\tilde{S}_k \mapsto S_k$, $k = 1, \ldots, l$, to a Lie algebra isomorphism of $\tilde{\mathfrak{t}}$ onto $\mathfrak{t}$ and to an isomorphism ω of $\mathfrak{U}(\tilde{\mathfrak{t}})$ onto $\mathfrak{U}(\mathfrak{t})$. If an analytic function φ is defined on an open set containing $\hat{T}$, then

$$(\omega X)(x)\varphi = X(x)(\varphi \mid \hat{T}) \tag{1.5}$$

for $X \in \mathfrak{U}(\tilde{\mathfrak{t}})$ and $x \in \hat{T}$. We write $\tilde{\mathfrak{S}}_n = \omega^{-1}(\mathfrak{S}_n)$ and $\tilde{\mathfrak{S}} = \omega^{-1}(\mathfrak{S}_n)$.

A character $s \in \hat{T}_{\mathbf{C}}$ is called regular if $s(\tau(\check{b})) \neq 1$ for all $b \in \Sigma_1$.

Let $\mathrm{Ind}(s \mid P, G)$ be the admissible representation of G induced by $s \in \hat{T}_{\mathbf{C}}$. It is the right regular representation of G on the space of all locally constant functions $f\colon G \to \mathbf{C}$ such that $f(pg) = (s\delta^{1/2})(p)f(g)$ for all $p \in P$, $g \in G$. The contragredient of $\mathrm{Ind}(s \mid P, G)$ is isomorphic to $\mathrm{Ind}(s^{-1} \mid P, G)$, and the canonical form on $\mathrm{Ind}(s \mid P, G) \times \mathrm{Ind}(s^{-1} \mid P, G)$ is given by

$$(f, \tilde{f}) \mapsto \int_K \tilde{f}(k) f(k)\, dk.$$

Define $\Phi_s\colon G \to \mathbf{C}$ by setting $\Phi_s(g) = (s\delta^{1/2})(h(g))$. Then Φ_s is a non-trivial K-invariant vector in $\mathrm{Ind}(s \mid P, G)$. Let

$$\begin{aligned} \Gamma_s &= \int_K \Phi_s(kg)\Phi_{s^{-1}}(k)\, dk = \int_K \Phi_s(kg)\, dk \\ &= \int_K (s\delta^{1/2})(h(kg))\, dk \end{aligned} \tag{1.6}$$

for $g \in G$. Then Γ_s is the matrix coefficient and zonal spherical function, or, simply, the spherical function corresponding to s. Since the contragredient of $\mathrm{Ind}(s^{-1} \mid P, G)$ is $\mathrm{Ind}(s \mid P, G)$, it is easily seen that

$$\Gamma_s(g) = \Gamma_{s^{-1}}(g^{-1}) \quad \text{for all } s \in \hat{T}_{\mathbf{C}} \text{ and } g \in G. \tag{1.7}$$

W acts on T; thus W also acts on $\hat{T}_{\mathbf{C}}$ and $\hat{T}$. For $w \in W$ and $s \in \hat{T}_C$ we have

$$\Gamma_s = \Gamma_{ws} \tag{1.8}$$

([**3**], Proposition 4.1). The converse will be proved in the next lemma.

1.2. Lemma. *If $s_1, s_2 \in \hat{T}_C$ and $\Gamma_{s_1} = \Gamma_{s_2}$, then there exists $w \in W$ such that $s_1 = ws_2$.*

Proof. Let us denote by V the subrepresentation of $\mathrm{Ind}(s_1 \mid P, G)$ generated by Φ_{s_1}. Clearly, there exists a maximal subrepresentation V_1 of V such that $\Phi_{s_1} \notin V_1$. By ([**2**], Corollary 6.3.7), there exists $w_1 \in W$ such that V/V_1 is isomorphic to a subrepresentation V_2 of $\mathrm{Ind}(w_1 s_1 \mid P, G)$ and V_2 is generated by $\Phi_{w_1 s_1}$. The mapping $f \mapsto \bar{f}$ from V_2 into $C^\infty(G)$ is defined by

$$\bar{f}(g) = \int_K f(kg)\, dk.$$

Obviously, $\overline{\Phi}_{w_1 s_1} = \Gamma_{w_1 s_1}$. The mapping $f \mapsto \bar{f}$ is a monomorphism of G-representations where G acts on $C^\infty(G)$ by right translations. Since $\Gamma_{w_1 s_1} = \Gamma_{s_1}$, we have seen that there is a composition factor V_2 of $\mathrm{Ind}(s_1 \mid P, G)$ such that V_2 is isomorphic to the subrepresentation of $C^\infty(G)$ generated by Γ_{s_1}. Thus, $\mathrm{Ind}(s_1 \mid P, G)$ and $\mathrm{Ind}(s_2 \mid P, G)$ have a common composition factor. By ([**2**], Theorem 6.3.3) there exists $w \in W$ such that $s_1 = ws_2$.

Relation (1.6) implies $s \mapsto \Gamma_s(g)$ is an analytic function on $\hat{T}_{\mathbf{C}}$ for every fixed $g \in G$. Each of these functions is W-invariant.

If s is a regular unramified character and $\alpha \in \Sigma$, we introduce

$$c(\alpha, s) = \frac{\left(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} s(a_\alpha)^{-1}\right)\left(1 + q_{\alpha/2}^{-1/2} s(a_\alpha)^{-1}\right)}{1 - s(a_\alpha)^{-2}}. \tag{1.9}$$

If $\alpha/2 \notin \Sigma_1$ then

$$c(\alpha, s) = \frac{1 - q_\alpha^{-1} s(a_\alpha)^{-1}}{1 - s(a_\alpha)^{-1}}.$$

Set

$$c(s) = \prod_{\alpha \in \Sigma^+} c(\alpha, s).$$

1.3. THEOREM. (*Macdonald*, [**2**], *Theorem* 4.2): *Let*

$$Q = \sum_{w \in W} q(w)^{-1}.$$

If $s \in \hat{T}_{\mathbf{C}}$ is regular, then for all $m \in M^-$ we have

$$\Gamma_s(m) = Q^{-1} \sum_{w \in W} c(ws)\big((ws)\delta^{1/2}\big)(m). \tag{1.10}$$

