COMPACT CONNECTED LIE GROUPS ACTING ON SIMPLY CONNECTED 4-MANIFOLDS

Hae Soo Oh
COMPACT CONNECTED LIE GROUPS
ACTING ON SIMPLY CONNECTED 4-MANIFOLDS

Hae Soo Oh

Suppose a compact connected Lie group $G$ acts effectively on a simply connected 4-manifold $M$. Then we show that $G$ is one of the groups $\text{SO}(5)$, $\text{SU}(3)/\text{Z}(G)$, $\text{SO}(3) \times \text{SO}(3)$, $\text{SO}(4)$, $\text{SO}(3) \times T^1$, $(\text{SU}(2) \times T^1)/D$, $\text{SU}(2)$, $\text{SO}(3)$, $T^2$, $T^1$, and that the representatives of the conjugacy classes of the principal isotropy groups for these groups on $M$ are, respectively, $\text{SO}(4)$, $U(2)$, $T^2$, $\text{SO}(3)$, $S^1$, $S^1$, $\tilde{\text{SO}}(2)$ or $e$, $\text{SO}(2)$ or $D_{2n}$, $e$, and $e$. We also show that in each of these cases $M$ is a connected sum of copies of $S^4$, $S^2 \times S^2$, $\mathbb{CP}^2$, and $-\mathbb{CP}^2$ (except when $G$ is $T^1$, see Theorem 2.6).

1. Introduction. All manifolds in this paper are assumed to be closed, connected and orientable. Also all actions are assumed to be effective and locally smooth. Orlik-Raymond [O-R] showed that if a simply connected 4-manifold admits an action of the two-dimensional torus group $T^2$, then $M$ is a connected sum of copies of $S^4$, $S^2 \times S^2$, $\mathbb{CP}^2$, and $-\mathbb{CP}^2$. Fintushel [F] proved that if $M$ admits a circle action and the orbit space $M^*$ is not a counterexample of Poincaré’s conjecture, then $M$ is also a connected sum of copies of these manifolds.

In this paper we determine all Lie groups which can act on a simply connected 4-manifold $M$, and dually we classify all simply connected 4-manifolds which admit an action of a given compact connected Lie group $G$.

An isotropy group $H$ is a principal if $H$ is conjugate to a subgroup of each isotropy group (that is, $G/H$ is a maximum orbit type for $G$ on $M$). One denotes by $G(x)$ the orbit of $G$ through $x$, and by $G_x$ the isotropy group at $x$. A maximal torus $T$ is a compact connected abelian Lie subgroup which is not properly contained in any larger such subgroup. We denote the normalizer of $G$ by $N(G)$, and the centralizer of $G$ by $Z(G)$. Let $\chi(M)$ denote the euler characteristic of a space $M$. Then it is well known that $\chi(G/T)$ is the order of $N(T)/T$.

2. The rank of a Lie group $G$ which can act on a simply connected 4-manifold $M$. Suppose $K$ is a subgroup of $G$ which acts on a topological space $X$. Then the action of $G$ on $X$ may not be effective even if the action restricted to $K$ is effective. But the maximal torus theorem gives rise to the following.
LEMMA 2.1. A compact connected Lie group $G$ acts effectively on a topological space $X$ if and only if the action restricted to a maximal torus $T$ of $G$ is effective.

Proof. Suppose $G$ does not act effectively. Then there exists at least one element $g \neq e$ in $G$ such that $gx = x$, for all $x \in X$. It follows from the maximal torus theorem that there exists an element $h \in G$ such that $g \in hTh^{-1}$. Hence $h^{-1}gh \in T$. Thus we have $(h^{-1}gh)x = h^{-1}g(hx) = h^{-1}hx = x$, for all $x \in X$, which says that the action restricted to $T$ is not effective. $\square$

By the rank of a Lie group $G$, we mean the dimension of a maximal torus of $G$.

LEMMA 2.2. If a compact connected Lie group $G$ acts on a simply connected 4-manifold $M$, then the rank of $G$ is less than 3.

Proof. Suppose the rank of $G$ is $\geq 3$. Then $M$ admits an effective $T^3$-action. By [P], $M$ is homeomorphic to either $T^4$ or $L(p, q) \times T^1$, which contradicts the simple connectivity of $M$. $\square$

It is known that every compact connected Lie group of dimension $\leq 6$ can be represented as a factor group $G/F$, where $G = G_1 \times G_2 \times \cdots \times G_n$ is a product; each factor $G_i$ is either $SO(2)$ or $SU(2)$ ($= S^3$), and $F$ is a finite subgroup of the center of $G$.

