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EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES OF A WEAKLY PSEUDOCONVEX DOMAIN

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Let D be a weakly pseudoconvex domain in C^n with C^∞-boundary and Δ be a hypersurface in D which intersects ∂D transversally. If $\partial\Delta$ consists of strictly pseudoconvex boundary points of D, then any bounded holomorphic function in Δ can be extended to a bounded holomorphic function in D.

1. Introduction. G. M. Henkin [5] proved that any bounded holomorphic function defined on an analytic closed submanifold in general position in a strongly pseudoconvex domain can be continued to a bounded holomorphic function in the entire domain. The related results have been given by the author [1] and J. E. Fornaess [4]. In this paper, we extend this problem to the weakly pseudoconvex case. Our proof depends on the integral formula constructed by E. L. Stout [8], and the kernel function constructed by F. Beatrous, Jr. [3] which was used to obtain a Hölder estimate for solutions to $\bar\partial$-problem in weakly pseudoconvex domains.

2. Let Ω be a bounded domain in C^{N+1} with C^∞-boundary. We shall denote by $O(\Omega)$ the space of holomorphic functions in Ω. We shall also denote by $H^\infty(\Omega)$ the space of bounded holomorphic functions on Ω and by $A(\Omega)$ the subspace of $H^\infty(\Omega)$ of functions which extend continuously to $\bar\Omega$.

DEFINITION 1. (R. M. Range [7]) A point $\lambda \in \partial\Omega$ is a strictly pseudoconvex boundary point if there are a neighborhood U of λ and a C^∞ function ϕ: $U \to R$ such that:

(a) $U \cap \Omega = \{z \in U: \phi(z) < 0\}$;

(b) $\Sigma(\partial^2\phi(\lambda)/\partial z_i \partial \bar z_j) w_i \bar w_j > 0$ for all $w \in C^{N+1} - (0)$;

(c) $d\phi(\lambda) \neq 0$.

The set of strictly pseudoconvex boundary points of Ω will be denoted by $S(\Omega)$. It follows from Definition 1 that $S(\Omega)$ is an open subset of the boundary $\partial\Omega$.

Let D be a pseudoconvex domain in C^{N+1} with C^∞-boundary. We fix a function $F \in O(\bar{D})$, $F \not\equiv 0$. Then F is holomorphic in a domain $\tilde{D}$ with $\bar{D} \subset \tilde{D}$. We set $\tilde{\Delta} = \{z \in \tilde{D}\colon F(z) = 0\}$ and $\Delta = \tilde{\Delta} \cap D$. We make the following assumptions:

(a) Δ is a non-empty connected set;

(b) $dF \neq 0$ on $\partial\Delta$;

(c) $\tilde{\Delta}$ meets ∂D transversally;

(d) $\partial\Delta \subset S(D)$.

In this setting, we have the following:

THEOREM. *Under hypotheses* (a)–(d), *there exists a continuous linear extension operator* $L\colon H^\infty(\Delta) \to H^\infty(D)$. *Moreover if* Δ *has no singular points then* $L(A(\Delta)) \subset A(D)$.

In order to prove this theorem, we use the function $\Phi(\zeta, z)$ in the following proposition which was constructed by F. Beatrous, Jr. ([**3**], Theorem 2.1).

PROPOSITION 1. *Let k be a positive integer* ($k \geq 3$). *There are a neighborhood U of $\partial\Delta$, a smooth positive function r on U, and a C^k function Φ on $U \times \bar{D}$ with the following properties*:

(i) *For each* $\zeta \in U$, $\Phi(\zeta, \cdot) \in C^k(\bar{D}) \cap O(D)$.

(ii) $G(\zeta, z) = \Phi(\zeta, z)/T(\zeta, z)$ *is a non-vanishing C^k function on* $\{(\zeta, z) \in U \times \bar{D}\colon |\zeta - z| \leq r(z)\}$.

(iii) $\Phi(\zeta, z) \neq 0$ *if* $|\zeta - z| \geq r(z)$.