From (1.10) we obtain

$$\Gamma_s(g) = Q^{-1} \sum_{w \in W} c(ws)\big((ws)\delta^{1/2}\big)(\sigma(g)) \tag{1.11}$$

for $g \in G$. Relation (1.7) implies

$$\Gamma_s(g^{-1}) = Q^{-1} \sum_{w \in W} c(ws^{-1})\big((ws^{-1})\delta^{1/2}\big)(\sigma(g)). \tag{1.12}$$

The set of all regular characters is an open dense subset of $\hat{T}_{\mathbf{C}}$. Since the functions $s \mapsto \Gamma_s(g)$ are analytic on $\hat{T}_{\mathbf{C}}$, we can get $\Gamma_s(g)$ as the limit of $\Gamma_{s_1}(g)$ with s_1 regular. In this way we can obtain the formula for Γ_s for any s. In such a way I. G. Macdonald derived the formula for any spherical function ([**5**], Proposition (4.6.2)).

Set $\Xi = \Gamma_1$. There exists a polynomial function p on $\mathcal{a}$ such that

$$\Xi(m) = \delta^{1/2}(m)p(\nu(m)(0)) \tag{1.13}$$

for all $m \in M^-$ and the degree of p is less than or equal to r_0 ([5], Proposition (4.6.1)).

For $s \in \hat{T}$, Γ_s is a positive definite spherical function. Note that every positive definite spherical function f on G satisfies

$$\overline{f(g)} = f(g^{-1}), \tag{1.14}$$

$$|f(g)| \leq f(1) \tag{1.15}$$

for all $g \in G$.

2. The algebras $\mathcal{C}^\gamma(G, K_0)$. Our first aim is to describe a connection between the asymptotic behaviour of Ξ and δ. Let $\| \ \|$ be the norm on $\mathcal{a}$ corresponding to the fixed scalar product on $\mathcal{a}$.

2.1. PROPOSITION. *There exist $c_1, c_2 > 0$ such that*

$$c_1\delta^{1/2}(\sigma(g)) \leq \Xi(g) \leq c_2(1 + \|\sigma(g)(0)\|)^{r_0}\delta^{1/2}(\sigma(g)) \tag{2.1}$$

for all $g \in G$.

Proof. The right inequality of (2.1) is an immediate consequence of (1.13).

For $g \in G$ choose $m \in M^-$ such that $\sigma(x) = \nu(m)$. Then

$$\begin{aligned}\Xi(g) = \Xi(m) = \Xi(m^{-1}) &= \int_K \delta^{1/2}(h(km^{-1}))\,dk \\ &\geq \delta^{1/2}(m^{-1})\mu(K \cap m^{-1}Km) = \delta^{1/2}(m^{-1})\mu(Km^{-1}K)^{-1} \\ &= \delta^{1/2}(m^{-1})\mu(KmK)^{-1}.\end{aligned}$$

Now Lemma 1.1 implies the left inequality of (2.1).

Fix γ, $0 < \gamma \leq 2$, and fix an open compact subgroup K_0 of G. Set

$$v_r^\gamma(f) = \sup\left\{|f(g)|\Xi^{-2/\gamma}(g)(1 + \|\sigma(g)(0)\|)^r;\ g \in G\right\} \tag{2.2}$$

for $r \in \mathbf{R}$ and $f \in C(G, K_0)$. Let

$$\mathcal{C}^\gamma(G, K_0) = \{f \in C(G, K_0);\ v_r^\gamma(f) < \infty \text{ for all } r \in \mathbf{R}\}.$$

The vector space $\mathcal{C}^\gamma(G, K_0)$ is topologized by means of the set of semi-norms v_r^γ, $r \in \mathbf{R}$. Then $\mathcal{C}^\gamma(G, K_0)$ is a metrizable locally convex space.

Obviously, $v_r^{\gamma_2}(f) \leq v_r^{\gamma_1}(f)$ for $\gamma_1 \leq \gamma_2$, $r \in \mathbf{R}$, and $f \in C(G, K_0)$. Therefore, $\mathcal{C}^{\gamma_1}(G, K_0) \subseteq \mathcal{C}^{\gamma_2}(G, K_0)$. The last inclusion is continuous and dense.

Set

$$u_r^\gamma(f) = \sup\left\{|f(g)|\delta^{-1/\gamma}(\sigma(g))\big(1 + \|\sigma(g)(0)\|\big)^r ; g \in G\right\} \tag{2.3}$$

for $r \in \mathbf{R}, f \in C(G, K_0)$. According to Proposition 2.1. we have

$$\mathcal{C}^\gamma(G, K_0) = \{f \in C(G, K_0); u_r^\gamma(f) < \infty \text{ for all } r \in \mathbf{R}\}.$$

The family of seminorms u_r^γ, $r \in \mathbf{R}$, induces the same topology as the preceding family v_r^γ, $r \in \mathbf{R}$.

2.2. Proposition. *The space $\mathcal{C}^\gamma(G, K_0)$ is a Fréchet space, $C_0(G, K_0)$ is a dense subspace of $\mathcal{C}^\gamma(G, K_0)$, and the inclusion $\mathcal{C}^\gamma(G, K_0) \to L^\gamma(G, K_0)$ is continuous.*

Proof. Choose $c > 0$ such that $v_g \leq c\delta(\sigma(g))^{-1}$ (Lemma 1.1), and $r \in \mathbf{R}$, such that $r\gamma > l$. Then

$$\int_G |f(g)|^\gamma \, dg \leq c\big(u_r^\gamma(f)\big)^\gamma \sum_{t \in T^-} \big(1 + \|t(0)\|\big)^{-r\gamma}$$

for all $f \in \mathcal{C}^\gamma(G, K_0)$. Since $\Sigma_{t \in T^-}(1 + \|t(0)\|)^{-r\gamma}$ is finite, we have $f \in L^\gamma(G, K_0)$ and the inclusion is continuous.

The proof of the rest is standard so we omit it.

The next technical lemma plays a crucial role in proving the fact that $\mathcal{C}^\gamma(G, K_0)$ is a topological algebra. This result is analogous to the corresponding assertion for real semisimple Lie groups ([7], Ch. 1 Theorem 1).