From now on $G$ is a compact connected Lie group acting on a simply connected 4-manifold $M$, and $H$ is a principal isotropy group for $G$ on $M$. (Note: any two principal isotropy groups are conjugate to each other. Actually $H$ denotes a representative group of the conjugacy class of principal isotropy groups.)

LEMMA 2.3. Suppose the rank of $G$ is 2 and the rank of $H$ is 0. Then $G$ is the two-dimensional torus group $T^2$.

Proof. From [B, p. 195], we have the following inequality:

\[ (*) \quad \dim M - \dim G/H - (\text{rank } G - \text{rank } H) \leq \dim M - 2 \text{ rank } G. \]

Since we assumed rank $H = 0$, then $\dim G/H \leq 4$. Hence the inequality gives rise to $4 \geq \dim G/H \geq \text{rank } G = 2$. Since $\dim G - \text{rank } G$ should be an even integer, $\dim G (= \dim G/H)$ must be either 4 or 2.
If \( \dim G \) is 4, then \( G \) acts transitively on \( M \) (that is, \( M = G/H \)). Since a compact connected Lie group of dimension 4 and of rank 2 is either \( \text{SU}(2) \times \text{SO}(2) \) or a factor group of this by a finite subgroup, \( G/H \) cannot be simply connected. We thus have \( \dim G = 2 = \text{rank } G \). Hence \( G \) is \( T^2 \).

**Lemma 2.4.** If a compact connected Lie group \( G \) acts on a simply connected 4-manifold \( M \), then we have the following:

(i) if \( \text{rank } G = 2 \) and \( \text{rank } H = 2 \), then \( \dim G \) is 10, 8, or 6;

(ii) if \( \text{rank } G = 2 \) and \( \text{rank } H = 1 \), then \( \dim G/H = 3 \) and \( \dim G \) is either 6 or 4;

(iii) if \( \text{rank } G = 2 \) and \( \text{rank } H = 0 \), then \( G = T^2 \);

(iv) the orbit space \( M^* \) is a simply connected manifold with boundary.

**Proof.** From [B, p. 195], we have an inequality,

\[
(*) \quad 4 \geq \dim G/H \geq \text{rank } G + \text{rank } H.
\]

It is known [M-Z] that if the maximal dimension of any orbit is \( k \), then \( \dim G \leq k(k+1)/2 \). Thus \( \dim G \leq 10 \). Since \( \dim G - \text{rank } G \) is an even integer, \( \dim G \) is 10, 8, 6, 4, or 2, provided \( \text{rank } G \) is 2.

(i) If \( \text{rank } G = 2 \) and \( \text{rank } H = 2 \), then it follows from inequality \((*)\) that \( \dim G/H = 4 \). Hence \( \dim G \geq 6 \).

(ii) If \( \text{rank } G = 2 \) and \( \text{rank } H = 1 \), then by \((*)\), \( \dim G/H \) is either 3 or 4. Suppose \( \dim G/H = 4 \). Then \( \dim H = \dim G - \dim G/H \) is 6, 4, or 2. On the other hand, \( \text{rank } H = 1 \) implies that the identity component of \( H \) is \( \text{SO}(2), \text{SO}(3), \) or \( \text{SU}(2) \). Hence \( \dim G/H \) should be 3. By [M-Z], \( \dim G \leq \frac{1}{2}(\dim G/H)(\dim G/H + 1) = 6 \).

(iii) was shown in Lemma 2.3.

(iv) If \( \text{rank } G = 2 \) and \( \text{rank } H \geq 1 \), then \((*)\) implies that \( \dim G/H \) is either 4 or 3. If \( \text{rank } G = 2 \), and \( \text{rank } H = 0 \), then by (iii), we have \( G = T^2 \).

Thus if \( \text{rank } G = 2 \), the orbit space \( M^* \) is \( D^0, D^1, \) or \( D^2 \) (cf. Lemma 5.1 [O-R]). If \( \text{rank } G = 1 \), then \( G \) is \( \text{SO}(2), \text{SO}(3), \) or \( \text{SU}(2) \). Since any proper subgroups of \( \text{SO}(3) \) and \( \text{SU}(2) \) are of dimension \( \leq 1 \), if \( G \) is either \( \text{SO}(3) \) or \( \text{SU}(2) \), then \( \dim G/H \) should be \( \geq 2 \). Hence \( M^* \) is \( D^1, D^2, \) or \( S^2 \).

