

Pacific Journal of Mathematics

ON SYLVESTER'S PROBLEM AND HAAR SPACES

PETER B. BORWEIN

ON SYLVESTER'S PROBLEM AND HAAR SPACES

PETER B. BORWEIN

Given a finite set of points in the plane (with distinct x coordinates) must there exist a polynomial of degree n that passes through exactly $n + 1$ of the points? Provided that the points do not all lie on the graph of a polynomial of degree n then the answer to this question is yes. This generalization of Sylvester's Problem (the $n = 1$ case) is established as a corollary to a version of Sylvester's Problem that holds for certain finite dimensional Haar spaces of continuous functions.

If E is a finite set of points in the plane then there exists a line through exactly two points of E unless all the points of E are colinear. This attractive result was posed as a problem by J. J. Sylvester in 1893 and was proved in 1933 by T. Gallai (see [3]). A particularly simple solution of Sylvester's Problem, due to L. M. Kelly, may be found in [1]. We ask the following question: If V_n is an n -dimensional vector space of real-valued continuous functions of a real variable and if E is a finite set in the plane, must there exist $g \in V_n$ so that the graph of g passes through exactly n points of E ? We show that the answer to the above question is affirmative if V_n is a uni-modal Haar space of dimension n . (See Theorem 1.)

A Haar space H_n of dimension n on an interval $[a, b]$ is an n -dimensional real vector space of real-valued continuous functions with the additional property that if $g \in H_n$ and g has n distinct zeros then g is identically zero. Haar spaces are often also called Chebychev spaces. A Haar space H_n of dimension n is uni-modal if it satisfies the following: if $g \in H_n$ has $n - 1$ distinct zeros at $a \leq \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} \leq b$ then g has a single change of monotonicity on each of the intervals

$$[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \dots, [\alpha_{n-2}, \alpha_{n-1}]$$

and g is monotonic on $[a, \alpha_1]$ and $[\alpha_{n-1}, b]$.

The algebraic polynomials of degree less than n form a uni-modal Haar space of dimension n on any interval. The following are other examples of uni-modal Haar spaces of dimension n on $[a, b]$:

(a) The space spanned by

$$\{1, e^{\alpha_1 x}, \dots, e^{\alpha_{n-1} x}\}$$

where $\alpha_1, \dots, \alpha_{n-1}$ are distinct non-zero real numbers.

(b) The space spanned by

$$\{1, x, x^2, \dots, x^{n-2}, f(x)\}$$

where $f^{(n-1)}(x) > 0$ on $[a, b]$.

(c) The space spanned by

$$\{1, x^2, x^4, \dots, x^{2n-2}\}$$

on an interval $[a, b]$ where $a > 0$.

We say that a finite set E contained in the plane R^2 in $\text{co}(H_n)$ if all the points of E lie on (the graph of) g where g is a single element of H_n .

We shall now prove:

THEOREM 1. *Suppose that E is a finite set of points in the strip $\{(x, y) | a \leq x \leq b\}$ and suppose that no two points of E lie on the same vertical line. Suppose that H_n is a uni-modal Haar space of dimension $n \geq 2$ on $[a, b]$. Then either there exists $g \in H_n$ so that g passes through exactly n points of E or E is $\text{co}(H_n)$.*

Proof. Our proof is motivated by L. M. Kelly's proof of Sylvester's Problem. We first note that if x_1, x_2, \dots, x_n are n distinct numbers in $[a, b]$ and if y_1, \dots, y_n are real numbers then there exists a unique $h \in H_n$ so that

$$h(x_i) = y_i \quad \text{for } i = 1, \dots, n.$$

This interpolation property is an easy consequence of the fact that H_n is a Haar space of dimension n . (For further discussion of Haar spaces see [2, p. 23].) We assume that E is not $\text{co}(H_n)$ and, hence, that E contains at least $n + 1$ points. We know that there is an element of H_n that passes through any n points of E . We assume, for the sake of deriving a contradiction, that any such element in fact passes through at least $n + 1$ points of E . Let $K \subset H_n$ denote the set of elements of H_n that pass through at least n points of E . Since H_n is Haar there is a unique element of H_n passing through any n points of E . Since E is finite K must be finite also.

Let P be a point in E that is vertically closest to, though not on, the graph of an element g in K . Since K and E are finite such a pair P and g must exist. We assumed that g was an element of K , thus there exist $n + 1$ points $(x_1, y_1), \dots, (x_{n+1}, y_{n+1})$ in E through which g passes. We may suppose that

$$a \leq x_1 < x_2 < \dots < x_{n+1} \leq b.$$

Write $P = (x^*, y^*)$ and suppose that the vertical distance from P to g is δ .

Case 1. $x_i < x^* < x_{i+1}$ where $2 \leq i \leq n - 1$. Let $f \in K$ pass through the n points $(x_1, y_1), \dots, (x_{i-1}, y_{i-1}), (x^*, y^*), (x_{i+2}, y_{i+2}), \dots, (x_{n+1}, y_{n+1})$. Consider $f - g \in H_n$. The function $f - g$ has $n - 1$ distinct zeros at $x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{n+1}$. Since H_n is a uni-modal Haar space, $f - g$ has at most a single change of monotonicity on the interval $[x_{i-1}, x_{i+2}]$. Since, $x_{i-1} < x_i < x^* < x_{i+1} < x_{i+2}$ we must have either

$$0 < |f(x_i) - g(x_i)| < |f(x^*) - g(x^*)| = \delta$$

or

$$0 < |f(x_{i+1}) - g(x_{i+1})| < |f(x^*) - g(x^*)| = \delta.$$

This implies the contradiction that either (x_i, y_i) or (x_{i+1}, y_{i+1}) is vertically too close to $f \in K$.

