COLORINGS OF HYPERMAPS AND A CONJECTURE OF BRENNER AND LYNDON

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In this paper the following result is obtained: Let $\alpha$ and $\beta$ be two permutations such that $\alpha\beta$ is transitive and $\alpha^p = \beta^q = 1$ (where $p$ and $q$ are distinct primes). Then the set of all permutations commuting both with $\alpha$ and $\beta$ is either reduced to the identity or one of the three cyclic groups $C_p$, $C_q$ or $C_{pq}$.

Introduction. In this paper we answer a question raised by J. L. Brenner and R. C. Lyndon in [1]. They consider a pair of permutations $(\alpha, \beta)$ acting on a finite set of $n$ elements such that $\alpha^3 = \beta^2 = 1$ and $\alpha\beta$ is transitive. Such a pair may be considered as a (combinatorial) map with exactly one face in the terminology of [2], [4], [6] and [8], Brenner and Lyndon computed the automorphism group of such a map (which is necessarily a cyclic group) for $n \leq 12$. The groups they find are $1$, $C_2$, $C_3$ and $C_6$ and they conjectured that no other groups can arise.

In what follows we prove a more general result and show that if $\alpha\beta$ is transitive and if $p$ and $q$ are primes ($p \neq q$) such that $\alpha^p = \beta^q = 1$ then the automorphism group of $(\alpha, \beta)$ is one of $1$, $C_p$, $C_q$, $C_{pq}$. It remains an open question to know whether $C_{pq}$ can be found for arbitrary large values of $n$ ($n \gg pq$).

Our main tool is the introduction of the concept of colorings of a hypermap. These colorings count in a certain way the number of fixed points of an automorphism of $(\alpha, \beta)$ when it acts on the set of cells (i.e. orbits of $\alpha$, $\beta$ and $\alpha\beta$). One step in the proof is to show that an automorphism of prime order cannot have exactly one fixed point in the set of cells: such a result is well known in the theory of Riemann surfaces ([5], p. 266).

All the permutations we consider act on a finite set $\Omega$ of $n$ elements. We will also use the following conventions:

The product $\alpha\beta$ of two permutations $\alpha$ and $\beta$ is the permutation defined by $\alpha\beta(x) = \alpha(\beta(x))$; for a subset $\Omega'$ of $\Omega$, $\alpha\Omega'$ denotes the set $\{\alpha x | x \in \Omega\}$, which has the same cardinality as $\Omega'$; a permutation $\alpha$ is regular if all its orbits have the same length, which is also the order of $\alpha$; the number of orbits of the permutation $\theta$ will be denoted by $z(\theta)$; a permutation is transitive if $z(\theta) = 1$. 

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A hypermap is a pair \((\alpha, \beta)\) of permutations such that the group \(\langle \alpha, \beta \rangle\) generated by them is transitive on \(\Omega\). The orbits of \(\alpha, \beta\) and \(\alpha\beta\) are the cells of the hypermap.

An automorphism of \((\alpha, \beta)\) is an element \(\varphi\) of \(\text{Sym}(\Omega)\) that commutes with \(\alpha\) and \(\beta\). By the transitivity of \(\langle \alpha, \beta \rangle\) for any \(x, y\) in \(\Omega\) there exists \(\theta\) in \(\langle \alpha, \beta \rangle\) such that \(x = \theta y\) and as for any integer \(k\), \(\varphi^k(x) = \theta \varphi^k(x)\) we have

\[
\varphi^k x = x \quad \text{if and only if} \quad \varphi^k y = y;
\]

hence an automorphism of \((\alpha, \beta)\) is a regular permutation.

In order to study the automorphism group of a hypermap we are led to examine for a given permutation \(\theta\) the set of regular permutations \(\varphi\) commuting with \(\theta\). This will be done in detail in the next paragraph.

1. **Commuting permutations.** We state here for later use some elementary facts about a pair of commuting permutations \(\alpha\) and \(\beta\) of a finite set. Throughout this section it will be assumed that \(\alpha, \beta\) act on a finite set \(\Omega\) of \(n\) elements and that the group \(\langle \alpha, \beta \rangle\) generated by \(\alpha\) and \(\beta\) is abelian.

We write \(\Omega/\alpha\) for the set of \(\alpha\)-orbits. As \(\alpha\) and \(\beta\) commute, the actions of \(\alpha, \beta\) on \(\Omega\) induce actions of \(\alpha\) on \(\Omega/\beta\) and of \(\beta\) on \(\Omega/\alpha\).

**Lemma 1.1.** If \(G = \langle \alpha, \beta \rangle\) is transitive, then any element \(\theta\) of \(G\) is regular.

**Proof.** For any \(x, y\) in \(\Omega\) there exists \(\varphi\) in \(G\) such that \(y = \varphi x\), since \(\theta^m x = x\) and as \(\langle \alpha, \beta \rangle\) is abelian, \(\theta^m y = \varphi \theta^m x = y\).

