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In this paper the following result is obtained: Let α and β be two permutations such that $\alpha\beta$ is transitive and $\alpha^p = \beta^q = 1$ (where p and q are distinct primes). Then the set of all permutations commuting both with α and β is either reduced to the identity or one of the three cyclic groups C_p, C_q or C_{pq}.

Introduction. In this paper we answer a question raised by J. L. Brenner and R. C. Lyndon in [1]. They consider a pair of permutations (α, β) acting on a finite set of n elements such that $\alpha^3 = \beta^2 = 1$ and $\alpha\beta$ is transitive. Such a pair may be considered as a (combinatorial) map with exactly one face in the terminology of [2], [4], [6] and [8], Brenner and Lyndon computed the automorphism group of such a map (which is necessarily a cyclic group) for $n \leq 12$. The groups they find are 1, C_2, C_3 and C_6 and they conjectured that no other groups can arise.

In what follows we prove a more general result and show that if $\alpha\beta$ is transitive and if p and q are primes ($p \neq q$) such that $\alpha^p = \beta^q = \mathbf{1}$ then the automorphism group of (α, β) is one of $\mathbf{1}$, C_p, C_q, C_{pq}. It remains an open question to know whether C_{pq} can be found for arbitrary large values of n ($n \gg pq$)

Our main tool is the introduction of the concept of colorings of a hypermap. These colorings count in a certain way the number of fixed points of an automorphism of (α, β) when it acts on the set of cells (i.e. orbits of α, β and $\alpha\beta$). One step in the proof is to show that an automorphism of prime order cannot have exactly one fixed point in the set of cells: such a result is well known in the theory of Riemann surfaces ([5], p. 266).

All the permutations we consider act on a finite set Ω of n elements. We will also use the following conventions:

The product $\alpha\beta$ of two permutations α and β is the permutation defined by $\alpha\beta(x) = \alpha(\beta(x))$; for a subset Ω' of Ω, $\alpha\Omega'$ denotes the set $\{\alpha x \mid x \in \Omega'\}$, which has the same cardinality as Ω'; a permutation α is *regular* if all its orbits have the same length, which is also the order of α; the number of orbits of the permutation θ will be denoted by $z(\theta)$; a permutation is transitive if $z(\theta) = 1$.

A hypermap is a pair (α, β) of permutations such that the group $\langle \alpha, \beta \rangle$ generated by them is transitive on Ω. The orbits of α, β and $\alpha\beta$ are the cells of the hypermap.

An automorphism of (α, β) is an element φ of $\mathrm{Sym}(\Omega)$ that commutes with α and β. By the transitivity of $\langle \alpha, \beta \rangle$ for any x and y in Ω there exists θ in $\langle \alpha, \beta \rangle$ such that $x = \theta y$ and as for any integer k, $\varphi^k(x) = \theta\varphi^k(x)$ we have

$$\varphi^k x = x \quad \text{if and only if} \quad \varphi^k y = y;$$

hence an automorphism of (α, β) is a regular permutation.

In order to study the automorphism group of a hypermap we are led to examine for a given permutation θ the set of regular permutations φ commuting with θ. This will be done in detail in the next paragraph.

I. Commuting permutations. We state here for later use some elementary facts about a pair of commuting permutations α and β of a finite set. Throughout this section it will be assumed that α, β act on a finite set Ω of n elements and that the group $\langle \alpha, \beta \rangle$ generated by α and β is abelian.

We write Ω/α for the set of α-orbits. As α and β commute, the actions of α, β on Ω induce actions of α on Ω/β and of β on Ω/α.

LEMMA I.1. *If $G = \langle \alpha, \beta \rangle$ is transitive, then any element θ of G is regular.*

Proof. For any x and y in Ω there exists φ in G such that $y = \varphi x$, since $\theta^m x = x$ and as $\langle \alpha, \beta \rangle$ is abelian, $\theta^m y = \varphi\theta^m x = y$.