2.3. Lemma. *Choose $r_1 \in \mathbf{R}$ such that $r_1 > 2r_0 + l$. Then the function*

$$g \mapsto \Xi^2(g)\big(1 + \|\sigma(g)(0)\|\big)^{-r_1} \tag{2.4}$$

is integrable. For $0 < \gamma \leq 2$ there exists $c > 0$ such that, for $p, q \in \mathbf{R}$,

$$\begin{aligned} &\int_G \Xi^{2/\gamma}(x)\Xi^{2/\gamma}(x^{-1}y)\big(1 + \|\sigma(x)(0)\|\big)^{-p}\big(1 + \|\sigma(x^{-1}y)(0)\|\big)^{-q} \, dx \\ &\quad \leq c\Xi^{2/\gamma}(y)\big(1 + \|\sigma(y)(0)\|\big)^{-q+(2/\gamma-1)r_0} \end{aligned} \tag{2.5}$$

for all $y \in G$ as soon as

$$q - p \leq -r_1 - 2r_0\,(2/\gamma - 1) \quad \textit{and} \quad q \geq r_0\,(2/\gamma - 1).$$

Proof. Let c_1, c_2 be the real positive numbers from Proposition 2.1. Choose $c_0 > 0$ such that $v_g \le c_0\delta(\sigma(g))^{-1}$ (Lemma 1.1). Then

$$\int_G \Xi^2(g)(1+\|\sigma(g)(0)\|)^{-r_1}\,dg \le c_0c_2^2 \sum_{t\in T^-}(1+\|t(0)\|)^{2r_0-r_1}.$$

As $2r_0 - r_1 < -l$, the series on the right side converges. The integral of the function (2.4) is denoted by c_3.

From the definition of σ and from (1.2) we get

$$\sigma(x)(0) + \sigma(x^{-1}y)(0) - \sigma(y)(0) \in -C_0^{\perp}.$$

Since $\sigma(x)(0), \sigma(x^{-1}y)(0), \sigma(y)(0) \in -\overline{C}_0$, we have

$$\|\sigma(y)(0)\| \le \|\sigma(x)(0) + \sigma(x^{-1}y)(0)\| \le \|\sigma(x)(0)\| + \|\sigma(x^{-1}y)(0)\|.$$

Thus

$$(1+\|\sigma(y)(0)\|) \le (1+\|\sigma(x)(0)\|)(1+\|\sigma(x^{-1}y)(0)\|). \tag{2.6}$$

Relation (1.4) implies $\delta(t) \le 1$ for all $t \in T_-$. We obtain

$$\delta(\sigma(x))\delta(\sigma(x^{-1}y)) \le \delta(\sigma(y)). \tag{2.7}$$

From the well-known equality

$$\int_K \Xi(xky)\,dk = \Xi(x)\Xi(y)$$

follows the identity

$$\int_G \Xi(x)\Xi(x^{-1}y)(1+\|\sigma(x)(0)\|)^{q-p}\,dx = \Xi(y)\int_G \Xi^2(x)(1+\|\sigma(x)(0)\|)^{q-p}\,dx. \tag{2.8}$$

Set $c_4 = c_1^{1-2/\gamma}c_2^{2(2/\gamma-1)}$. Relations (2.1) and (2.7) imply

$$\Xi^{2/\gamma-1}(x)\Xi^{2/\gamma-1}(x^{-1}y) \le c_4\Xi^{2/\gamma-1}(y)\big((1+\|\sigma(x)(0)\|)(1+\|\sigma(x^{-1}y)(0)\|)\big)^{r_0(2/\gamma-1)}.$$

Now, from the last inequality, (2.6) and (2.8), we obtain (2.5).

2.4. Theorem. *For $0 < \gamma \le 2$, $\mathcal{C}^\gamma(G, K_0)$ is a Fréchet algebra under the convolution.*

Proof. For $r > 0$ choose $q, p \in \mathbf{R}$ satisfying $q \geq r + r_0(2/\gamma - 1)$ and $q - p \leq -r_1 - 2r_0(2/\gamma - 1)$. The definition of v_r^γ and (2.5) imply

$$|(f_1 * f_2)(y)| \leq c\Xi^{2/\gamma}(y)(1 + \|\sigma(y)(0)\|)^{-q+(2/\gamma-1)r_0} v_p^\gamma(f_1) v_q^\gamma(f_2)$$

for all $f_1, f_2 \in \mathcal{C}^\gamma(G, K_0)$ and $y \in G$. Now, it is obvious that $v_r^\gamma(f_1 * f_2) \leq c v_p^\gamma(f_1) v_q^\gamma(f_2)$.

The proof of the preceding theorem is simpler for $\gamma = 2$ ([**8**], Theorem 4.4.2).

Let $K_0 \subseteq K_1$ be open compact subgroups of G. Then $\mathcal{C}^\gamma(G, K_1) \subseteq \mathcal{C}^\gamma(G, K_0)$, the inclusion being continuous and even an isomorphism onto the image. The family $\mathcal{C}^\gamma(G, K_0)$, with K_0 an open compact subgroup of G, is an inductive system of locally convex spaces. The inductive limit $\mathcal{C}^\gamma(G)$ of this family is the union of all $\mathcal{C}^\gamma(G, K_0)$ over all K_0. The space $\mathcal{C}^\gamma(G)$ is a complete locally convex space.

3. Asymptotic behaviour of spherical functions. For $s \in \hat{T}_{\mathbf{C}}$ the mapping $t(0) \mapsto \ln|s(t)|$, $t \in T$, is a homomorphism from the lattice $\{t(0); t \in T\}$ in $\mathcal{C}$ into the additive group of real numbers. This mapping is uniquely extended to the linear form s_1 on $\mathcal{C}$. Set $\eta(s) = \Lambda^{-1}(s_1)$. If $x_1, \ldots, x_l$ is a basis of A contained in $\{t(0); t \in T\}$ and $\tilde{x}_1, \ldots, \tilde{x}_l$ the basis of $\mathcal{C}$ biorthogonal to $x_1, \ldots, x_l$ with respect to the fixed scalar product on $\mathcal{C}$, then

$$\eta(s) = \sum_{j=1}^{l} \ln|s(\tau(x_j))|\tilde{x}_j. \tag{3.1}$$

The function η is a continuous, open and closed surjection from $\hat{T}_{\mathbf{C}}$ onto $\mathcal{C}$. If X is a compact subset of $\mathcal{C}$, $\eta^{-1}(X)$ is a compact subset of $\hat{T}_{\mathbf{C}}$. Note that η commutes with the action of W.