If \( G = \text{SO}(2) \), then by Lemma 3.1 [F], \( M^* \) is a simply connected 3-manifold with boundary.

If an abelian group \( G \) acts effectively on a manifold \( M \), then the principal isotropy group \( H \) is trivial. We have shown that if \( \text{rank } G = 2 \)
and rank $H = 0$, then $G$ is $T^2$, hence $H$ is trivial. In this case, the manifolds are determined by the following theorem.

**Theorem 2.5. [O-R]** If $M$ is a simply connected 4-manifold supporting an effective $T^2$-action, then $M$ has $k \geq 2$-fixed points, and

$$M \approx \begin{cases} S^4, & \text{if } k = 2; \\ CP^2 \text{ or } -CP^2, & \text{if } k = 3; \\ S^2 \times S^2, CP^2#CP^2, CP^2#-CP^2, & \text{or } -CP^2#-CP^2, \text{ if } k = 4; \\ \text{a connected sum of copies of these spaces,} & \text{if } k > 4. \end{cases}$$

**Theorem 2.6. [F$_2$]** Let $SO(2)$ act locally smoothly and effectively on the simply connected 4-manifold $M$, and suppose the orbit space $M^*$ is not a counterexample to the 3-dimensional Poincaré conjecture. Then $M$ is a connected sum of copies of $S^4$, $CP^2$, $-CP^2$, and $S^2 \times S^2$.

Suppose a compact Lie group $G$ acts on a compact connected manifold $M$ so that the orbit space $M/G$ is a closed interval $[0,1]$, and let $G(x)$ and $G(y)$ be the orbits corresponding to 0 and 1 respectively. Then $G(x)$ and $G(y)$ are singular orbits and all other orbits are principal orbits of type $G/H$. Moreover, we may assume $H \subset G_x$ and $H \subset G_y$. The following lemma was proved by Mostert [Mo].

**Lemma 2.7. [Mo]** If a Lie group $G$ acts locally smoothly and effectively on a manifold $M$ so that $M/G$ is a closed interval, then $G_x/H$ and $G_y/H$ are spheres.

3. **The case of rank $G = 2$.**

3A. Suppose rank $G = 2$ and rank $H = 2$. Then by Lemma 2.4, dim $G$ is 10, 8, or 6. Inequality (**) implies dim $G/H = 4$ and hence $M$ is a homogeneous space.

(i) It follows from [E, p. 239] that if dim $G = 10$, then $M$ is $S^4$ or $RP^4$. Since $M$ is simply connected, $M$ is $S^4$. Hence [Wo, p. 282] gives rise to $G = SO(5)$ and $H = SO(4)$.

(ii) It is known [Wa] that if $n(n - 1)/2 + 1 < \text{dim } G < n(n + 1)/2$, $n = \text{dim } M$, then $n = 4$. Mann [Ma] proved that the effective action of $SU(3)/Z(SU(3))$ of dimension 8 on the complex projective plane $CP^2 = SU(3)/U(2)$ is the only exceptional possibility for $n = 4$.

(iii) If dim $G = 6$, then dim $H$ should be 2. Since $G$ is assumed to be connected and $G/H = M$ is assumed to be simply connected, the homotopy exact sequence of the fibre bundle implies that $H$ is also connected,
hence $H$ is $T^2$. The Lie group $G$ of dimension 6 and of rank 2 is either $SU(2) \times SU(2)$, or a factor group of this by a finite subgroup. Since $Z(SU(2) \times SU(2)) = \{(1, 1), (-1, 1), (1, -1), (-1, -1)\}$ is contained in a maximal torus (and hence in $H$), $SU(2) \times SU(2)$ is not admissible. For similar reasons, $SO(3) \times SU(2)$, $SU(2) \times SO(3)$, and $SO(4)$ are not admissible. Hence $(SU(2) \times SU(2))/\text{the center} = SO(3) \times SO(3)$ is the only admissible group. Hence $M$ is $S^2 \times S^2$.

We recall some properties of $SO(3)$ (see [R]).