LEMMA I.2. *If $G = \langle \alpha, \beta \rangle$ is transitive on Ω, then α is transitive on Ω/β, and G is also transitive on the set of all intersections $A \cap B$ for $A \in \Omega/\alpha$, $B \in \Omega/\beta$. Therefore these intersections all have the same cardinality.*

Proof. The first statement is clear. If $A, A' \in \Omega/\alpha$ and $B, B' \in \Omega/\beta$, then $A' = \beta^k A$ and $B' = \alpha^h B$ for some h and k in Z. Then

$$\alpha^h\beta^k(A \cap B) = \alpha^h(A' \cap B) = A' \cap B'.$$

LEMMA I.3. *Let r be the common value of $|A \cap B|$, $n = |\Omega|$, let a, b be the orders of α and β. Then there exist a_1, b_1 such that $n = a_1 b_1 r$, $a = a_1 r$, $b = b_1 r$. If b is prime then $|\Omega/\alpha| = 1$ or b.*

Proof. As any A and B are both unions of $A_i \cap B_j$, r divides a and b, so that $a = a_1 r$, $b = b_1 r$. Since α and β are regular $|\Omega/\alpha| = n/a$, $|\Omega/\beta| = n/b$ and there are n^2/ab disjoint intersections $A \cap B$. Thus $n = r \cdot (n^2/ab)$ and $n = ab/r = a_1 b_1 r$. If b is prime then $r = 1$ or b and $n/a = b$ or 1.

LEMMA I.4. *If $\langle \alpha, \beta \rangle$ is transitive, and a, b, r are as above, then there exists an integer k relatively prime with r such that $\alpha^{n/b} = \beta^{nk/a}$.*

Proof. Since α is transitive on Ω/β, and $|\Omega/\beta| = n/b$ then $\alpha^{n/b}$ stabilizes each $B \in \Omega/\beta$; it also stabilizes each $A \cap B$ as $\alpha A = A$. As α is transitive on A of length a, $\alpha^{n/b} = \alpha^{a/r}$ is transitive on $C = A \cap B$. Similarly $\beta^{n/a}$ is transitive on C. For a particular C the restrictions of $\alpha^{n/b}$ and $\beta^{n/a}$ to C generate the same cyclic group of order r, then for some k such that $(k, r) = 1$, $\alpha^{n/b}$ and $\beta^{nK/a}$ have the same action on C. Thus the element $\alpha^{n/b}\beta^{-nk/a}$ of $\langle \alpha, \beta \rangle$ has at least one fixed point by I.1, it is the identity.

II. Colorings. Throughout this section we assume that φ is a regular permutation of order m acting on a finite set Ω of n elements.

A *coloring* on the set Ω is a map λ defined on Ω with values in an abelian group R. For any permutation α and any coloring λ of Ω we define another coloring $D_\alpha \lambda$ by setting

$$D_\alpha \lambda(x) = \lambda(\alpha(x)) - \lambda(x).$$

A coloring is said to be *orthogonal* to α if $D_\alpha \lambda$ is constant on Ω. In this case $\lambda(\alpha^k(x)) = \lambda(x) + k \cdot u$ where u is the constant value of $D_\alpha \lambda$. The length l of an orbit of α must verify $lu = 0$ in the abelian group. As we will only consider colorings orthogonal to φ, we will assume that R is the additive group Z/mZ. Thus the relation $mu = 0$ is satisfied for any u.

We are now interested in the extension of a coloring vanishing on a transversal T of Ω/φ, and having a given value v on an element x not in T. For such an x there exists a unique $\bar{x}$ in T and an integer h ($1 \leq h \leq m$) such that $\varphi^h(\bar{x}) = x$.