For $s \in \hat{T}_{\mathbf{C}}$, the character $t \mapsto |s(t)|$, $T \to \mathbf{R}_+^*$ is denoted by $|s|$.

3.1. LEMMA. (i) *For $s \in \hat{T}_{\mathbf{C}}$ and $g \in G$ one has*

$$|\Gamma_s(g)| \leq \Gamma_{|s|}(g). \tag{3.2}$$

(ii) *Let $s \in \hat{T}_{\mathbf{C}}$. If $\eta(s) \in \overline{C}_0$, then*

$$|\Gamma_s(m^{-1})| \leq \Xi(m)|s|^{-1}(m) \quad \textit{for all } m \in M^-. \tag{3.3}$$

Proof. (i) is obvious. By the definition of h and (1.1) one has

$$\nu(m^{-1})(0) - h(km^{-1})(0) \in C_0^{\perp}$$

for $k \in K$. Thus $|s(m^{-1})| \geq |s(h(km^{-1}))|$. From this inequality we obtain

$$|\Gamma_s(m^{-1})| \leq \Gamma_{|s|}(m^{-1}) = \int_K |s|(h(km^{-1}))\delta^{1/2}(h(km^{-1}))\,dk$$

$$\leq |s|^{-1}(m)\Xi(m).$$

3.2. COROLLARY. *There exists $c > 0$ such that*

$$|\Gamma_s(m^{-1})| \leq c|s|^{-1}(m)\delta^{1/2}(m)(1 + \|\nu(m)(0)\|)^{r_0} \tag{3.4}$$

for all $m \in M^-$ and $s \in \hat{T}_{\mathbf{C}}$ with $\eta(s) \in \overline{C}_0$.

The next technical lemma is necessary for a description of the asymptotics of the derivative of spherical functions. This is an analogue of Lemmas 3.3.2.2. and 3.3.2.3. of [**12**].

3.3. LEMMA. *Let $w_0 \in W$ and $w_0C_0 = -C_0$. Then one has*

$$\nu(m)(0) - h(km)(0) \in C_0^{\perp}, \tag{3.5}$$

$$h(km)(0) - w_0(\nu(m)(0)) \in C_0^{\perp} \tag{3.6}$$

for all $k \in K$ and $m^{-1} \in M^-$.

Proof. The first relation is clear. Fix $k \in K$, $m^{-1} \in M^-$ and $w \in K$ such that $\nu(w) = w_0$. Choose $u_1 \in U$, $m_1 \in M$ and $k_1 \in K$ such that $km = u_1m_1k_1$. By definition one has $\nu(m_1) = h(km)$. Since $\nu(wm^{-1}w^{-1})(0) = -w_0(\nu(m)(0))$, one now has $(wm^{-1}w^{-1})^{-1} \in M^-$. (3.5) implies $-w_0(\nu(m)(0)) - h(k_1m^{-1})(0) \in C_0^{\perp}$. We can find elements $u_2 \in U$, $m_2 \in M$ and $k_2 \in K$ such that $k_1m^{-1} = u_2m_2k_2$. Obviously, $k = u_1(m_1u_2m_1^{-1})m_1m_2k_2$. Since M normalizes U, we have $\nu(m_1) = \nu(m_2)^{-1}$. Now (3.6) immediately follows.

If $X \in \mathfrak{U}(\mathfrak{t})$ and $g \in G$, then the result of the action of X on the function $s \mapsto \Gamma_s(g)$ will be denoted by $\Gamma_s(g)^X$. Analogously, for $t \in T$, $s(t)^X$ will denote the result of the action of X on the function $s \mapsto s(t)$.

In the first section we have fixed a basis $b_1, \ldots, b_l$ of T. Let $\tilde{b}_1, \ldots, \tilde{b}_l$ be the basis of A biorthogonal to the basis $b_1(0), \ldots, b_l(0)$. One has

$$s(t)^{S_j} = i(\tilde{b}_j, t(0))s(t) \quad \text{for } t \in T, j = 1, \ldots, l. \tag{3.7}$$

For $X \in \mathfrak{S}$ there exist $n_1, \ldots, n_l \in \mathbf{Z}_+$ such that $X = S_1^{n_1} \cdots S_l^{n_l}$. (3.7) implies

$$s(t)^X = s(t) \prod_{j=1}^{l}{}' \left(i(b_j, t(0))\right)^{n_j}, \tag{3.8}$$

where the product runs over all j with $n_j \neq 0$.

3.4. LEMMA. *For $n \in \mathbf{Z}_+$ there exists $c > 0$ such that*

$$\left|\Gamma_s(m^{-1})^X\right| \leq c\left(1 + \|\nu(m)(0)\|\right)^n \Gamma_{|s|}(m^{-1}) \tag{3.9}$$

for all $X \in \mathfrak{S}_n$ and $m \in M^-$.

Proof. Let $\{\tilde{\alpha};\ \alpha \in \Delta\}$ be the basis of $\mathfrak{a}$ biorthogonal to the basis $\{\check{\alpha};\ \alpha \in \Delta\}$. The norm $\|\ \|_\infty$ on $\mathfrak{a}$ is defined by

$$\|x\|_\infty = \max\{|(\tilde{\alpha}, x)|\,;\ x \in \Delta\}.$$

Choose $w_0 \in W$ such that $w_0 C_0 = -C_0$. Relations (3.5) and (3.6) imply $\|h(km^{-1})(0)\|_\infty \leq \|\nu(m)(0)\|$ for $m \in M^-$ and $k \in K$. Choose $c_2 \geq 1$ such that $|(\tilde{b}_j, x)| \leq c_2 \|x\|_\infty$ for all $x \in \mathfrak{a}$, $j = 1, \ldots, l$. Pick $c_3 \geq 1$ such that $\|x\|_\infty \leq c_3 \|x\|$ for all $x \in \mathfrak{a}$.