(iv) $\operatorname{Re} T(\zeta, z) > \rho(\zeta) - \rho(z) + r(z)|\zeta - z|^2$ *if* $|\zeta - z| \leq r(\zeta)$,

where ρ is the defining function for the domain D constructed by F. Beatrous, Jr., and

$$T(\zeta, z) = -2\sum_i \frac{\partial\rho}{\partial z_i}(\zeta)(z_i - \zeta_i) - \sum_{i,j} \frac{\partial^2\rho}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j).$$

Moreover we can extend $\Phi(\zeta, z)$ to a C^k function on a neighborhood of $\partial D \times \bar{D}$, holomorphic in z such that $\Phi(\zeta, z)$ satisfies $\Phi(\zeta, z) = \sum_{j=1}^{N+1} P_j(\zeta, z)(\zeta_j - z_j)$, *and* $\Phi(\zeta, z) \neq 0$ *if* $\rho(\zeta) > \rho(z)$, *where $P_j(\zeta, z)$ is a C^k function on a neighborhood of $\partial D \times \bar{D}$, holomorphic in z.*

Let $D_\nu = \{z \in D\colon \rho(z) < -\varepsilon_\nu\}$ and $\Delta_\nu = \Delta \cap D_\nu$, where $\{\varepsilon_\nu\}$ is a sequence of sufficiently small strictly decreasing positive numbers converging to 0. By E. L. Stout [**8**], we have the following:

PROPOSITION 2. *If $f \in H^\infty(\Delta)$, then the following formula holds for all $z \in \Delta_\nu$ and all sufficiently large ν:*

$$(1)\qquad f(z) = \int_{\partial\Delta_\nu} f(\zeta) \frac{\tilde{\Psi}(\zeta, z)\tilde{\omega}_F}{\Phi(\zeta, z)^N \|\text{grad } F(\zeta)\|} = \int_{\partial\Delta_\nu} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N},$$

where $\tilde{\Psi}(\zeta, z)$ is a $C^{k-1}(0, N-1)$ form in a neighborhood of $\partial D \times \bar{D}$ and, for each ζ near ∂D, coefficients of $\tilde{\Psi}(\zeta, \cdot)$ are holomorphic in D. One could arrange for $\tilde{\Psi}(\zeta, \cdot)$ to be holomorphic on $\bar{D}$ if $\bar{D}$ were assumed to have a pseudoconvex neighborhood basis. $\tilde{\omega}_F$ is given by

$$\tilde{\omega}_F = \sum_{j=1}^{N+1} (-1)^{j-1} \bar{F}_j \, d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_{N+1},$$

where $F_j = \partial F/\partial z_j$, $j = 1, \ldots, N+1$, and $\wedge$ means the symbol is to be omitted. Therefore $K(\zeta, z)$ is a $C^{k-1}(N, N-1)$ form on a neighborhood of $\partial D \times \bar{D}$ and for each ζ near ∂D, coefficients of $K(\zeta, \cdot)$ are holomorphic in D.

We set

$$H_\nu(z) = \int_{\partial\Delta_\nu} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} \quad \textit{for } z \in \bar{D}_\nu \,|\, \partial\Delta_\nu,$$

and

$$L(f)(z) = H(z) = \lim_{\nu \to \infty} H_\nu(z) \quad \textit{for } z \in \bar{D} \,|\, \partial\Delta.$$

LEMMA 1. *$H(z)$ is holomorphic on D and $H(z) = f(z)$ for all $z \in \Delta$.*

Proof. For $z \in W \Subset D_{\nu_0}$, $\nu > \mu \geq \nu_0$, we have

$$H_\nu(z) - H_\mu(z) = \int_{\partial(\Delta_\nu - \Delta_\mu)} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} = \int_{\Delta_\nu - \Delta_\mu} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^N}\right).$$

Since the form $\bar{\partial}_\zeta(K(\zeta, z)/\Phi(\zeta, z)^N)$ is bounded for $\zeta \in \Delta_\nu - \Delta_\mu$ and $z \in W$, there exists a constant K such that

$$|H_\nu(z) - H_\mu(z)| \leq K \sup_{\zeta \in \Delta} |f(\zeta)| \text{Vol}(\Delta_\nu - \Delta_\mu).$$

Hence $H_\nu(z)$ converges locally uniformly on D. Therefore $H(z)$ is holomorphic in D. By Proposition 2, $H(z) = f(z)$ for all $z \in \Delta$. Therefore Lemma 1 is proved.