LEMMA II.1. *For v in Z/mZ, there exists a coloring λ orthogonal to φ, vanishing on T and such that $\lambda(x) = v$ if and only if the equation in u, $hu \equiv v$, has a solution in Z/mZ.*

Proof. If $D_\alpha \lambda$ is a constant u, then $\lambda(x) = \lambda(\bar{x}) + hu$ so that $hu = v$. If this equation has a solution u_0 say, then for any y in Ω there exists $\bar{y}$ in T such that $y = \varphi^l(\bar{y})$; setting $\lambda(y) = lu_0$ we obtain the coloring λ. $\square$

LEMMA II.2. *Let* $\langle \varphi, \alpha \rangle$ *be abelian and* λ *be a coloring orthogonal to* φ. *Then* $D_\alpha \lambda$ *is constant on the orbits of* φ.

Proof. We have to show that $D_\alpha \lambda(\varphi x) = D_\alpha \lambda(x)$. But as $D_\alpha \lambda(\varphi(x)) = \lambda \alpha \varphi x - \lambda \varphi x$ and since α and φ commute:

$$\begin{aligned} D_\alpha \lambda \varphi(x) &= \lambda \varphi \alpha x - \lambda \alpha x + \lambda \alpha x - \lambda x + \lambda x - \lambda \varphi x \\ &= D_\varphi \lambda(\alpha x) + D_\alpha \lambda(x) - D_\varphi \lambda(x). \end{aligned}$$

As $D_\varphi \lambda$ is constant, also the result follows. Remark that $D_\alpha \lambda$ defines a coloring on Ω/φ. For A in Ω/φ, $D_\alpha \lambda(A)$ denotes the common value of $D_\alpha \lambda(x)$ for x in A.

LEMMA II.3. *Let* $\langle \varphi, \alpha \rangle$ *be abelian and transitive on* Ω. *Then there exists a coloring* λ *orthogonal to* φ, *such that*

$$\sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) \equiv z(\alpha) \quad \text{in } Z/mZ.$$

Proof. Let $|\Omega| = n$, α have order a, and let r be the cardinality of the intersection of an orbit of α with one of φ. As α is transitive on Ω/φ there exists x such that $T = \{x, \alpha x, \ldots, \alpha^{n/m-1} x\}$ is a transversal of Ω/φ. Let $y = \alpha^{n/m} x$; we claim that there exists λ vanishing on T and such that $\lambda(y) = z(\alpha) = n/a$.

By Lemma I.4 there exists k such that $\varphi^{n/a \cdot k} = \alpha^{n/m}$; then $y = \varphi^{n/a \cdot k}(x)$. By II.1 such a λ exists if the equation

$$nku/a \equiv n/a$$

has a solution in Z/mZ.

But since $(k, r) = 1$ there exist u, v, such that $uk + vr = 1$. Then

$$nku/a + nvr/a = n/a$$

and as $nr/a = m$ (I.3), we are done.

LEMMA II.4. *Let* $G = \langle \varphi, \alpha \rangle$ *be abelian. Then there exists a coloring* λ *such that* $D_\varphi \lambda$ *is constant on G-orbits and such that*

$$\sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) = z(\alpha).$$

Moreover if φ *is of prime order and fixes only one orbit of* α *then* λ *can be found orthogonal to* φ.

Proof. By Lemma II.3 for any G-orbit Ω_h there exists a coloring λ_h such that $D_\varphi \lambda_h$ is constant on Ω_h and

$$\sum_{A \in \Omega_h/\varphi} D_\alpha \lambda_h(A) = z(\alpha_h)$$

where α_h is the restriction of α to Ω_h. Taking for λ the union of the λ_h we have the result, since $z(\alpha) = \Sigma\, z(\alpha_h)$. If φ is of prime order, then by I.3 $|\Omega_h/\alpha| = 1$ or m. In the first case, Ω_h is an α orbit fixed by φ. This occurs only once, for h_0 say; the equation to solve in Ω_{h_0} is $ku \equiv 1 \pmod m$ which gives $u = k^{-1}$ in Z/mZ. In the second case $|\Omega_h/\alpha| = m$ and the equation to solve is $mk'u \equiv m \pmod m$ which is satisfied by any u, in particular for $u = k^{-1}$. We thus can choose λ such that $D_\alpha \lambda = u$ on any $\Omega_h \cdot D_\alpha \lambda$ is thus constant.

