Pacific Journal of Mathematics

THE DETERMINANTAL IDEALS OF LINK MODULES. II

Lorenzo Traldi

Vol. 109, No. 1 May 1983

PACIFIC JOURNAL OF MATHEMATICS
Vol. 109, No. 1, 1983

THE DETERMINANTAL IDEALS OF LINK MODULES. II

LORENZO TRALDI

Let H be the multiplicative free abelian group of rank $m \geq 1$. Suppose $0 \to B \to A \to IH \to 0$ is a short exact sequence of $\mathbf{Z}H$-modules, and the module A is finitely generated. Then B is also a finitely generated $\mathbf{Z}H$-module, and for any $k \in \mathbf{Z}$ the determinantal ideals of A and B satisfy the equality

$$E_k(A):(IH)^p = E_{k-1}(B):(IH)^q$$

for all sufficiently large values of p and q. Furthermore, if this exact sequence is the link module sequence of a tame link of m components in S^3, then

$$E_k(A) = E_{k-1}(B):(IH)^{\binom{m-1}{2}}$$

whenever $k \geq m$.

1. Introduction. Let H be the multiplicative free abelian group of rank $m \geq 1$, and $\mathbf{Z}H$ its integral group ring; if ε: $\mathbf{Z}H \to \mathbf{Z}$ is the augmentation map then its kernel is the augmentation ideal IH of $\mathbf{Z}H$. Following **[6]**, we will call a short exact sequence

$$(1) \qquad 0 \to B \xrightarrow{\phi} A \xrightarrow{\psi} IH \to 0$$

of $\mathbf{Z}H$-modules and homomorphisms an *augmentation sequence*, provided that the $\mathbf{Z}H$-module A is finitely generated. The module B is then also finitely generated, and so for any $k \in \mathbf{Z}$ there are well-defined determinantal ideals $E_k(A)$, $E_k(B) \subseteq \mathbf{Z}H$.

In **[6]** we discussed the relationship between the product ideals $E_k(A)\cdot(IH)^p$ and $E_{k-1}(B)\cdot(IH)^q$ for various values of k, p, and q. In the present paper, instead, we will consider the relationship between the various quotient ideals $E_k(A):(IH)^p$ and $E_{k-1}(B):(IH)^q$. (We recall the definition: if $U, V \subseteq \mathbf{Z}H$ are ideals then the quotient ideal $U:V$ is $\{x \in \mathbf{Z}H \mid xV \subseteq U\}$.)

At first glance, it may seem that if $U \subseteq \mathbf{Z}H$ is an ideal the quotient ideals $U:(IH)^p$ and the various product ideals $U\cdot(IH)^q$ are, in some sense, "duals" of each other, but this is not so. For the descending sequence

$$U = U\cdot(IH)^0 \supseteq U\cdot(IH)^1 \supseteq U\cdot(IH)^2 \supseteq \cdots$$

of ideals of $\mathbf{Z}H$ need not terminate, in general, while since $\mathbf{Z}H$ is noetherian the ascending sequence

$$U = U\colon(IH)^0 \subseteq U\colon(IH)^1 \subseteq U\colon(IH)^2 \subseteq \cdots$$

must, that is, there is a (unique least) $\rho(U)$ such that

$$U\colon(IH)^{\rho(U)} = U\colon(IH)^r \quad \forall\, r \geq \rho(U).$$

We will devote most of our attention to this terminal quotient ideal.

THEOREM (1.1). *If* (1) *is an augmentation sequence then for any* $k \in \mathbf{Z}$

$$E_k(A) : (IH)^{\rho(E_k(A))} = E_{k-1}(B) : (IH)^{\rho(E_{k-1}(B))}.$$

It is of interest, then, to determine the integers $\rho(E_k(A))$ and $\rho(E_{k-1}(B))$. Though this seems impracticable in general, we will prove

THEOREM (1.2). *If* (1) *is an augmentation sequence,* $n \in \mathbf{Z}$, *and* $\varepsilon E_n(A) = \mathbf{Z}$, *then* $\rho(E_k(A)) = 0$ *whenever* $k \geq n$. *Furthermore,* $\rho(E_{k-1}(B)) = 0$ *whenever* $k \geq n + \binom{m-1}{2}$, *and* $\rho(E_{k-1}(B)) \leq n + \binom{m-1}{2} - k$ *whenever* $n \leq k \leq n + \binom{m-1}{2}$. *Consequently,* $\rho(E_{k-1}(B)) \leq \binom{m-1}{2}$ *whenever* $k \geq n$.