We want to show that $H(z) \in H^\infty(D)$. Let $S_{z^0,\sigma} = \{z: |z - z^0| < \sigma\}$. Let $z^0 \in \partial\Delta$. Then there exist a constant $\sigma_1 > 0$ and a biholomorphic change of coordinates on a neighborhood of z^0 such that ρ is strictly convex in a neighborhood of $\bar{D} \cap \bar{S}_{z^0,\sigma_1}$, $\Delta \cap S_{z^0,\sigma_1} = \{z \in S_{z^0,\sigma_1}: z_{N+1} = 0\}$ and $\partial\rho(z^0)/\partial z_1 \neq 0$. Let $0 < \sigma_2 < \sigma_1$. Let $z \in S_{z^0,\sigma_2} \cap D_\nu$. We write

$$H_\nu(z) = \int_{\partial\Delta_\nu \cap S_{z^0,\sigma_1}} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} + \int_{\partial\Delta_\nu | S_{z^0,\sigma_1}} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N}.$$

Then

$$\left| \int_{\partial\Delta_\nu | S_{z^0,\sigma_1}} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} \right| \leq \gamma_1 \sup_{\zeta \in \Delta_\nu} |f(\zeta)|,$$

where γ_1 depends only on D and Δ. We set

$$\tilde{H}_\nu(z) = \int_{\partial\Delta_\nu \cap S_{z^0,\sigma_1}} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N}.$$

Then it is sufficient to show that $|\tilde{H}_\nu(z)| \leq \gamma_2 \sup_{\zeta \in \Delta} |f(\zeta)|$, where γ_2 depends only on D and Δ.

We consider the system of equations for $\zeta^0 = (\zeta_1^0, \ldots, \zeta_{N+1}^0)$ of the following form for $z \in S_{z^0,\sigma_2}$:

$$\tag{2} \begin{aligned} &\sum_{i=1}^{N+1} \frac{\partial\rho}{\partial\zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0, \\ &\zeta_i^0 = z_i \qquad (i = 2, 3, \ldots, N), \\ &\zeta_{N+1}^0 = 0 \end{aligned}$$

Then we have the following lemma which was proved by G. M. Henkin [5]. But we give the proof of E. Amar [2] which is simpler than Henkin's.

LEMMA 2. *There exist positive constants σ_3 $(< \sigma_2)$, γ_3 and γ_4, depending only on D and Δ, such that for any $\sigma \leq \sigma_3$ and any $z \in S_{z^0,\sigma/2}$ there exists a unique solution $\zeta^0 = \zeta^0(z)$ of system* (2) *which belongs to the set $S_{z^0,\sigma} \cap \tilde{\Delta}$. Here the point $\zeta^0 = \zeta^0(z)$ has the following properties*:

$$\tag{3} |z - \zeta^0|^2 \leq \frac{1}{\gamma_3}[\rho(z) - \rho(\zeta^0)],$$

$$\tag{4} |z - \zeta^0|^2 \geq |z_{N+1}|^2 \geq \gamma_4[\rho(z) - \rho(\zeta^0)],$$

$$\zeta^0 = z \quad \text{for any } z \in S_{z^0,\sigma/2} \cap \tilde{\Delta}.$$

Proof. From (2), we have

$$z_1 = \zeta_1 - \frac{(\partial\rho(\zeta)/\partial z_{N+1})z_{N+1}}{\partial\rho(\zeta)/\partial z_1} = \zeta_1 - a(\zeta)z_{N+1},$$

where $a(\zeta)$ is C^∞ in a neighborhood of z^0. There exists $\sigma_3 > 0$ such that for any $\zeta \in B(z^0, \sigma_3)$, $z \in B(z^0, \sigma_3)$ we have $|\nabla a(\zeta)||z_{N+1}| \le \frac{1}{2}$. We set by recurrence that

$$\begin{aligned} \zeta_1^{(1)} &= z_1, \\ \zeta^{(j)} &= \left(\zeta_1^{(j)}, z_2, \dots, z_N, z_{N+1}^0\right), \\ \zeta_1^{(j)} &= z_1 + a\left(\zeta^{(j-1)}\right)z_{N+1}. \end{aligned}$$