1. Every subgroup of $SO(3)$ is conjugate to one of the following: $SO(2)$, $N(SO(2))$, the cyclic group $Z_k$ of order $k$, the dihedral group $D_n$ of order $2n$, the groups $T$, $O$, $I$ of all rotational symmetries of the tetrahedron, octahedron, and icosahedron, respectively.

2. If $V$ is a finite subgroup of $SO(3)$, then $SO(3)/V$ is an orientable 3-manifold with $H_2(SO(3)/V) = 0$. Using the double covering $\pi: SU(2) \rightarrow SO(3)$ we can calculate the first homology group of $SO(3)/V$:

\[
\begin{align*}
H_1(SO(3)/Z_k) &= Z_{2k}, \\
H_1(SO(3)/D_{2n}) &= Z_2 + Z_2, \\
H_1(SO(3)/D_{2n+1}) &= Z_4, \\
H_1(SO(3)/T) &= Z_3, \\
H_1(SO(3)/O) &= Z_2, \\
H_1(SO(3)/I) &= 0.
\end{align*}
\]

In the following $\tilde{K}$ denotes the preimage of $K \subset SO(3)$ under the covering map.

3B. Suppose rank $G = 2$ and rank $H = 1$. Then by Lemma 2.4, $\dim G$ is either 6 or 4 and $\dim G/H$ is 3.

1. If $\dim G = 4$, then $G$ is $SU(2) \times T^1$ or a factor group of this by a finite subgroup. Since $\dim G/H$ is 3, $\dim H$ is 1. Since any 1-dimensional subgroup of $SU(2) \times T^1$ contains a non-trivial element of $Z(SU(2) \times T^1) = \{(1, 1), (-1, -1)\} \times T^1$, it is not admissible. The remaining possibilities are $SO(3) \times T^1$ and $(SU(2) \times T^1)/D$, where $D = \{(1, 1), (-1, -1)\}$.

1a. Suppose $G$ is $(SU(2) \times T^1)/D$. Then the identity component $H_0$ of $H$ cannot be included in $(SU(2) \times 1)/D$ since $(SO(2) \times 1)/D$ contains $(-1, 1)/D$ ($\in Z(G)$). Nor can $H_0$ be $(1 \times T^1)/D$ since $(1 \times T^1)/D$ is a subgroup of $Z(G)$. Hence by using an argument similar to that of 8.1 of [R], we can show that $H$ is included in a maximal torus of $G$.

Since $\dim G/H$ is 3, the orbit space $M^*$ is a closed interval $[0, 1]$. That is, the orbit space $M^*$ is as shown below.

\[
\begin{array}{c}
\vdots \\
H \\
\vdots \\
G_x \\
\rightarrow \\
G_y
\end{array}
\]

By Lemma 2.7, $G_x/H$ and $G_y/H$ are spheres. But $((\widetilde{SO(2)} \times T^1)/D)/H$ is not a sphere. Hence $G_x$ (and also $G_y$) must be $(\widetilde{SO(2)} \times T^1)/D$ or $G$.  

(i) If $G_x$ and $G_v$ are maximal tori, then the number of fixed points of the action restricted to $G_x$ is either 2 or 4 since the order of $N(G_x)/G_x$ is $\chi(G/G_x) = 2$. Now it follows from Theorem 2.5 that $M$ is $S^4$ or an $S^2$-bundle over $S^2$ according as the number of fixed points is 2 or 4. Let $A = p^{-1}([0, \frac{1}{2}])$ and $B = p^{-1}([0, \frac{1}{2}])$, where $p: M \to M^* = [0, 1]$ is the orbit map. From the Mayer-Vietoris sequence for $(M, A, B)$, we have

$$0 \to H_3(M) \to H_2(G/H) \to Z \oplus Z \to H_2(M) \to H_1(G/H) \to 0.$$ 

Now we have

$$(G/H)/\left\{\left[(\widetilde{SO}(2) \times T^1)/D\right]/H\right\}$$

$$\approx \left\{\left[(SU(2) \times T^1)/D\right]/\left[(\widetilde{SO}(2) \times T^1)/D\right]\right\}$$

$$\approx (SU(2) \times T^1)/(\widetilde{SO}(2) \times T^1) \approx S^2$$

(see [B, p. 87]). Since $[(\widetilde{SO}(2) \times T^1)/D]/H$ is a topological group, the fundamental group of this is abelian. From a homotopy exact sequence of the fibre bundle $[(\widetilde{SO}(2) \times T^1)/D]/H \to G/H \to S^2$, we can see that $\pi_1(G/H)$ is abelian, hence $H_1(G/H) = \pi_1(G/H)$.