(*Here* $\binom{m-1}{2}$ *is the binomial coefficient, and in particular* $\binom{0}{2} = \binom{1}{2} = 0$.)

If (1) is the module sequence of a tame link $L \subseteq S^3$ of m components (described, e.g., in [**1**]) then it is known [**5**] that $\varepsilon E_m(A) = \mathbf{Z}$. (Note: in [**5**] the notation $E_k(A) = E_k(L)$ is used in this case.) Combining this with Theorems (1.1) and (1.2), we obtain

COROLLARY (1.3). *If* (1) *is the module sequence of a tame link* $L \subseteq S^3$, *then*

$$E_k(A) = E_{k-1}(B)$$

whenever $k > \binom{m}{2}$, *and*

$$E_k(A) = E_{k-1}(B) : (IH)^{m+\binom{m-1}{2}-k}$$

whenever $m \leq k \leq \binom{m}{2}$. *Consequently,*

$$E_k(A) = E_{k-1}(B) : (IH)^{\binom{m-1}{2}}$$

whenever $k \geq m$.

A special case of this is particularly pleasant: if (1) is the module sequence of a tame two-component link $L \subseteq S^3$ then $E_k(A) = E_{k-1}(B)$ whenever $k \geq 2$. Since $E_1(A) = E_0(B) \cdot IH$, and $E_k(A) = E_{k-1}(B) = 0$ whenever $k \leq 0$, it follows that for any $k \in \mathbf{Z}$ $E_k(A)$ and $E_{k-1}(B)$ are equivalent as invariants of L, that is, each ideal determines the other. In this respect, the behavior of these invariants for two-component links is analogous to their behavior for knots. (Recall that if $m = 1$ and (1) is any augmentation sequence then [**6**] $E_k(A) = E_{k-1}(B)$ for every value of k.)

For links of three or more components in S^3, the relationship between the determinantal ideals of the modules A and B appearing in the link module sequence is more complex; we will discuss this further in §3.

Another result, analogous to Theorem (1.2) (though seemingly of less use in the application to the module sequences of tame links), is

THEOREM (1.4). *If* (1) *is an augmentation sequence*, $n \in \mathbf{Z}$, *and* $\varepsilon E_{n-1}(B) = \mathbf{Z}$, *then* $\rho(E_{k-1}(B)) = 0$ *whenever* $k \geq n$. *Furthermore*, $\rho(E_k(A)) = 0$ *whenever* $k \geq n + m - 1$, *and* $\rho(E_k(A)) \leq n + m - 1 - k$ *whenever* $n \leq k \leq n + m - 1$. *Consequently*, $\rho(E_k(A)) \leq m - 1$ *whenever* $k \geq n$.

The author would like to express his gratitude to William S. Massey, for his stimulating correspondence, and to Sharon Richter, for her drawing.

2. Proofs.

PROPOSITION (2.1). *Let U and V be ideals of* $\mathbf{Z}H$. *Then* $U:(IH)^{\rho(U)} = V:(IH)^{\rho(V)}$ *if, and only if, there are integers* $p, q \geq 0$ *such that* $U \cdot (IH)^p \subseteq V$ *and* $V \cdot (IH)^q \subseteq U$.

Proof. First, suppose that $U:(IH)^{\rho(U)} = V:(IH)^{\rho(V)}$. Then $U \cdot (IH)^{\rho(V)} \subseteq (U:(IH)^{\rho(U)}) \cdot (IH)^{\rho(V)} = (V:(IH)^{\rho(V)}) \cdot (IH)^{\rho(V)} \subseteq V$, and similarly $V \cdot (IH)^{\rho(U)} \subseteq U$.

Suppose, instead, that there are non-negative integers p and q as described. Then $(U:(IH)^{\rho(U)}) \cdot (IH)^{p+\rho(U)} \subseteq U \cdot (IH)^p \subseteq V$, and hence $U:(IH)^{\rho(U)} \subseteq V:(IH)^{p+\rho(U)} \subseteq V:(IH)^{\rho(V)}$. Similarly, $V:(IH)^{\rho(V)} \subseteq U:(IH)^{\rho(U)}$, so these two ideals coincide. □

Theorem (1.1) follows immediately from Proposition (2.1) and Theorem (1.1) of [**6**].