If z and $\zeta^{(j)}$ are in $B(z^0, \sigma_3)$, then

$$\left|\zeta_l^{(j)} - \zeta_l^{(j-1)}\right| < |z_{N+l}|\,|\nabla a|\,\left|\zeta_l^{(j-1)} - \zeta_l^{(j-2)}\right| < \tfrac{1}{2}\left|\zeta_l^{(j-1)} - \zeta_l^{(j-2)}\right|.$$

Therefore $\zeta^{(j)}$ converges. Then $\lim_{\nu\to\infty}\zeta^{(j)} = \zeta^0$ is the soution of (2). The strict convexity of the function ρ and equations (2) imply the inequalities:

$$\text{(5)} \quad \rho(\zeta^0) - \rho(z) + \gamma_3\left|\zeta^0 - z\right|^2 \le 2\,\mathrm{Re}\sum_{i=1}^{N+1}\frac{\partial\rho}{\partial\zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0,$$

$$\text{(6)} \quad \rho(\zeta^0) - \rho(z) + \gamma_4'\left|\zeta^0 - z\right|^2 \ge 2\,\mathrm{Re}\sum_{i=1}^{N+1}\frac{\partial\rho}{\partial\zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0,$$

where $z \in S_{z^0,\sigma_3/2}$. From (5) we have (3). From (6) we have

$$\left|\zeta^0 - z\right|^2 \ge \frac{1}{\gamma_4'}\left[\rho(z) - \rho(\zeta^0)\right].$$

But

$$\left|\zeta^0 - z\right|^2 \ge \left|z_{N+1}\right|^2 + \left|\zeta_1^0 - z_1\right|^2 \le \gamma_4''\left|z_{N+1}\right|^2.$$

Therefore we have $|z_{N+1}|^2 \ge (1/\gamma_4)[\rho(z) - \rho(\zeta^0)]$. Therefore Lemma 2 is proved.

For any $z \in \overline{D}_\nu \cap S_{z^0,\sigma_2} | \partial\Delta_\nu$ and any vector $w = (w_1, \dots, w_{N+1}) \neq 0$, we have

$$\text{(7)} \quad \left.\frac{d\tilde{H}_\nu(z + \lambda w)}{d\lambda}\right|_{\lambda=0} = \int_{\partial\Delta_\nu\cap S_{z^0,\sigma_1}} \frac{f(\zeta)\sum_{j=1}^{N+1}(\partial/\partial z_j)K(\zeta, z)w_j}{\Phi(\zeta, z)^{N+1}} - \int_{\partial\Delta_\nu\cap S_{z^0\,\sigma_1}} \frac{N\sum_{j=1}^{N+1}(\partial\Phi(\zeta, z)/\partial z_j)w_jK(\zeta, z)}{\Phi(\zeta, z)^{N+1}}.$$

LEMMA 3. *Let* $f(\zeta) \in H^\infty(\Delta)$. *Then for any point* $z^0 \in \partial\Delta$ *and any point* $z \in \partial(S_{z^0,\sigma} \cap D_\nu) \,|\, \partial\Delta_\nu$ $(\sigma < (\sigma_3/2))$, *we have*

$$\left| \frac{d\tilde{H}_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \Bigg| \leq \gamma_5 \sup_{\zeta \in \Delta} |f(\zeta)|,$$

where $\zeta^0 = \zeta^0(z)$ *and* γ_5 *depends only on* D *and* Δ.