From the homotopy sequence of the fibre bundle $H \to G \to G/H$, we have

$$0 \to \pi_2(G/H) \to Z \to Z \to \pi_1(G/H) \to \pi_0(H) \to 0.$$ 

If $M$ is $S^4$, then from the homology sequence we have $\pi_2(G/H) = Z \oplus Z$ which contradicts the homotopy sequence. Hence the number of fixed points must be 4. Therefore we have $G_x = G_v$ which implies $\pi_1(G/H) = 1$, and hence $H$ is connected. Thus $H$ is $S^1$ and $M$ is either $S^2 \times S^2$ or $CP^2 \# -CP^2$.

(ii) If $G_x$ and $G_v$ are $G$ (i.e. $x$ and $y$ are fixed points), then the homotopy exact sequence of a fibre bundle $H \to G \to G/H = G_v/H \approx S^3$ yields $H \approx S^1$. Furthermore, the number of fixed points of the action restricted to $(\widetilde{SO}(2) \times T^1)/D$ is two. Hence, by Theorem 2.5, we have $M = S^4$ (alternatively,

$$M \approx p^{-1}([0, \frac{1}{2}]) \cup p^{-1}([\frac{1}{2}, 1]) \approx D^4 \cup D^4 \approx S^4.$$ 

(iii) If $G_x$ is $G$ and $G_v$ is a maximal torus, then by an argument similar to that used in (ii), $H$ is connected and hence $H$ is $S^1$. The number of fixed points of the action restricted to $G_v$ is 3 and hence it follows from Theorem 2.5 that $M$ is $CP^2$.

(Ib) Suppose $G$ is $SO(3) \times T^1$. Then by 8.1 of [R], $H$ is contained in a maximal torus or conjugate to either $SO(2) \times 1$ or $N(SO(2)) \times 1$. But
(SO(3) × 1)/(N(SO(2)) × 1) = RP^2 × S^1 is not orientable and hence by [B, p. 188], H cannot be N(SO(2)) × 1. (1) If H is contained in a maximal torus, then neither G_x nor G_y can be G since (SO(3) × T)/H is not a sphere. Hence by an argument similar to that of (Ia), H is S^1 and M is S^2 × S^2 or CP^2#−CP^2. (2) If H is SO(2) × 1, then by Lemma 2.7, there are three possibilities:

(i) G_x ≈ SO(2) × T ≈ G_y, which implies M = S^2 × S^2.
(ii) G_x ≈ SO(3) and G_y ≈ SO(2) × T^1, which implies M = [(S^2 × D^2) ∪ (D^3 × S^1)] = S^4.
(iii) G_x ≈ SO(3) ≈ G_y, which implies M = S^3 × S^1, not admissible.

(II) If dim G = 6, then dim H should be 3. Since the rank of G is 2, G is SU(2) × SU(2), SO(3) × SU(2), SU(2) × SO(3), or (SU(2) × SU(2))/D, where D = {(1, 1), (−1, −1)}.

Assertion. Suppose H_0 is the identity component of a 3-dimensional subgroup H of SU(2) × SU(2) and let p_i be the projection onto the i-th factor, for i = 1, 2. Then p_i|H_0, the restriction of p_i to H_0, is either a trivial map or an isomorphism.

To prove this Assertion, first of all we have to show that p_i|H_0 is either trivial or surjective. Suppose p_i|H_0 is neither surjective nor trivial. Then p_i(H_0) should be either SO(2) or N(SO(2)), and hence the kernel of p_i|H_0 is a two-dimensional normal subgroup of H_0. This is impossible. Hence p_1|H_0 or p_2|H_0 must be surjective. Suppose p_1|H_0 is surjective and let K be the kernel of p_1|H_0. Then H_0/K ≈ SU(2). Since SO(3) is simple, H_0 cannot be SO(3). If H_0 is SU(2), then K = τ(H_0/K) = τ(SU(2)) = 1. Thus p_1|H_0 is an isomorphism and H_0 ≈ SU(2).