LEMMA II.5. *Let $G = \langle \varphi, \alpha \rangle$ be abelian and such that the intersection $A_i \cap B_j$ of an orbit of φ with one of α contains at most one element. Then for any coloring λ orthogonal to φ we have*

$$\sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) = 0.$$

Proof. It suffices to show that the sum vanishes on each α orbit in Ω/φ. Let C be such an orbit; under the hypothesis of the lemma, there exists an orbit Γ in Ω of length $|C|$ and $\Sigma_{c \in C} D_\alpha \lambda(c) = \Sigma_{x \in \Gamma} D_\alpha \lambda(x)$. But $\Gamma = \{x, \alpha x, \dots, \alpha^k x\}$ and the last sum is $\Sigma_{i=0}^k (\lambda \alpha^{i+1} x - \lambda \alpha^i x)$ which vanishes as $\alpha^{k+1} x = x$.

LEMMA II.6. *Let α and β be any two permutations commuting with φ, then for any coloring λ orthogonal to φ one has*

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta} \lambda(A) = \sum_{A \in \Omega/\varphi} D_\alpha \lambda(A) + \sum_{A \in \Omega/\varphi} D_\beta \lambda(A).$$

Let Γ be any subset of Ω having exactly one element in each cycle of φ. Then

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta} \lambda(A) = \sum_{x \in \Gamma} D_{\alpha\beta} \lambda(x).$$

Since $D_{\alpha\beta}\lambda(a) = \lambda(\alpha\beta(a)) - \lambda(\beta(a)) + \lambda(\beta(a)) - \lambda(a)$ we have

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta} \lambda(A) = \sum_{x' \in \beta(\Gamma)} D_\alpha \lambda(x') + \sum_{x \in \gamma} D_\beta \lambda(x).$$

But $\beta(\Gamma)$ is also a subset of Ω having one element in each cycle of φ and the result follows.

III. The main theorems.

THEOREM 1. *Let $H = (\alpha, \beta)$ be a hypermap φ an automorphism of H of prime order p. Then the number of cells fixed by φ is necessarily different from one.*

Proof. Let us show that the assumption that φ fixes exactly one cell leads to a contradiction. Suppose that this cell is an orbit of α (a similar proof holds for an orbit of β or $\alpha\beta$). If φ fixes no other cycle of α then $z(\alpha) - 1$ is clearly divisible by p. Then by Lemma II.4 there exists a coloring orthogonal to φ such that $\Sigma_{A \in \Omega/\varphi} D_\alpha\lambda(A) \equiv 1 \pmod p$, but by Lemma II.6.

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) = \sum_{A \in \Omega/\varphi} D_\alpha\lambda(A) + \sum_{A \in \Omega/\varphi} D_\beta\lambda(A)$$

and Lemma II.5 insures the nullity of $\Sigma_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A)$ and $\Sigma_{A \in \Omega/\varphi} D_\beta\lambda(A)$. As no cycle of either β or $\alpha\beta$ is fixed by φ, we have thus found a contradiction and Theorem II.1 is proved.

LEMMA III.1 *Let φ be a permutation of order p^2 commuting with α of order p. Then for any λ orthogonal to φ*

$$p \sum_{A \in \Omega/\varphi} D_\alpha\lambda(A) \equiv 0 \pmod{p^2}.$$

Proof. We can assume that $\langle \varphi, \alpha \rangle$ acts transitively on Ω; the general case is then obtained by summing over the orbits of $\langle \varphi, \alpha \rangle$.