For $X \in \mathfrak{S}_n$ choose $n_1, \ldots, n_l \in \mathbf{Z}_+$ such that $X = S_1^{n_1} \cdots S_l^{n_l}$. Then $n_1 + \cdots + n_l \leq n$. From Definitions (1.6) and (3.8) we have

$$\begin{aligned}\left|\Gamma_s(m^{-1})^X\right| &\leq \int_K \left(\prod_{j=1}^{l}{}' \left|(\tilde{b}_j, h(km^{-1})(0))\right|^{n_j}\right)(s\delta^{1/2})(h(km^{-1}))\, dk \\ &\leq (c_1 c_2 c_3)^n \left(1 + \|\nu(m)(0)\|\right)^n \Gamma_{|s|}(m^{-1}).\end{aligned}$$

4. The spherical transformation on $\mathcal{C}^2(G, K)$. Let $C^\infty(\hat{T})$ be the vector space of all infinitely differentiable complex-valued functions on $\hat{T}$. We write

$$p_n(\varphi) = \max\left\{|(X\varphi)(x)|;\ X \in \tilde{\mathfrak{S}}_n \text{ and } x \in \hat{T}\right\} \tag{4.1}$$

for $\varphi \in C^\infty(\hat{T})$ and $n \in \mathbf{Z}_+$. The seminorms p_n, $n \in \mathbf{Z}_+$, topologize $C^\infty(\hat{T})$ such that $C^\infty(\hat{T})$ is a Fréchet space; moreover, it is a topological algebra under pointwise multiplication. The group W acts on $C^\infty(\hat{T})$; let $C_W(\hat{T})$ be the subalgebra of all W-invariant elements of $C^\infty(\hat{T})$. Then $C_W^\infty(\hat{T})$ is a Fréchet algebra.

The definition of $\mathcal{C}^2(G, K)$, (2.4), and (3.2) imply that the function $g \mapsto f(g)\Gamma_s(g^{-1})$ is integrable for $s \in \hat{T}$ and $f \in \mathcal{C}^2(G, K)$. One now defines $\hat{f}$: $\hat{T} \to \mathbf{C}$ by

$$\hat{f}(s) = \int_G f(g)\Gamma_s(g^{-1})\, dg. \tag{4.2}$$

We call $\hat{f}$ a spherical transform of f and the mapping $f \mapsto \hat{f}$ is called a spherical transformation.

4.1. THEOREM. *One has $\hat{f} \in C_W^\infty(\hat{T})$ for $f \in \mathcal{C}^2(G, K)$. The spherical transformation $\hat{}$: $\mathcal{C}^2(G, K) \to C_W^\infty(\hat{T})$ is a morphism of topological algebras with unit.*

Proof. Let $f \in \mathcal{C}^2(G, K)$. Then (1.8) implies $\hat{f}(s) = \hat{f}(ws)$ for $s \in \hat{T}$ and $w \in W$. Fix $n \in \mathbf{Z}_+$ and $r_1 > 2r_0 + l$. Choose $c > 0$ satisfying (3.9). Let c_1 be the integral of (2.4). One now has

$$\left|f(g)\Gamma_s(g^{-1})^X\right| \le c v_{n+r_1}^2(f)\left(1 + \|\sigma(g)(0)\|\right)^{-r_1}\Xi(g)$$

for $g \in G$ and $X \in \tilde{\mathfrak{S}}_n$. Since the function on the right side is integrable, we obtain that $f \in C_W^\infty(\hat{T})$ and

$$(X\hat{f})(s) = \int_G f(g)\Gamma_s(g^{-1})^X\, dx. \tag{4.3}$$

Thus $p_n(\hat{f}) \le c c_1 v_{n+r_1}^2(f)$.

It is easy to see that $(f_1 * f_2)\hat{} = \hat{f}_1\hat{f}_2$ for $f_1, f_2 \in \mathcal{C}^2(G, K)$.

The cardinal number of W is denoted by $|W|$. For $\varphi \in C_W^\infty(\hat{T})$ one defines $\check{\varphi}$: $G \to \mathbf{C}$ by

$$\check{\varphi}(g) = \frac{Q}{|W|}\int_{\hat{T}} \varphi(s)\Gamma_s(g)|c(s)|^{-2}\, ds. \tag{4.4}$$

where ds is the normalized Haar measure on $\hat{T}$. Clearly, $\check{\varphi} \in C(G, K)$.

Using Macdonald's formula, we obtain

$$\check{\varphi}(g) = \delta^{1/2}(\sigma(g))\int_{\hat{T}} \varphi(s^{-1})c(s)^{-1}\overline{s(\sigma(g))}\, ds. \tag{4.5}$$

Note that the function $s \mapsto c(s)^{-1}$ belongs to $C^\infty(\hat{T})$.

4.2. LEMMA. *The mapping $\varphi \mapsto \check{\varphi}$ from $C_W^\infty(\hat{T})$ into $C(G, K)$ is injective.*

Proof. Set $X = C_0(G, K)\hat{}$. Then X is spanned by the set of functions $s \mapsto \Gamma_s(g)$, $g \in G$. Using Lemma 1.2. and applying the Stone-Weierstrass Theorem on $C(W\backslash\hat{T})$, we conclude that X is a dense subalgebra of $C_W^\infty(\hat{T})$ with respect to the supremum norm. One now sees that $\check{\varphi} = 0$ implies $\varphi = 0$.