LEMMA (2.2). *Let U and V be ideals of $\mathbf{Z}H$, and suppose that $\varepsilon(U) = \mathbf{Z}$. Then $U + V = U + V \cdot (IH)^k$ for any $k \geq 0$.*

Proof. Since $(IH)^0 = \mathbf{Z}H$, certainly $U + V = U + V \cdot (IH)^0$.

Since $\varepsilon(U) = \mathbf{Z}$, $U + IH = \mathbf{Z}H$, and hence $U + V = (U + V) \cdot (U + IH) \subseteq U + V \cdot IH \subseteq U + V$. Thus $U + V = U + V \cdot IH$.

Proceeding inductively, suppose $k \geq 1$ and $U + V = U + V \cdot (IH)^k$. Then $U + V = U + V \cdot (IH)^k = (U + V \cdot (IH)^k) \cdot (U + IH) \subseteq U + V \cdot (IH)^{k+1} \subseteq U + V$, and hence $U + V = U + V \cdot (IH)^{k+1}$. □

COROLLARY (2.3). *Let $U \subseteq \mathbf{Z}H$ be an ideal with $\varepsilon(U) = \mathbf{Z}$. Then $\rho(U) = 0$.*

Proof. By definition, $(U : (IH)^{\rho(U)}) \cdot (IH)^{\rho(U)} \subseteq U$, and hence $U = U + (U : (IH)^{\rho(U)}) \cdot (IH)^{\rho(U)}$. By the preceding lemma, then, $U = U + (U : (IH)^{\rho(U)})$, that is, $U \supseteq U : (IH)^{\rho(U)}$. Since $U \subseteq U : (IH)^{\rho(U)}$, it follows that $U = U : (IH)^{\rho(U)}$, and hence $\rho(U) = 0$. □

We may now proceed to the proof of Theorem (1.2); suppose (1) is an augmentation sequence and $\varepsilon E_n(A) = \mathbf{Z}$.

If $m = 1$, then by Theorem $(1.1)_1$ of **[6]** $E_k(A) = E_{k-1}(B)$ for any value of k. Also, if $k \geq n$ then $E_k(A) \supseteq E_n(A)$, so $\varepsilon E_k(A) = \mathbf{Z}$, so by Corollary (2.3) $\rho(E_k(A)) = 0$.

If $m = 2$, then by Theorem $(1.1)_2$ of **[6]** $E_{k-1}(B) \cdot IH \subseteq E_k(A) \subseteq E_{k-1}(B)$ for any value of $k \in \mathbf{Z}$. If $k \geq n$ then $E_k(A) \supseteq E_n(A)$, so by Corollary (2.3) $\rho(E_k(A)) = 0$. Furthermore, since $E_{k-1}(B) \cdot IH \subseteq E_k(A)$, $E_k(A) = E_k(A) + E_{k-1}(B) \cdot IH$, so by Lemma (2.2) $E_k(A) = E_k(A) + E_{k-1}(B)$, that is, $E_k(A) \supseteq E_{k-1}(B)$; since $E_k(A) \subseteq E_{k-1}(B)$, it follows that $E_k(A) = E_{k-1}(B)$.

If $m \geq 3$ and $k \geq n$ then $\mathbf{Z} = \varepsilon E_n(A) = \varepsilon E_k(A)$, so by Corollary (2.3) $\rho(E_k(A)) = 0$. As shown in §3 of **[6]**,

$$E_{k-1}(B) \supseteq \sum_i E_{i+m}(X) E_{k-i-1}(A),$$

where X is a $\mathbf{Z}H$-module with $E_{m-2}(X) = 0$, $E_j(X) = (IH)^{\binom{m}{2}-j}$ for $m - 1 \leq j < \binom{m}{2}$, and $E_{\binom{m}{2}}(X) = \mathbf{Z}H$.