(Iia) If either p_1|H_0 or p_2|H_0 is trivial, then H ≈ SU(2) × V, for a finite subgroup V, which contains a normal subgroup of SU(2) × SU(2). This is a conjecture subgroup. In fact, suppose K is a subgroup of SU(2) × SU(2) such that p_1(H) = SU(2) = p_2(H).

If x* and y* are the endpoints of a closed interval M/SO(4), then x and y should be fixed points so that G_x (and also G_y) could contain H as a conjecture subgroup. In fact, suppose K is a subgroup of SU(2) × SU(2) such that H ⊆ K and dim K ≥ 4. Then dim K is either 4 or 6 since rank G is 2. If dim K is 4, then the kernel of P_1 is an 1-dimensional subgroup of K. So K contains 1 × SO(2). For any g ∈ SU(2), there exists h ∈ SU(2)
such that \((h, g) \in K\). Moreover, \((h, g)^{-1}(1 \times SO(2))(h, g) = 1 \times g^{-1}SO(2)g \in K\). By the maximal torus theorem, we have \(1 \times SU(2) \subset K\). Similarly, \(SU(2) \times 1 \subset K\). Hence \(K = SU(2) \times SU(2)\). Since \(G/(H/D)\) must be \(S^3\) (by Theorem 2.7), by a homotopy exact sequence of \(H/D \to G \to S^3\), \(H/D\) is connected. Since \(\mathcal{H}/D \approx SU(2)/D \approx SO(3)\), \(H/D\) is \(SO(3)\) and hence \(M\) is \(S^4\).

(Iic) If \(G\) is \((SO(3) \times SO(3))\), then by an argument similar to the Assertion, we can show that \(x\) and \(y\) should be fixed points so that \(G_x\) (and \(G_y\)) can contain a non-normal 3-dimensional subgroup \(H\) as a conjugate subgroup. But \((SO(3) \times SO(3))/H\) cannot be a sphere. Hence \((SO(3) \times SO(3))\) is not admissible.

(IId) If \(G\) is \((SU(2) \times SO(3))\), then by an argument similar to that used in the proof of the Assertion, \(P_1|H_0\) is either a trivial map or an isomorphism. If \(P_1|H_0\) is trivial, then \(H\) is \(V \times SO(3)\) for a finite subgroup \(V\) of \(SU(2)\), which contains a normal subgroup \(1 \times SO(3)\). If \(P_1|H_0\) is an isomorphism, then \(H\) contains \((-1, 1), (1, 1)\) \((\subset Z(G))\). Hence \((SU(2) \times SO(3))\) is not admissible.

As a summary we have the table:

<table>
<thead>
<tr>
<th>(\dim G)</th>
<th>rank (H)</th>
<th>(G)</th>
<th>(H)</th>
<th>(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
<td>SO(5)</td>
<td>SO(4)</td>
<td>(S^4)</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>(SU(3)/Z(G))</td>
<td>(U(2)/Z(G))</td>
<td>(CP^2)</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>(SO(3) \times SO(3))</td>
<td>(T^2)</td>
<td>(S^2 \times S^2)</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>SO(4)</td>
<td>SO(3)</td>
<td>(S^4)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>((SO(3) \times T^1))</td>
<td>(S^1)</td>
<td>(S^4 \times S^2, CP^2 # -CP^2)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>((SU(2) \times T^1)/D)</td>
<td>(S^1)</td>
<td>(S^4, CP^2, S^2 \times S^2, CP^2 # -CP^2)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(T^2)</td>
<td>(e)</td>
<td>Theorem 2.5</td>
</tr>
</tbody>
</table>

Here \(S^1\) is a circle subgroup and \(D\) is the two element central subgroup \((-1, -1)\).

4. **The case of rank \(G = 1\)**. If a compact connected Lie group \(G\) is of rank 1, then \(G\) is \(T^1, SO(3)\), or \(SU(2)\), and the rank of \(H\) must be either 1 or 0.

4A. Suppose rank \(H = 1\). Then \(G\) is either \(SO(3)\) or \(SU(2)\).