Proof. We set $\varepsilon = |z_{N+1}|$, where

$$z = (z_1, \ldots, z_{N+1}) \in \partial(S_{z^0,\sigma} \cap D_\nu) \,|\, \partial\Delta_\nu.$$

Then Lemma 2 implies the inequalities

$$\varepsilon \leq |\zeta^0 - z| \leq \left\{ \frac{\rho(z) - \rho(\zeta^0)}{\gamma_3} \right\}^{1/2} \leq \frac{\varepsilon}{(\gamma_3\gamma_4)1/2}.$$

Since $\Sigma_{i=1}^{N+1} (\partial\rho/\partial\zeta_i)(\zeta^0)(\zeta_i^0 - z_i) = 0$, it follows that

$$\left| \sum_{i=1}^{N+1} \frac{\partial\Phi}{\partial z_i}(\zeta, z)(\zeta_i^0 - z_i) \right| = \left| \sum_{i=1}^{N+1} \left(\frac{\partial\Phi(\zeta, z)}{\partial z_i} + 2\frac{\partial\rho}{\partial\zeta_i}(\zeta^0) \right)(\zeta_i^0 - z_i) \right|$$

$$\leq \left| \sum_{i=1}^{N+1} \left(\frac{\partial\Phi}{\partial z_i}(\zeta, z) - \frac{\partial\Phi}{\partial z_i}(\zeta^0, z) + O(|\zeta^0 - z|) \right)(\zeta_i^0 - z_i) \right|$$

$$\leq \gamma_6\varepsilon(|\zeta - z| + \varepsilon).$$

Here we have used the equation

$$\frac{\partial\Phi}{\partial z_i}(\zeta^0, z) = -2\frac{\partial\rho}{\partial\zeta_i}(\zeta^0) + O(|\zeta^0 - z|).$$

By (7), we have

$$\left| \frac{d\tilde{H}_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=1} \Bigg| = \left| \frac{d\tilde{H}_\nu(z + \lambda(z - \zeta^0))}{d\lambda} \right|_{\lambda=0} \Bigg|$$

$$\leq \gamma_7 \int_{\partial\Delta_\nu \cap S_{z^0,\sigma_1}} \frac{|f(\zeta)||z - \zeta^0|}{|\Phi(\zeta, z)|^{N+1}} d\lambda + \gamma_8 \int_{\partial\Delta_\nu \cap S_{z^0,\sigma_1}} \frac{|f(\zeta)|\varepsilon(|\zeta - z| + \varepsilon)}{|\Phi(\zeta, z)|^{N+1}} d\lambda.$$

We can choose coordinates $(\eta_1(\zeta), \ldots, \eta_{N+1}(\zeta))$ in S_{z^0,σ_3} such that $\eta_1(\zeta) = \rho(\zeta) - \rho(z) + i \operatorname{Im} \Phi(\zeta, z)$. Then

$$|\Phi(\zeta, z)| \geq \gamma_9 \left[\left(t_1 + |\zeta - z|^2 \right)^2 + t_2^2 \right]^{1/2}$$

and

$$|\zeta - z| \geq \gamma_{10}\left(t_1^2 + \cdots + t_{2N}^2 + \varepsilon^2\right)^{1/2} \geq \gamma_{11}|\zeta - z|,$$

where we have written $\eta_i(\zeta) = t_{2i-1} + \sqrt{-1}\, t_{2i}$ ($i = 1, 2, \ldots, N+1$). Then we have

$$\left|\frac{dF(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\Big|_{\lambda=1}\right| \leq \gamma_{12} \sup_{\zeta \in \Delta} |f(\zeta)|$$

$$\times \left\{ \varepsilon \int_{\substack{t_2^2 + \cdots + t_{2N}^2 \leq 1 \\ t_1 \geq 0}} \frac{dt_2 dt_3 \cdots dt_{2N}}{\left[\left(t_1 + t_2^2 + \cdots + t_{2N}^2 + \varepsilon^2\right)^2 + t_2^2\right]^{N/2}} \right.$$

$$+ \varepsilon \int_{\substack{t_2^2 + \cdots + t_{2N}^2 \leq 1 \\ t_1 \geq 0}} \frac{\left(t_1^2 + t_2^2 + \cdots + t_{2N}^2 + \varepsilon^2\right)^{1/2} dt_2 \cdots dt_{2N}}{\left[\left(t_1 + t_2^2 + \cdots t_{2N}^2 + \varepsilon^2\right)^2 + t_2^2\right]^{N+1/2}}$$