In particular, if $k \geq n + \binom{m-1}{2}$ then $E_{k-1}(B) \supseteq E_{\binom{m}{2}}(X) E_{k-\binom{m-1}{2}}(A) = E_{k-\binom{m-1}{2}}(A) \supseteq E_n(A)$, so $\varepsilon E_{k-1}(B) = \varepsilon E_n(A) = \mathbf{Z}$, so by Corollary (2.3) $\rho(E_{k-1}(B)) = 0$.

If $n \leq k < n + \binom{m-1}{2}$, then

$$\begin{aligned} E_{k-1}(B) &\supseteq E_{k-n-1+m}(X)E_n(A) + E_{m-1}(X)E_k(A) \\ &= (IH)^{\binom{m-1}{2}+n-k} \cdot E_n(A) + (IH)^{\binom{m-1}{2}} \cdot E_k(A) \\ &= (IH)^{\binom{m-1}{2}+n-k} \cdot \left(E_n(A) + (IH)^{k-n} \cdot E_k(A)\right). \end{aligned}$$

Since $\varepsilon E_n(A) = \mathbf{Z}$, it follows from Lemma (2.2) that $E_n(A) + (IH)^{k-n} \cdot E_k(A) = E_n(A) + E_k(A)$, so since $E_n(A) \subseteq E_k(A)$ (and hence $E_n(A) = E_n(A) + E_k(A)$) we conclude that

$$E_{k-1}(B) \supseteq (IH)^{\binom{m-1}{2}+n-k} \cdot E_k(A).$$

Since $\rho(E_k(A)) = 0$ (as noted earlier), it follows from this and Theorem (1.1) that

$$E_{k-1}(B) \supseteq (IH)^{\binom{m-1}{2}+n-k} \cdot \left(E_{k-1}(B) : (IH)^{\rho(E_{k-1}(B))}\right),$$

and hence

$$E_{k-1}(B) : (IH)^{\rho(E_{k-1}(B))} \subseteq E_{k-1}(B) : (IH)^{\binom{m-1}{2}+n-k}.$$

That $\rho(E_{k-1}(B)) \leq \binom{m-1}{2} + n - k$ follows immediately.

This completes the proof of Theorem (1.2).

Turning to Theorem (1.4), suppose (1) is an augmentation sequence and $\varepsilon E_{n-1}(B) = \mathbf{Z}$.

If $m = 1$, then by Theorem $(1.1)_1$ of **[6]** $E_k(A) = E_{k-1}(B)$ for any value of k. If $k \geq n$ then $E_{k-1}(B) \supseteq E_{n-1}(B)$, and so $\varepsilon E_{k-1}(B) = \mathbf{Z}$; by Corollary (2.3), then, $\rho(E_{k-1}(B)) = 0$.

If $m \geq 2$ and $k \geq n$ then $\mathbf{Z} = \varepsilon E_{n-1}(B) = \varepsilon E_{k-1}(B)$, so by Corollary (2.3) $\rho(E_{k-1}(B)) = 0$. Also, by Lemma (2.1) of **[6]**

$$E_k(A) \supseteq \sum_i E_{k-i}(B)E_i(IH).$$

In **[2]** it is shown that $E_0(IH) = E_0(N_2(m)) = 0$, $E_j(IH) = E_j(N_2(m)) = (IH)^{m-j}$ for $1 \leq j < m$, and $E_m(IH) = E_m(N_2(m)) = \mathbf{Z}H$. ($N_2(m)$ is a presentation matrix for IH, studied in **[2]**.)

In particular, if $k \geq n + m - 1$ then $E_k(A) \supseteq E_{k-m}(B)E_m(IH) = E_{k-m}(B) \supseteq E_{n-1}(B)$, so $\varepsilon E_k(A) = \mathbf{Z}$, and hence by Corollary (2.3) $\rho(E_k(A)) = 0$.