(i) If \(G = SO(3)\), then \(H\) is either \(SO(2)\) or \(N(SO(2))\). Since \(SO(3)/N(SO(2)) = RP^2\) is not orientable, \(H\) should be \(SO(2)\). Since \(SO(3)/SO(2)\) is \(S^2\), the orbit space \(M^*\) is either \(S^2\) or \(D^2\). If \(M^*\) is \(S^2\), then \(M\) is an \(S^2\)-bundle over \(S^2\). If \(M^*\) is \(D^2\), then \(\partial D^2\) corresponds to the fixed points and \(\text{int } D^2\) corresponds to the principal orbits. Hence \(M\) is \(S^4\).
(ii) If $G = SU(2)$, then by an argument similar to that used in (i), $H$ is $SO(2)$, and $M$ is either $S^4$ or an $S^2$-bundle over $S^2$.

4B. Suppose rank $H = 0$. Then $G$ is $T^1$, $SO(3)$, or $SU(2)$.

(i) If $G$ is $T^1$, then $H$ must be trivial and $M$ was described in Theorem 2.6.

(ii) If $G = SO(3)$ and $x^*$ and $y^*$ are the endpoints of $M^*$, then $G_x$ and $G_y$ are conjugate to $SO(2)$, $N(SO(2))$, or $SO(3)$. By Lemma 2.7, none of $x$ and $y$ are fixed points and $G_x$ should be conjugate to $G_y$.

(iia) If $G_x$ and $G_y$ are conjugate to $N(SO(2))$, then $H$ is a dihedral group $Z_2$ (since $G_x/H$ and $G_y/H$ must be spheres). Richardson [R] showed that $S^4$ admits an action of $SO(3)$ such that $S^4/\text{SO}(3) = [x^*, y^*]$, a closed interval, $H = D_{2n}$, $(SO(3))(x) = RP^2 = (SO(3))(y)$. Since the orbit maps $M \to M/G$ and $S^4 \to S^4/\text{SO}(3)$ have cross-sections ([Mo], [R]), $M$ is equivariantly homeomorphic to $S^4$.

(iib) If $G_x$ and $G_y$ are conjugate to $SO(2)$, then $H$ should be a cyclic group $Z_k$ and $M$ is the space $[0, 1] \times SO(3)/Z_k$ with $0 \times SO(3)/Z_k$ collapsed to $SO(3)/G_x$ ($\approx S^2$) and $1 \times SO(3)/Z_k$ collapsed to $SO(3)/G_y$ ($\approx S^2$). Let $p$ be the orbit map. Let $A = p^{-1}([0, \frac{1}{2}])$ and $B = p^{-1}([\frac{1}{2}, 1])$.

From the Mayer-Vietoris sequence for $(M, A, B)$, we have $H_3(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and hence $\chi(M) = 4$. Now we consider the action restricted to $G_x$ ($\approx T^1$). The set of fixed points under the restricted action is contained in $G(x) \cup G(y)$. Since $N(G_x)/G_x$ is $Z_2$, there are only two fixed points for $G_x$ on $G(x)$, and hence there are at most four fixed points under the restricted action. Let $F(G_x, M)$ denote the set of fixed points. Then it is well known that $\chi(F(G_x, M)) = \chi(M) = 4$. Therefore there are four fixed points for $G_x$ on $M$, which implies $G_x = G_y$. Since $H^3(M; Z) = H_3(M; Z) = 0$, $H_2(M; Z)$ is torsion free and hence $H_2(M; Z)$ is $Z \oplus Z$. The Mayer-Vietoris sequence gives rise to

$$0 \to Z \oplus Z \to Z \oplus Z \to H_1(SO(3)/Z_k; Z) \to 0.$$ 

Here $i_*$ and $j_*$ are induced by inclusions $i: A \to M$ and $j: B \to M$ respectively. Since $G_x = G_y$, $i$ and $j$ are virtually the same maps. Hence $(Z \oplus Z)/\text{im}(i_* \oplus j_*) = Z_n \oplus Z_n$ for an integer $n$, which contradicts $H_1(SO(3)/Z_k; Z) = Z_{2k}$.

(iii) If $G$ is $SU(2)$ and $\pi$ is the double covering map, then the only subgroups of $SU(2)$ which do not contain the kernel of $\pi$ are cyclic subgroups of odd order. Hence every subgroup $K$ of $SU(2)$ contains $Z(SU(2))$ unless $K$ is a cyclic group of odd order. Since the action was assumed to be effective, $H$ is either $Z_{2k+1}$ or $e$. By an argument similar to that used in (ii) of 4B, we can show that $H$ cannot be $Z_{2k+1}$. If $H$ is the
identity, then by Lemma 2.7, there are three possibilities:
(a) \(x\) and \(y\) are fixed points, which implies \(M \approx S^4\).
(b) \(G\) is conjugate to \(\widetilde{SO}(2)\) and \(y\) is a fixed point, which implies \(M \approx CP^2\#S^4\) (Recall: \(SU(2) \to SU(2)/\widetilde{SO}(2)\) is the Hopf bundle).
(c) \(G\) and \(G\) are conjugate to \(\widetilde{SO}(2)\), which implies \(M \approx CP^2\#CP^2\).