$$\left. + \varepsilon^2 \int_{\substack{t_2^2 + \cdots + t_{2N}^2 \leq 1 \\ t_1 \geq 0}} \frac{dt_2 \cdots dt_{2N}}{\left[\left(t_1 + t_2^2 + \cdots + t_{2N}^2 + \varepsilon^2\right)^2 + t_2^2\right]^{N+1/2}} \right\}$$

(by G. M. Henkin [5])

$$\leq \gamma_{13} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

We want to have

$$\sup_{z \in D_\nu} |H_\nu(z)| \leq \gamma_{14} \sup_{\zeta \in \Delta} |f(\zeta)|,$$

where γ_{14} depends only on D and Δ. We shall denote by $(\partial\Delta_\nu)_\sigma$ the σ-neighborhood of $\partial\Delta_\nu$. Since the function $H_\nu(z)$ is holomorphic at all points $z \in \overline{D}_\nu \,|\, \partial\Delta_\nu$, we have

$$\sup_{z \in D_\nu} |H_\nu(z)| \leq \sup_{z \in \partial D_\nu | (\partial\Delta_\nu)_\sigma} |H_\nu(z)| + \sup_{z \in [(\partial\Delta_\nu)_\sigma | \partial\Delta_\nu] \cap \partial D_\nu} |H_\nu(z)|.$$

We obtain

$$\sup_{z \in \partial D_\nu | (\partial\Delta_\nu)_\sigma} |H_\nu(z)|$$

$$\leq \gamma_{15} \left[\int_{t_2^2 + \cdots t_{2N}^2 \leq 1} \frac{dt_2 \cdots dt_{2N}}{\left[\left(t_2^2 + \cdots t_{2N}^2 + \sigma^2\right)^2 + t_2^2\right]^{N/2}} \right] \sup_{\zeta \in \Delta} |f(\zeta)|$$

$$\leq \gamma_{16} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Let $\sigma < 16\sigma_3$. We now fix $z \in [(\partial\Delta_\nu)_\sigma - \partial\Delta_\nu] \cap \partial D_\nu$. We take ν so large that one can find $z^0 \in \partial\Delta$ such that $z \in S_{z^0,2\sigma}$. Then by Lemma 2, there exists a solution $\zeta^0 = \zeta^0(z)$ of system (2) belonging to the set $S_{z^0,4\sigma} \cap \tilde{\Delta}$ and satisfying the inequalities

$$\gamma_3 |z_{N+1}|^2 \leq \rho(z) - \rho(\zeta^0) \leq |z_{N+1}|^2/\gamma_4. \tag{8}$$

Let $T_\nu = \{\lambda \in C\colon z(\lambda) = \zeta^0 + \lambda(z - \zeta^0) \in D_\nu \cap S_{z^0,4\sigma}\}$. T_ν is a convex domain containing $\lambda = 0$. For any λ we have

$$\sum_{i=1}^{N+1} \frac{\partial\rho}{\partial\zeta_i}(\zeta^0)(\zeta_i^0 - z_i(\lambda)) = 0.$$

From this we have

$$|z(\lambda) - \zeta^0|^2 \leq \frac{1}{\gamma_{17}}\{\rho(z(\lambda)) - \rho(\zeta^0)\}.$$

Hence for $\lambda \in \partial T_\nu$ with $z(\lambda) \in \partial D_\nu$, we obtain

$$\begin{aligned} |z(\lambda) - z^0| &\leq |z(\lambda) - \zeta^0| + |\zeta^0 - z^0| \\ &\leq \frac{1}{\sqrt{\gamma_{17}}}(\rho(z(\lambda)) - \rho(\zeta^0))^{1/2} + \frac{\sigma_3}{4} \\ &= \frac{1}{\sqrt{\gamma_{17}}}(\rho(z) - \rho(\zeta^0))^{1/2} + \frac{\sigma_3}{4} \\ &\leq \frac{\varepsilon}{\sqrt{\gamma_4\gamma_{17}}} + \frac{\sigma_3}{4} \leq \frac{\sigma}{\sqrt{\gamma_4\gamma_{17}}} + \frac{\sigma_3}{4}. \end{aligned}$$