If $n \leq k < n + m - 1$, then

$$\begin{aligned} E_k(A) &\supseteq E_{n-1}(B)E_{k-n+1}(IH) + E_{k-1}(B)E_1(IH) \\ &= (IH)^{m-k+n-1} \cdot E_{n-1}(B) + (IH)^{m-1} \cdot E_{k-1}(B) \\ &= (IH)^{m-k+n-1} \cdot \left(E_{n-1}(B) + (IH)^{k-n} \cdot E_{k-1}(B)\right). \end{aligned}$$

Since $\varepsilon E_{n-1}(B) = \mathbf{Z}$, it follows from Lemma (2.2) that

$$E_{n-1}(B) + (IH)^{k-n} \cdot E_{k-1}(B) = E_{n-1}(B) + E_{k-1}(B) = E_{k-1}(B);$$

hence

$$E_k(A) \supseteq (IH)^{n+m-1-k} \cdot E_{k-1}(B).$$

Since $\rho(E_{k-1}(B)) = 0$, it follows from this and Theorem (1.1) that

$$E_k(A) \supseteq (IH)^{n+m-1-k} \cdot \left(E_k(A) : (IH)^{\rho(E_k(A))}\right).$$

We may conclude from this that $\rho(E_k(A)) \leq n + m - 1 - k$.

This completes the proof of Theorem (1.4).

We may note here, without going into detail, that Theorems (1.1), (1.2), and (1.4) hold in a broader context, with $\mathbf{Z}H$ replaced by an arbitrary noetherian commutative ring with unity R, and IH replaced by the ideal of R generated by the elements of some R-sequence $\{r_1, \ldots, r_m\}$. (The hypotheses $\varepsilon E_n(A) = \mathbf{Z}$ and $\varepsilon E_{n-1}(B) = \mathbf{Z}$ of Theorems (1.2) and (1.4) should be replaced by the equivalent hypotheses $\mathbf{Z}H = E_n(A) + IH$ and $\mathbf{Z}H = E_{n-1}(B) + IH$, respectively, prior to any such generalization.) An analogous generalization is discussed, in greater depth, in §5 of [**6**].

3. Links of three or more components. A simple consequence of Corollary (1.3) is: if (1) is the module sequence of a tame link of m components in S^3, then for $k \geq m$ the ideal $E_k(A)$ is determined by $E_{k-1}(B)$. A natural question to ask, especially in view of the cases $m = 1$ and $m = 2$ (discussed in §1) is: does $E_k(A)$, in turn, determine $E_{k-1}(B)$, for $k \geq m$? That the answer to this question is "no" may be seen by considering the three-component links 6_2^3 and 8_5^3 (as they are named in Appendix C of [**4**]). As W. S. Massey has shown, if (1) is the link module sequence of the former then $E_3(A) = \mathbf{Z}H$ and $E_2(B) = IH$ [**3**, Example 1], while if (1) is the link module sequence of the latter then $E_3(A) = \mathbf{Z}H = E_2(B)$ [**3**, Example 2].

Another natural question is: can the result of Theorem (1.1) be made more definitive for $1 < k < m$, as it can for $k \geq m$ (Corollary (1.3)) and

$k = 1$ [**2**]? Though we shall not answer this question, we will consider an example of a three-component link for which the relationship between $E_2(A)$ and $E_1(B)$ is particularly complex.

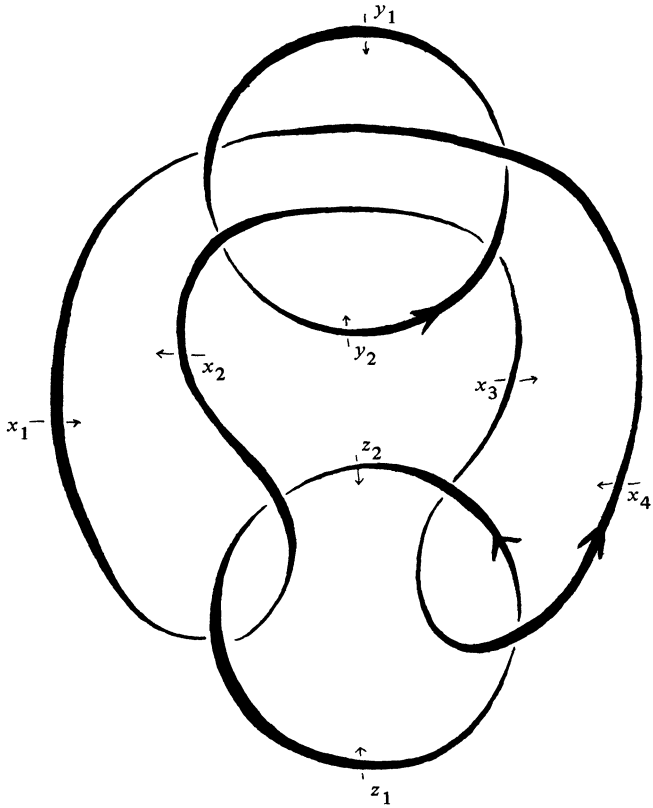

FIGURE 1

Pictured in Figure 1 is the link 8^3_{10} [**4**, Appendix C]. The Wirtinger presentation [**4**, p. 56] of the fundamental group G of the complement of this link in S^3 is