For $\varphi \in C^\infty(\hat{T})$ and $t \in T$ we write

$$\varphi_t = \int_{\hat{T}} \varphi(s)\overline{s(t)}ds. \tag{4.6}$$

Let $\tilde{b}_1, \ldots, \tilde{b}_l$ be the basis of $\mathfrak{a}$ biorthogonal to $b_1(0), \ldots, b_l(0)$. One has $(\tilde{S}_j\varphi)_t = i(\tilde{b}_j, t(0))\varphi_t$ for $j = 1, \ldots, l$. For $X \in \tilde{\mathfrak{S}}$ choose $n_1, \ldots, n_l \in \mathbf{Z}_+$ such that $X = (\tilde{S}_1)^{n_1} \cdots (\tilde{S}_l)^{n_l}$. We have

$$(X\varphi)_t = \varphi_t \prod_{j=1}^{l}{}' \left(i\left(\tilde{b}_j, t(0)\right)\right)^{n_j}. \tag{4.7}$$

4.3. Lemma. *To every $n \in \mathbf{Z}_+$ there corresponds $c_n > 0$ such that*

$$\left(1 + \|t(0)\|\right)^n |\varphi_t| \leq c_n p_{nl}(\varphi) \quad \textit{for all } \varphi \in C^\infty(\hat{T}) \textit{ and } t \in T. \tag{4.8}$$

Proof. The norm $\| \|_\infty$ on $\mathfrak{a}$ is defined by

$$\|x\|_\infty = \max\left\{\left|\left(\tilde{b}_j, x\right)\right| ; j = 1, \ldots, l\right\}.$$

Let c be an upper bound of the function $t \mapsto (1 + \|t(0)\|)^n \|t(0)\|_\infty^{-n}$ on the set $T\backslash\{\tau(0)\}$. Set

$$Y = \left\{j;\ 1 \leq j \leq l \text{ and } \left(\tilde{b}_j, t(0)\right) \neq 0\right\}.$$

Let $X = \Pi_{j \in Y}(\tilde{S}_j)^n$. Then (4.7) implies $p_{nl}(\varphi) \geq \|t(0)\|_\infty^n |\varphi_t|$. One now has $p_{nl}(\varphi) \geq c^{-1}(1 + \|t(0)\|)^n |\varphi_t|$ for $t \neq \tau(0)$.

4.4. Lemma. *One has $\check{\varphi} \in \mathcal{C}^2(G, K)$ for $\varphi \in C_W^\infty(\hat{T})$. The mapping $\check{}$: $C_W^\infty(\hat{T}) \to \mathcal{C}^2(G, K)$ is a monomorphism of topological vector spaces.*

Proof. For $n \in \mathbf{Z}_+$ let c_n satisfy (4.8). Then (4.5) and (4.8) imply $u_n^2(\check{\varphi}) \leq c_n p_{nl}(\varphi(s^{-1})c(s)^{-1})$ for $\varphi \in C_W^\infty(\hat{T})$. Clearly, there exists $c > 0$ such that for all $\varphi \in C_W^\infty(\hat{T})$ the following inequality holds true:

$$u_n^2(\check{\varphi}) \leq c p_{nl}\left(\varphi(s^{-1})c(s)^{-1}\right).$$

In the rest of the section we shall assume K is a standard maximal compact subgroup. The next lemma is a part of the proof of Theorem (5.1.2) of [5] and here we omit the proof.

4.5. Lemma. *For $m_1, m_2 \in M^-$ the integral*

$$\int_{\hat{T}} \Gamma_s(m_1)\overline{\Gamma_s(m_2)}|c(s)|^{-2}\, ds$$

is equal to $|W|Q^{-1}v_{m_1}^{-1}$ when $\nu(m_1) = \nu(m_2)$. Otherwise this integral is zero.

The immediate consequence of the previous lemma is the fact that $Q|W|^{-1}|c(s)|^{-2}\, ds$ is the Plancherel measure for G/K on $\hat{T}$.

4.6. Corollary. *One has $(\hat{f})\check{} = f$ for all $f \in \mathcal{C}^2(G, K)$.*

From the results of the previous lemmas we easily obtain the next theorem.

4.7. Theorem. *Let K be a standard maximal compact subgroup of G. The spherical transformation*

$$\hat{\ }: \mathcal{C}^2(G, K) \to C_W^\infty(\hat{T}), \qquad \hat{f}(s) = \int_G f(g)\Gamma_s(g^{-1})\, dg$$

is an isomorphism of topological algebras. The inverse formula is given by

$$f(g) = \frac{Q}{|W|}\int_{\hat{T}} \hat{f}(s)\Gamma_s(g)|c(s)|^{-2}\, ds. \tag{4.9}$$

5. The spherical transformation on $\mathcal{C}^\gamma(G, K)$, $\gamma < 2$. Fix γ, $0 < \gamma < 2$, for the whole section. We write

$$D(x) = \bigcap_{w \in W} w(x - C_0^\perp)$$

for $x \in \overline{C}_0$. It is a closed, convex, W-invariant subset of $\mathfrak{a}$. One has

$$D(x) \cap \overline{C}_0 = (x - C_0^\perp) \cap \overline{C}_0. \tag{5.1}$$

Thus, $D(x)$ is a compact subset of $\mathfrak{a}$.

By (1.4) one sees that $\eta(\delta^{1/\gamma - 1/2}) = (1/\gamma - 1/2)\eta(\delta) \in C_0$. The set $D(\eta(\delta^{1/\gamma - 1/2}))$ is denoted by D_γ. The interior of D_γ contains 0.

The interior of $\eta^{-1}(D_\gamma)$ is denoted by D^γ. It is an open, W-invariant set with compact closure and contains $\hat{T}$.

For φ an analytic function on D^γ and $n \in \mathbf{Z}_+$ set

$$p_n^\gamma(\varphi) = \sup\{|(X\varphi)(x)|;\ X \in \mathfrak{S}_n \text{ and } x \in D^\gamma\}. \tag{5.2}$$

The vector space of all analytic functions φ on D^γ such that $p_n^\gamma(\varphi) < \infty$ for all $n \in \mathbf{Z}_+$ is denoted by $\mathcal{H}(D^\gamma)$. The seminorms p_n^γ, $n \in \mathbf{Z}_+$, generate the structure of the Fréchet space on $\mathcal{H}(D^\gamma)$, and $\mathcal{H}(D^\gamma)$ is a topological algebra under pointwise multiplication. Denote by $\mathcal{H}_W(D^\gamma)$ the subalgebra of all W-invariant functions in $\mathcal{H}(D^\gamma)$. It is a Fréchet algebra.