We impose the further restriction that the constant $\sigma < \sigma_3\sqrt{\gamma_4\gamma_{17}}/4$. Then $|z(\lambda) - z^0| < \sigma_3/2$. Therefore $z(\lambda) \in S_{z^0,\sigma_3/2}$. Since the point $\zeta^0(z)$ satisfies system (2) with any $z(\lambda)$ satisfying $\lambda \in \partial T_\nu$ and $z(\lambda) \in \partial D_\nu$, it follows that $\zeta^0(z(\lambda)) = \zeta^0(z)$ for any $\lambda \in \partial T_\nu$ with $z(\lambda) \in \partial D_\nu$. Moreover

$$\begin{aligned} \frac{|\lambda|\varepsilon}{\gamma_3\gamma_4} &\geq \frac{|\lambda|}{\gamma_3}(\rho(z) - \rho(\zeta^0)) \geq |\lambda|\,|z - \zeta^0| = |z(\lambda) - \zeta^0| \\ &\geq (\gamma_4(\rho(z(\lambda)) - \rho(\zeta^0)))^{1/2} \\ &= [\gamma_4(\rho(z) - \rho(\zeta^0))]^{1/2} \geq (\gamma_3\gamma_4)^{1/2}\varepsilon. \end{aligned}$$

Therefore $|\lambda| \geq \gamma_3\gamma_4$ for any $\lambda \in \partial T_\nu$ with $z(\lambda) \in \partial D_\nu$. If $\lambda \in \partial T_\nu$ and $z(\lambda) \in S_{z^0,4\sigma}$, there exists $\gamma_{18} > 0$ such that $|\lambda| \geq \gamma_{18}$. Let $\gamma_{19} = \min(\gamma_3\gamma_4, \gamma_{18})$. Then

$$|\lambda| \geq \gamma_{19} \quad \text{for any } \lambda \in \partial T_\nu. \tag{9}$$

By Lemma 3, we have

$$(10)\qquad \left| \left. \frac{dH_\nu(\zeta^0 + t(z(\lambda) - \zeta^0))}{dt} \right|_{t=1} \right| \le \gamma_5 \sup_{\zeta \in \Delta} |f(\zeta)|$$

for any $\lambda \in \partial T_\nu$. We note that

$$\left. \frac{dH_\nu(\zeta^0 + t(z(\lambda) - \zeta^0))}{dt} \right|_{t=1} = \frac{dH_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}.$$

From (8), (9) and (10), we have

$$\left| \frac{dH_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right| \le \frac{\gamma_5}{|\lambda|} \sup_{\zeta \in \Delta} |f(\zeta)| \le \frac{\gamma_5}{\gamma_{19}} \sup_{\zeta \in \Delta} |f(\zeta)|$$

for any $\lambda \in \partial T_\nu$.Since the function $dH_\nu(\zeta^0 + \lambda(z - \zeta^0))/d\lambda$ is holomorphic in λ for all $\lambda \in \overline{T}_\nu$, it follows that

$$\sup_{\lambda \in T_\nu} \left| \frac{dH_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda} \right| \le \frac{\gamma_5}{\gamma_{19}} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Consequently

$$|H_\nu(z) - H_\nu(\zeta^0)| = \left| \int_0^1 \frac{d}{d\lambda} H_\nu(\zeta^0 + \lambda(z - \zeta^0))\, d\lambda \right| \le \frac{\gamma_5}{\gamma_{19}} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

From (8), $\zeta^0 \in \Delta_\nu$. Since $H_\nu(\zeta^0) = f(\zeta^0)$, we have

$$|H_\nu(z)| \le \left(\frac{\gamma_5}{\gamma_{19}} + 1 \right) \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Therefore

$$\sup_{z \in D_\nu} |H_\nu(z)| \le \gamma_{20} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