Since α is of order p, by Lemma I.3 the cardinality of the intersection of a cycle of φ and one of α is either 1 or p. If it is 1, then by Lemma II.4 we have

$$\sum_{A \in \Omega/\varphi} D_\alpha\lambda(A) \equiv 0 \pmod{p^2}.$$

If it is p, then Ω/φ has only one element. Let $\varphi = (b_1, b_2, \ldots, b_{p^2})$. The sum $\Sigma_{A \in \Omega/\varphi} D_\alpha\lambda(A)$ equals $D_\alpha\lambda(b_1)$ and we find

$$D_\alpha\lambda(b_1) = \lambda(\alpha(b_1)) - \lambda(b_1).$$

But as φ and α commute and φ is a cycle, α is a power of φ and $\alpha = \varphi^{ip}$, $0 \le i \le p - 1$. Thus as λ is orthogonal to φ, $\lambda(\alpha(b_1)) = \lambda(\varphi^{ip}(b_1)) = \lambda(b_1) + ipu$, so that $D_\alpha\lambda(b_1) = ipu$, as required.

We are now able to prove our main theorem.

THEOREM 2. *Let p and q be two distinct primes α and β be two permutations such that*

(1) *$\alpha\beta$ is a cycle,*

(2) $\alpha^q = \beta^p = 1$.

Then the automorphism group of (α, β) is either trivial or one of C_p, C_q, C_{pq}.

Proof. It is clear that $\mathrm{Aut}\langle \alpha, \beta \rangle$ is cyclic.

Let now φ be an automorphism of prime order, clearly φ fixes one cell of the hypermap (α, β): the unique cycle of $\alpha\beta$. By Theorem 1 it fixes one more cell, if this cell is of length one then φ is the identity, if it is of length p or q then clearly φ has orbits of length dividing p or q and φ is of order p, q or 1. This proves that $\mathrm{Aut}\langle \alpha, \beta \rangle$ is of order $p^u q^v$. To obtain the complete result we will show that assuming the existence of an automorphism of order $m = p^2$ (or $m = q^2$ similarly) we have a contradiction. Let φ be such an automorphism, let λ be the coloring constructed in Lemma II.3 for $\alpha\beta$, we have

$$\sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) \equiv z(\alpha\beta) \quad (\equiv 1) \pmod{p^2}.$$

But by Lemma II.6:

$$1 \equiv \sum_{A \in \Omega/\varphi} D_{\alpha\beta}\lambda(A) \equiv \sum_{A \in \Omega/\varphi} D_{\alpha}\lambda(A) + \sum_{A \in \Omega/\varphi} D_{\beta}\lambda(A) \pmod{p^2}$$

and

$$\sum_{A \in \Omega/\varphi} D_{\beta}\lambda(A) \equiv 0 \pmod{p^2}$$

as the cardinality r of the intersection of a cycle of φ and one of β is 0 or 1 (r dividing p^2 and q).

We thus have using Lemma III.1 and multiplying by p the above equality:

$$p \equiv p \sum_{A \in \Omega/\varphi} D_{\alpha}\lambda(A) \equiv 0 \pmod{p^2}.$$

Which is the contradiction we are looking for.

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References

[1] J. L. Brenner and R. C. Lyndon, *Permutations and cubic graphs*, Pacific J. Math., **104** (1983), 285–315.
[2] R. Cori, *Un code pour les graphes planaires et ses applications*, Astérisque, Société Mathématique de France, **27** (1975).
[3] R. Cori, A. Machí, J. G. Penaud and B. Vauquelin, *On the automorphism group of a planar hypermap*, European J. Combin., **2** (1981), 331–334.
[4] J. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices Amer. Math. Soc., **7** (1960), 646.
[5] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag (1980).
[6] A. Jacques, *Sur le genre d'une paire de substitutions*, C. R. Acad. Sci. Paris, série A.B, **267** (1968), 625–627.
[7] V. A. Liskovets, *A census of non isomorphic planar maps*, Colloquia Matematica Societatis János Bolyai (1–25), (1978)-Algebraic method in graph theory, p. 479–494.
[8] T. R. S. Walsh, *Hypermap versus bipartite maps*, J. Combinatorial Theory-Ser. B, **18** (1975), 155–163.

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