Let $f \in \mathcal{C}^\gamma(G, K)$ and $s \in D^\gamma$. We will show that the function $g \mapsto f(g)\Gamma_s(g^{-1})$ is integrable. Choose $w \in W$ such that $w\eta(s) \in \overline{C}_0$. It is easy to see that $(\delta^{1/\gamma - 1/2}|ws|^{-1})(m) \leq 1$ for all $m \in M^-$. By Corollary 3.2 there exists $c_1 > 0$ such that

$$|\Gamma_s(m^{-1})| \leq c_1\delta^{1-1/\gamma}(m)\big(1 + \|\nu(m)(0)\|\big)^{r_0}$$

for all $m \in M^-$. Choose $r \in \mathbf{Z}_+$ such that $r - r_0 > 2r_0 + l$. Then there exists $c_2 > 0$ such that

$$|f(m)\Gamma_s(m^{-1})| \leq c_2 u_r^\gamma(f)\Xi^2(m)\big(1 + \|\nu(m)(0)\|\big)^{r_0 - r} \tag{5.3}$$

for all $m \in M^-$. By (2.4), the right side is integrable, so is the left side.

For $f \in \mathcal{C}^\gamma(G, K)$ we define the function $\hat{f}$: $D^\gamma \to \mathbf{C}$ by

$$\hat{f}(s) = \int_G f(g)\Gamma_s(g^{-1})\, dg.$$

5.1. Theorem. *One has $\hat{f} \in \mathcal{H}_W(D^\gamma)$ for $f \in \mathcal{C}^\gamma(G, K)$. The spherical transformation $\hat{\ }$: $\mathcal{C}^\gamma(G, K) \to \mathcal{H}_W(D^\gamma)$ is a morphism of topological algebras with unit.*

Proof. Let $f \in \mathcal{C}^\gamma(G, K)$, $s \in D^\gamma$ and $n \in \mathbf{Z}_+$. Formulas (3.9) and (5.3) imply

$$|f(m)\Gamma_s(m^{-1})^X| \leq cc_2\big(1 + \|\nu(m)(0)\|\big)^{n + r_0 - r}\Xi^2(m)u_r^\gamma(f) \tag{5.4}$$

for $r - n - r_0 > 2r_0 + l$. Set $r_1 = r - n - r_0$, and the integral (2.4) is denoted by c_3. Since the function $g \mapsto f(g)\Gamma_s(g^{-1})^X$ is integrable, we have $\hat{f} \in \mathcal{H}_W(D^\gamma)$ and

$$(X\hat{f})(s) = \int_G f(g)\Gamma_s(g^{-1})^X\, dg.$$

(5.4) implies $p_n^\gamma(\hat{f}) \leq cc_2c_3u_r^\gamma(f)$.

In the rest of this section we assume K is a standard maximal compact subgroup.

For $\varphi \in \mathcal{H}_W(D^\gamma)$, $(\varphi\,|\,\hat{T})\check{}$ is again denoted by $\check{\varphi}$.

5.2. LEMMA. *If* $\varphi \in \mathcal{H}_W(D^\gamma)$, *then* $\check{\varphi} \in \mathcal{C}^\gamma(G, K)$. *The mapping* $\check{}$: $\mathcal{H}_W(D^\gamma) \to \mathcal{C}^\gamma(G, K)$ *is a monomorphism of topological vector spaces.*

Proof. The proof proceeds in several parts.

(1) Let Y_1 denote the set of zeros of the function $c(s)$. Since K is standard, one has $\eta(Y_1) \cap (-\overline{C}_0) = \varnothing$. $\eta(Y_1)$ is a closed subset of $\mathcal{C}$. There exists an open subset Y_2 of $\mathcal{C}$ containing $\overline{C}_0$ such that the closure of Y_2 is adjoint with $(-\eta(Y_1))$. Set $Y = \eta^{-1}(Y_2) \cap D^\gamma$. Then, for $X \in \mathcal{U}(\mathfrak{t})$, $X(c^{-1}(s^{-1}))$ is a bounded function on Y.

(2) The unit circle in $\mathbf{C}$ is denoted by Γ. One now has

$$(5.5) \qquad \int_{\hat{T}} \psi(s)\, ds = (2\pi i)^{-l} \int_\Gamma \cdots \int_\Gamma \psi(\Theta^{-1}(z_1, \ldots, z_l))\, \frac{dz_1}{z_1} \cdots \frac{dz_l}{z_l}$$

for $\psi \in C(\hat{T})$.

For an analytic function φ on Y and $n \in \mathbf{Z}_+$ we write

$$(5.6) \qquad \|\varphi\|_n = \sup\{|(X\varphi)(s)|;\ X \in \mathfrak{S}_n \text{ and } s \in Y\}.$$

For $t \in T$ we set $(\varphi \,|\, \hat{T})_t = \varphi_t$.

We first claim that

$$(5.7) \qquad |\varphi_t| \delta(t)^{1/\gamma - 1/2} \leq \|\varphi\|_0$$

for any analytic function φ on Y and $t \in T$.

If $\gamma < \gamma_1 < 2$, then $\delta^{1/\gamma_1 - 1/2} \in Y$. Let $\tilde{b}_1, \ldots, \tilde{b}_l$ be the basis of A biorthogonal to the basis $b_1(0), \ldots, b_l(0)$. Then

$$\eta(s) = \sum_{j=1}^{l} \ln|s(b_j)|\tilde{b}_j \quad \text{for } s \in \hat{T}_{\mathbf{C}}.$$

Since the line connecting 0 and $\eta(\delta^{1/\gamma_1 - 1/2})$ is contained in Y_2, and Y_2 is open, we can find points $y_0, \ldots, y_m \in Y_2$ such that $y_0 = 0$, $y_m = \eta(\delta^{1/\gamma_1 - 1/2})$ and such that the vector $y_j - y_{j-1}$ is collinear with an element of $\{\tilde{b}_1, \ldots, \tilde{b}_l\}$ for every j, $1 \leq j \leq m$. Choose $r_j \in \hat{T}_{\mathbf{C}}$ such that $\eta(r_j) = y_j$ and $r_j = |r_j|$, $j = 0, \ldots, m$. The positively oriented circle in C with center 0 and radius $r_n(b_j)$ is denoted by Γ_j^n.