Hence

$$\sup_{z \in D} |H(z)| \le \gamma_{20} \sup_{\zeta \in \Delta} |f(\zeta)|.$$

The next step is to show that if $f \in A(\Delta)$, then also $H(z) = L(f)(z) \in A(D)$. In this case we have assumed that Δ has no singular points. Therefore by N. Kerzman [6], there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of functions holomorphic in a neighborhood of $\overline{\Delta}$ in $\tilde{\Delta}$ such that $\|f_k - f\|_\Delta \to 0$ when $k \to \infty$. By the continuity of L it suffices to prove that each Lf_k is in $A(D)$. Hence we can suppose f is holomorphic in $\overline{\Delta}'$ ($\overline{\Delta} \subset \Delta' \subset \overline{\Delta} \subset \tilde{\Delta}$).

Let $z^0 \in \partial\Delta$ and let $z \in S_{z^0,\sigma/2} \cap (\bar{D}_\nu \mid \partial\Delta_\nu)$. By Stokes' formula, we have

$$\begin{aligned}
H_\nu(z) &= \int_{\partial\Delta_\nu} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} \\
&= \int_{\partial\Delta'} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} - \int_{\Delta'-\Delta_\nu} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^N}\right) \\
&= \int_{\partial\Delta'} \frac{f(\zeta)K(\zeta, z)}{\Phi(\zeta, z)^N} - \int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^N}\right) \\
&\quad - \int_{(\Delta'-\Delta_\nu)|S_{z^0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^N}\right).
\end{aligned}$$

The first and the third term on the left are continuous in z^0. Therefore it is sufficient to show that, if we set

$$F_\nu(z) = \int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\bar{\partial}_\zeta\left(\frac{K(\zeta, z)}{\Phi(\zeta, z)^N}\right),$$

then $F_\nu(z)$ is continuous at z^0.

LEMMA 4. *Let $z \in S_{z^0,\sigma/2} \cap (\bar{D}_\nu \mid \partial\Delta)$. Then*

$$\left|\frac{dF_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\bigg|_{\lambda=1}\right| \le \gamma_{21}\varepsilon|\log\varepsilon| \sup_{\zeta\in\tilde{\Delta}} |f(\zeta)|.$$

Proof. We can write

$$\begin{aligned}
F_\nu(z) &= \int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} f(\zeta)\frac{A(\zeta, z)}{\Phi(\zeta, z)^N} \\
&\quad + \int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} \frac{f(\zeta)\Sigma_{j=1}^{N+1}(\zeta_j - z_j)B_j(\zeta, z)}{\Phi(\zeta, z)^{N+1}}
\end{aligned}$$

where $A(\zeta, z)$ and $B_j(\zeta, z)$ are (N, N) forms which are continuous in ζ and holomorphic in z. Therefore

$$\begin{aligned}
\left|\frac{dF_\nu(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\bigg|_{\lambda=1}\right| &\le \gamma_{22}\int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} \frac{\varepsilon}{\Phi(\zeta, z)^{N+1}}\,d\lambda \\
&\quad + \gamma_{23}\int_{(\Delta'-\Delta_\nu)\cap S_{z^0,2\sigma}} \frac{|\zeta - z|\varepsilon(|\zeta - z| + \varepsilon)}{|\Phi(\zeta, z)|^{N+2}}\,d\lambda
\end{aligned}$$

(by the estimates of G. M. Henkin [**5**])

$$\leq \gamma_{24}\varepsilon|\log \varepsilon| \sup_{\zeta\in\tilde{\Delta}} |f(\zeta)|.$$

Therefore Lemma 4 is proved.

Using the method of Henkin [**5**], we have

$$|F_\nu(z) - F_\nu(z^0)| \leq \gamma_{25}\sigma|\log \sigma| \sup_{\zeta\in\Delta'} |f(\zeta)| + \gamma_{26}\sigma \sup_{\zeta\in\Delta'} |\text{grad}\, f(\zeta)|.$$

Therefore $F_\nu(z)$ is continuous at z^0. Therefore the theorem is proved.

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