$$\langle x_1, x_2, x_3, x_4, y_1, y_2, z_1, z_2; x_1z_1 = z_1x_2, y_2x_2 = x_3y_2, x_3z_2 = z_2x_4, y_1x_4 = x_1y_1, x_2y_1 = y_2x_2, x_4y_2 = y_1x_4, z_1x_4 = x_4z_2, z_2x_2 = x_2z_1\rangle.$$

Since any one of the relations in this presentation is redundant, we may simply delete the seventh. Also, we may remove the fourth relation and the generator x_1, replacing any occurrence of x_1 in another relation by an occurrence of $y_1x_4y_1^{-1}$; similarly, we may remove the third relation and the generator x_3, replacing x_3 by $z_2x_4z_2^{-1}$ in the remaining relations. What results, after some simple rewriting of relations, is the presentation

$$\langle x_2, x_4, y_1, y_2, z_1, z_2;\ x_4 = y_1^{-1}z_1x_2z_1^{-1}y_1,\ y_1 = x_2^{-1}y_2x_2,\\ x_2^{-1}y_2^{-1}z_2x_4z_2^{-1}y_2 = 1,\ x_4y_2x_4^{-1}y_1^{-1} = 1,\ z_1 = x_2^{-1}z_2x_2\rangle.$$

After deleting the first relation and the generator x_4, and replacing x_4 by $y_1^{-1}z_1x_2z_1^{-1}y_1$ in the remaining relations, we may delete the second and fifth relations and the generators y_1 and z_1, substituting $x_2^{-1}y_2x_2$ for y_1 and $x_2^{-1}z_2x_2$ for z_1, and obtain the presentation

$$\langle x_2, y_2, z_2;\ x_2^{-1}y_2^{-1}z_2x_2^{-1}y_2^{-1}z_2x_2z_2^{-1}y_2x_2z_2^{-1}y_2 = 1,\\ y_2^{-1}z_2x_2z_2^{-1}y_2x_2y_2x_2^{-1}y_2^{-1}z_2x_2^{-1}z_2^{-1} = 1\rangle.$$

The Alexander matrix M of this presentation [1, §3] is the transpose of the matrix

$$\begin{pmatrix} (1+t_1^{-1}t_2^{-1}t_3)(t_1^{-1}t_2^{-1}t_3 - t_1^{-1}) & (1-t_2)(t_1+t_2^{-1}t_3) \\ (1-t_1^{-1})(t_2^{-1}+t_1^{-1}t_2^{-2}t_3) & (t_1-1)(t_1+t_2^{-1}) \\ (t_1^{-1}-1)(t_2^{-1}+t_1^{-1}t_2^{-2}t_3) & (t_1-1)(1-t_2^{-1}) \end{pmatrix}.$$

(Here t_1, t_2, and t_3 are the elements of $G/G' = H$ determined by the elements of G represented by x_2, y_2, and z_2, respectively.) If (1) is the module sequence of the link 8_{10}^3, then M is a presentation matrix for the $\mathbf{Z}H$-module A [1, §3], and hence, in particular, the ideal of $\mathbf{Z}H$ generated by the entries of M is

$$E_2(A) = (1+t_1^{-1}t_2^{-1}t_3)\cdot IH + (t_1+1, t_2-1)\cdot(t_1-1).$$

The matrix M can be factored as a product $M = M'\cdot N_2(3)$, where

$$N_2(3) = \begin{pmatrix} 1-t_2 & t_1-1 & 0 \\ 1-t_3 & 0 & t_1-1 \\ 0 & 1-t_3 & t_2-1 \end{pmatrix}$$

and

$$M' = \begin{pmatrix} t_1^{-1}t_2^{-1}(1+t_1^{-1}t_2^{-1}t_3) & -t_1^{-1}t_2^{-1}(1+t_1^{-1}t_2^{-1}t_3) & 0 \\ t_1+t_2^{-1}t_3 & 0 & t_2^{-1}(t_1-1) \end{pmatrix}.$$