We summarize these in the following table:

<table>
<thead>
<tr>
<th>(\dim G)</th>
<th>(\text{rank } H)</th>
<th>(G)</th>
<th>(H)</th>
<th>(M^*)</th>
<th>(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>(SO(3))</td>
<td>(SO(2))</td>
<td>(D^2)</td>
<td>(S^2 \times S^2, CP^2#CP^2)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(SU(2))</td>
<td>(\widetilde{SO}(2))</td>
<td>(D^2)</td>
<td>(S^2 \times S^2, CP^2#CP^2)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>(SO(3))</td>
<td>(D_{2n})</td>
<td>(D^1)</td>
<td>(S^4)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>(SU(2))</td>
<td>(e)</td>
<td>(D^1)</td>
<td>(S^4, CP^2, CP^2#CP^2)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(T^1)</td>
<td>(e)</td>
<td></td>
<td>Theorem 2.6</td>
</tr>
</tbody>
</table>

5. Conclusion. Suppose a compact connect Lie group \(G\) acts on a simply connected 4-manifold \(M\). Then it was shown in §2 that the rank of \(G\) is either 1 or 2. Let \(H\) denote a representative subgroup of the conjugacy class of principal isotropy groups. Then \(G\), \(M\), and \(H\) are completely determined in §§3 and 4 in the cases of \(\text{rank } G = 2\) and \(\text{rank } G = 1\), respectively. Thus we have proved the following.

**Theorem 5.1.** If a Lie group \(G\), a subgroup \(H\), and a manifold \(M\) are those denoted above, then \(G\), \(H\), and \(M\) must be one of the cases represented in Table I (§3) and Table II (§4).

The author would like to thank professors P. E. Conner and Robert Perlis of the Louisiana State University for their encouragement and helpful discussions.

**References**


Received May 11, 1981.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $132.00 a year (6 Vol., 12 issues). Special rate: $66.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Copyright © 1983 by Pacific Journal of Mathematics
Tibor Bisztriczky, On the singularities of almost-simple plane curves ...... 257
Peter B. Borwein, On Sylvester’s problem and Haar spaces ................. 275
Emilio Bujalance, Cyclic groups of automorphisms of compact
nonorientable Klein surfaces without boundary ............................... 279
Robert Jay Daverman and John J. Walsh, Acyclic decompositions of
manifolds ................................................................................. 291
Lester Eli Dubins, Bernstein-like polynomial approximation in higher
dimensions ............................................................................... 305
Allan L. Edelson and Jerry Dee Schuur, Nonoscillatory solutions of
$(rx^n)^n \pm f(t, x)x = 0$ .................................................................. 313
Akira Endô, On units of pure quartic number fields ............................ 327
Hector O. Fattorini, A note on fractional derivatives of semigroups and
cosine functions ......................................................................... 335
Ronald Fintushel and Peter Sie Pao, Circle actions on homotopy spheres
with codimension 4 fixed point set ................................................. 349
Stephen Michael Gagola, Jr., Characters vanishing on all but two
conjugacy classes ......................................................................... 363
Saverio Giulini, Singular characters and their $L^p$ norms on classical Lie
groups .......................................................................................... 387
Willy Govaerts, Locally convex spaces of non-Archimedean valued
continuous functions .................................................................... 399
Wu-Chung Hsiang and Bjørn Jahren, A remark on the isotopy classes of
diffeomorphisms of lens spaces .................................................. 411
Hae Soo Oh, Compact connected Lie groups acting on simply connected
4-manifolds .................................................................................. 425
Frank Okoh and Frank A. Zorzitto, Subsystems of the polynomial
system .......................................................................................... 437
Knut Øyma, An interpolation theorem for $H^\infty_E$ ......................... 457
Nikolaos S. Papageorgiou, Nonsmooth analysis on partially ordered vector
spaces. II. Nonconvex case, Clarke’s theory ................................. 463