Case 2. Either $x^* < x_2$ or $x_n < x^*$. We treat the case $x^* < x_2$. The other case is virtually identical. Let $f \in K$ pass through the n points

$$(x^*, y^*), (x_3, y_3), \dots, (x_{n+1}, y_{n+1}).$$

Since $f - g \in H_n$ has $n - 1$ distinct zeros at x_3, x_4, \dots, x_{n+1} we know that $f - g$ is monotonic on $[a, x_3]$. This leads to the contradiction that

$$0 < |f(x_2) - g(x_2)| < |f(x^*) - g(x^*)| = \delta. \quad \square$$

We get a solution of Sylvester's Problem by taking H_2 in Theorem 1 to be the uni-modal Haar space of lines (it may be necessary to rotate E first to ensure that no two points of E lie on the same vertical line). We also have

COROLLARY 1. *Let E be a finite set in R^2 with no two points on the same vertical line. Suppose that the points of E do not all lie on a polynomial of degree less than $n + 1$. Then there exists:*

- (a) *a line through exactly two points of E .*
- (b) *a parabola through exactly three points of E .*
- (c) *a cubic through exactly four points of E .*

(n) *a polynomial of degree n through exactly $n + 1$ points of E .*

We can construct a Haar space H'_n on $[a, b)$ where we demand that each $g \in H'_n$ be periodic with period $b - a$. To make H'_n uni-modal we

require that: if $g \in H'_n$ has $n - 1$ distinct zeroes at $a \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < b$ then g has a single change of monotonicity on each of the $n - 1$ intervals

$$[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \dots, [\alpha_{n-2}, \alpha_{n-1}], [\alpha_{n-1}, b - a + \alpha_1].$$

THEOREM 2. *Suppose that E is a finite set of points in the strip $\{(x, y) | a \leq x < b\}$ and suppose that no two points of E lie on the same vertical line. Suppose that H'_n is a uni-modal Haar space of dimension $n \geq 2$ on $[a, b)$. Then either there exists $g \in H'_n$ so that g passes through exactly n points of E or E is $\text{co}(H'_n)$.*

Proof. The proof is exactly analogous to the proof of Theorem 1, Case 1. □

The trigonometric polynomials of degree n form a uni-modal Haar space of dimension $2n + 1$ on $[-\pi, \pi)$. We can now, of course, formulate a corollary similar to Corollary 1 for trigonometric polynomials.

REFERENCES

- [1] H. S. M. Coxeter, *A problem of collinear points*, Amer. Math. Monthly, **55** (1948), 26–28.
- [2] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.
- [3] Th. Motzkin, *The lines and planes connecting the points of a finite set*, Trans. Amer. Math. Soc., **70** (1951), 451–464.

Received June 15, 1981.

DALHOUSIE UNIVERSITY
 HALIFAX, NOVA SCOTIA
 CANADA, B3H 4H8

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906–1982)

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the *Pacific Journal of Mathematics*. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$132.00 a year (6 Vol., 12 issues). Special rate: \$66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to *Pacific Journal of Mathematics*, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The *Pacific Journal of Mathematics* ISSN 0030-8730 is published monthly by the *Pacific Journal of Mathematics* at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to *Pacific Journal of Mathematics*, P. O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1983 by Pacific Journal of Mathematics

Tibor Bisztriczky, On the singularities of almost-simple plane curves	257
Peter B. Borwein, On Sylvester's problem and Haar spaces	275
Emilio Bujalance, Cyclic groups of automorphisms of compact nonorientable Klein surfaces without boundary	279
Robert Jay Daverman and John J. Walsh, Acyclic decompositions of manifolds	291
Lester Eli Dubins, Bernstein-like polynomial approximation in higher dimensions	305
Allan L. Edelson and Jerry Dee Schuur, Nonoscillatory solutions of $(rx^n)^n \pm f(t, x)x = 0$	313
Akira Endô, On units of pure quartic number fields	327
Hector O. Fattorini, A note on fractional derivatives of semigroups and cosine functions	335
Ronald Fintushel and Peter Sie Pao, Circle actions on homotopy spheres with codimension 4 fixed point set	349
Stephen Michael Gagola, Jr., Characters vanishing on all but two conjugacy classes	363
Saverio Giulini, Singular characters and their L^p norms on classical Lie groups	387
Willy Govaerts, Locally convex spaces of non-Archimedean valued continuous functions	399
Wu-Chung Hsiang and Bjørn Jahren, A remark on the isotopy classes of diffeomorphisms of lens spaces	411
Hae Soo Oh, Compact connected Lie groups acting on simply connected 4-manifolds	425
Frank Okoh and Frank A. Zorzitto, Subsystems of the polynomial system	437
Knut Øyma, An interpolation theorem for H_E^∞	457
Nikolaos S. Papageorgiou, Nonsmooth analysis on partially ordered vector spaces. II. Nonconvex case, Clarke's theory	463