For $t \in T$ choose $n_1, \ldots, n_l \in \mathbf{Z}$ such that $t = b_1^{n_1} \cdots b_l^{n_l}$. Then one has

$$(5.8) \qquad \varphi_t = (2\pi i)^{-l} \int_{\Gamma_1^n} \cdots \int_{\Gamma_l^n} \varphi(\Theta^{-1}(z_1, \ldots, z_l))\, \frac{dz_1}{z_1^{1+n_1}} \cdots \frac{dz_l}{z_l^{1+n_l}}$$

for $n = 0$. Applying Cauchy's Theorem, we find that (5.8) is valid for $0 \leq n \leq m$. Now (5.8) with $n = m$ implies $|\varphi_t| \leq \|\varphi\|_0 \delta^{1/2 - 1/\gamma_1}(t)$ for all γ_1, $\gamma < \gamma_1 < 2$. Thus, we have proved (5.7).

(3) Fix $r \in \mathbf{Z}_+$ and let φ be an analytic function on Y such that $X\varphi$ is a bounded function on Y for every $X \in \mathcal{S}_{lr}$. Let $\| \ \|_\infty$ be the norm on $\mathcal{Q}$ defined by $\|x\|_\infty = \max\{|(\tilde{b}_n, x)|;\ 1 \le n \le l\}$. Let $c_1 > 1$ be an upper bound of the function $t \mapsto (1 + \|t(0)\|)^r \|t(0)\|_\infty^{-r}$ on $T \setminus \{\tau(0)\}$.

For $t \in T$ set $X = \{n;\ 1 \le n \le l \text{ and } (\tilde{b}_n, t(0)) \neq 0\}$. One now has

$$\left| \left(\left(\prod_{x \in X} S_n^r \right) \varphi \right)_t \right| \ge \|t(0)\|_\infty^r |\varphi_t|.$$

Using (5.7), we obtain

$$(1 + \|t(0)\|)^r |\varphi_t| \le c_1 \|\varphi\|_{lr} \delta(t)^{1/2 - 1/\gamma}. \tag{5.9}$$

(4) For $r \in \mathbf{Z}_+$ it is easy to see there exists $c_2 > 0$ such that

$$\left\| \varphi(s) c(s^{-1})^{-1} \right\|_{lr} \le c_2 p_{rl}^\gamma(\varphi) \quad \text{for all } \varphi \in \mathcal{H}(D^\gamma).$$

By what we have shown we obtain

$$u_r^\gamma(\check{\varphi}) \le c_1 c_2 p_{rl}^\gamma(\varphi) \quad \text{for all } \varphi \in \mathcal{H}_W(D^\gamma).$$

Thus, $\check{\varphi} \in \mathcal{C}^\gamma(G, K)$, and the mapping $\varphi \mapsto \check{\varphi}$ is continuous.

(5) According to Lemma 4.2, to prove that $\varphi \mapsto \check{\varphi}$ is injective, it is sufficient to show that $\varphi \,|\, \hat{T} = 0$ implies $\varphi = 0$. This is an immediate consequence of (1.5).

Now, the next theorem is proved.

5.3. THEOREM. *Let K be a standard maximal compact subgroup of G. The spherical transformation*

$$\hat{}: \mathcal{C}^\gamma(G, K) \to \mathcal{H}_W(D^\gamma), \qquad \hat{f}(s) = \int_G f(g) \Gamma_s(g^{-1})\, dg$$

is an isomorphism of topological algebras. The inversion formula is given by

$$f(g) = \frac{Q}{|W|} \int_{\hat{T}} \hat{f}(s) \Gamma_s(g) |c(s)|^{-2}\, ds$$

for all $f \in \mathcal{C}^\gamma(G, K)$.

6. The exception case. In this section we shall state without proofs the results about spherical transformations on $\mathcal{C}^2(G, K)$ with K an exceptional maximal compact subgroup. We restrict ourselves to G, the group of rational points of a simply connected almost simple algebraic group defined over k. We assume K is exceptional.

6.1. THEOREM. *Set*

$$I = \left\{ f \in \mathcal{C}^2(G, K);\ \int_G f(g)\Gamma_s(g^{-1})\, dg = 0 \text{ for all } s \in \hat{T} \right\},$$

$$J = \{ f \in \mathcal{C}^2(G, K); f * g = 0 \text{ for all } g \in I \}.$$

Then I and J are closed two-sided ideals in $\mathcal{C}^2(G, K)$ *and*

$$\mathcal{C}^2(G, K) = I \oplus J.$$

The spherical transformation

$$\hat{\ }: \mathcal{C}^2(G, K) \to C_W^\infty(\hat{T}), \qquad \hat{f}(s) = \int_G f(g)\Gamma_s(g^{-1})\, dg \tag{6.1}$$

is an epimorphism of topological algebras. The kernel of this epimorphism is I. The restriction of the spherical transformation (6.1) *to J is an isomorphism of topological algebras J and* $C_W^\infty(\hat{T})$. *For* $\varphi \in C_W^\infty(\hat{T})$ *one has* $\check{\varphi} \in J$ *where*

$$\check{\varphi}(g) = \frac{Q}{|W|} \int_{\hat{T}} \varphi(s)\Gamma_s(g)|c(s)|^{-2}\, ds.$$

If $\varphi \in C_W(\hat{T})$, *then* $(\check{\varphi})\hat{\ } = \varphi$. *The ideal J is given by* $J = \{\check{\varphi};\ \varphi \in C_W^\infty(\hat{T})\}$. *One has* $I \neq \{0\}$ *for K exceptional.*

6.2. *Note*. Suppose that G is a rank one group, i.e. $l = 1$. Then it can be shown that spherical transformation $\hat{\ }: \mathcal{C}^\gamma(G, K) \to \mathcal{H}_W(D^\gamma)$, $0 < \gamma < 2$, is an epimorphism of topological algebras and a statement analogous to Theorem 6.1 is true.

6.3. *Note*. We can determine $0 < \gamma_1 < 2$ depending on q_α, $\alpha \in \Sigma_1$, such that the assumption in Lemma 5.2 can be replaced by the assumption that $\gamma_1 < \gamma < 2$, and the statement and proof of Lemma are valid with this assumption.

Now, if $\gamma_1 < \gamma < 2$ the spherical transformation is an epimorphism of the topological algebra $\mathcal{C}^\gamma(G, K)$ onto $\mathcal{H}_W(D^\gamma)$ and the analogue of Theorem 6.1 is true.

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