(Here $N_2(3)$ is a matrix discussed by Crowell and Strauss [**2**], with columns corresponding to the integers 1, 2, and 3 (in order), and rows corresponding to the pairs 12, 13, and 23 (in order).) It follows [**2**, p. 106] that the module B of the link module sequence of 8^3_{10} has the presentation matrix

$$P = \begin{pmatrix} M' \\ N_3(3) \end{pmatrix}$$

$$= \begin{pmatrix} t_1^{-1}t_2^{-1}\left(1 + t_1^{-1}t_2^{-1}t_3\right) & -t_1^{-1}t_2^{-1}\left(1 + t_1^{-1}t_2^{-1}t_3\right) & 0 \\ t_1 + t_2^{-1}t_3 & 0 & t_2^{-1}(t_1 - 1) \\ t_3 - 1 & 1 - t_2 & t_1 - 1 \end{pmatrix}.$$

($N_3(3)$ is another matrix discussed in [**2**]; its columns correspond to the pairs 12, 13, and 23 (in order).) Thus the ideal of $\mathbf{Z}H$ generated by the determinants of the two-by-two submatrices of P is

$$E_1(B) = E_2(A) + \left(1 + t_1^{-1}t_2^{-1}t_3\right)^2.$$

In particular, the $\mathbf{Z}H$-modules A and B of the link module sequence of 8^3_{10} have the property that

$$\begin{aligned}(E_2(A) : IH)\cdot IH &= (E_1(B) : IH)\cdot IH \\ &\subset E_2(A) \subset E_1(B) \subset E_2(A) : IH = E_1(B) : IH,\end{aligned}$$

in which all three indicated inclusions are strict. The relationship between $E_2(A)$ and $E_1(B)$ does not, then, seem to fall into the pattern of the simple relationships between $E_k(A)$ and $E_{k-1}(B)$ for $k \neq 2$ (namely, $E_k(A) = E_{k-1}(B)\cdot IH$ for $k < 2$, and $E_k(A) = E_{k-1}(B) : IH$ for $k > 2$).

References

1. R. H. Crowell, *The derived module of a homomorphism*, Advances in Math., **6** (1971), 210–238.
2. R. H. Crowell and D. Strauss, *On the elementary ideals of link modules*, Trans. Amer. Math. Soc., **142** (1969), 93–109.
3. W. S. Massey, *Completion of link modules*, Duke Math. J., **47** (1980), 399–420.
4. D. Rolfsen, *Knots and Links*, Publish or Perish, Inc., Berkeley, California, 1976.
5. L. Traldi, *A generalization of Torres' second relation*, Trans. Amer. Math. Soc., **269** (1982), 593–610.
6. _____, *The determinantal ideals of link modules, I*, Pacific J. Math., **101** (1982), 215–222.

Received January 27, 1982.

LAFAYETTE COLLEGE
EASTON, PA 18042

Pacific Journal of Mathematics

Vol. 109, No. 1 May, 1983

Donald George Babbitt and V. S. Varadarajan, Formal reduction theory of meromorphic differential equations: a group theoretic view 1

Jo-Ann Deborah Cohen, Norms on $F(X)$ 81

Robert Fitzgerald, Witt kernels of function field extensions 89

Hervé Jacquet and Joseph Andrew Shalika, The Whittaker models of induced representations 107

Masakiti Kinukawa, Some generalizations of contraction theorems for Fourier series 121

Joseph Weston Kitchen, Jr. and David A. Robbins, Sectional representations of Banach modules 135

Victor Charles Pestien, Jr., Weak approximation of strategies in measurable gambling 157

Richard Scott Pierce and Charles Irvin Vinsonhaler, Realizing central division algebras 165

Walter Ricardo Ferrer Santos, Cohomology of comodules 179

Marko Tadić, Harmonic analysis of spherical functions on reductive groups over $\mathfrak{p}$-adic fields 215

Lorenzo Traldi, The determinantal ideals of link modules. II 237

Alain J. Valette, A remark on the Kasparov groups $\mathrm{Ext}^i(A, B)$ 247