**Lemma 1.2.** If \(G = \langle \alpha, \beta \rangle\) is transitive on \(\Omega\), then \(\alpha\) is transitive on \(\Omega/\beta\), and \(G\) is also transitive on the set of all intersections \(A \cap B\) for \(A \in \Omega/\alpha\), \(B \in \Omega/\beta\). Therefore these intersections all have the same cardinality.

**Proof.** The first statement is clear. If \(A, A' \in \Omega/\alpha\) and \(B, B' \in \Omega/\beta\), then \(A' = \beta^k A\) and \(B' = \alpha^h B\) for some \(h\) and \(k\) in \(\mathbb{Z}\). Then

\[
\alpha^h \beta^k (A \cap B) = \alpha^h (A' \cap B) = A' \cap B'.
\]

**Lemma 1.3.** Let \(r\) be the common value of \(|A \cap B|\), \(n = |\Omega|\), let \(a, b\) be the orders of \(\alpha\) and \(\beta\). Then there exist \(a_1, b_1\) such that \(n = a_1 b_1 r\), \(a = a_1 r\), \(b = b_1 r\). If \(b\) is prime then \(|\Omega/\alpha| = 1\) or \(b\).
Proof. As any $A$ and $B$ are both unions of $A_i \cap B_j$, $r$ divides $a$ and $b$, so that $a = a_1 r$, $b = b_1 r$. Since $\alpha$ and $\beta$ are regular $|\Omega/\alpha| = n/a$, $|\Omega/\beta| = n/b$ and there are $n^2/ab$ disjoint intersections $A \cap B$. Thus $n = r \cdot (n^2/ab)$ and $n = ab/r = a_1 b_1 r$. If $b$ is prime then $r = 1$ or $b$ and $n/a = b$ or 1.

**Lemma 1.4.** If $\langle \alpha, \beta \rangle$ is transitive, and $a$, $b$, $r$ are as above, then there exists an integer $k$ relatively prime with $r$ such that $\alpha^{n/b} = \beta^{nk/a}$.

**Proof.** Since $\alpha$ is transitive on $\Omega/\beta$, and $|\Omega/\beta| = n/b$ then $\alpha^{n/b}$ stabilizes each $B \in \Omega/\beta$; it also stabilizes each $A \cap B$ as $\alpha A = A$. As $\alpha$ is transitive on $A$ of length $a$, $\alpha^{n/b} = \alpha^{a/r}$ is transitive on $C = A \cap B$. Similarly $\beta^{n/a}$ is transitive on $C$. For a particular $C$ the restrictions of $\alpha^{n/b}$ and $\beta^{n/a}$ to $C$ generate the same cyclic group of order $r$, then for some $k$ such that $(k, r) = 1$, $\alpha^{n/b}$ and $\beta^{nk/a}$ have the same action on $C$. Thus the element $\alpha^{n/b} \beta^{-nk/a}$ of $\langle \alpha, \beta \rangle$ has at least one fixed point by I.1, it is the identity.

**II. Colorings.** Throughout this section we assume that $\varphi$ is a regular permutation of order $m$ acting on a finite set $\Omega$ of $n$ elements.

A coloring on the set $\Omega$ is a map $\lambda$ defined on $\Omega$ with values in an abelian group $R$. For any permutation $\alpha$ and any coloring $\lambda$ of $\Omega$ we define another coloring $D_\alpha \lambda$ by setting

$$D_\alpha \lambda(x) = \lambda(\alpha(x)) - \lambda(x).$$

A coloring is said to be orthogonal to $\alpha$ if $D_\alpha \lambda$ is constant on $\Omega$. In this case $\lambda(\alpha^k(x)) = \lambda(x) + k \cdot u$ where $u$ is the constant value of $D_\alpha \lambda$. The length $l$ of an orbit of $\alpha$ must verify $lu = 0$ in the abelian group. As we will only consider colorings orthogonal to $\varphi$, we will assume that $R$ is the additive group $\mathbb{Z}/m\mathbb{Z}$. Thus the relation $mu = 0$ is satisfied for any $u$.

We are now interested in the extension of a coloring vanishing on a transversal $T$ of $\Omega/\varphi$, and having a given value $v$ on an element $x$ not in $T$. For such an $x$ there exists a unique $\bar{x}$ in $T$ and an integer $h$ $(1 \leq h \leq m)$ such that $\varphi^h(\bar{x}) = x$.

**Lemma II.1.** For $v$ in $\mathbb{Z}/m\mathbb{Z}$, there exists a coloring $\lambda$ orthogonal to $\varphi$, vanishing on $T$ and such that $\lambda(x) = v$ if and only if the equation in $u$, $hu \equiv v$, has a solution in $\mathbb{Z}/m\mathbb{Z}$.

**Proof.** If $D_\alpha \lambda$ is a constant $u$, then $\lambda(x) = \lambda(\bar{x}) + hu$ so that $hu = v$. If this equation has a solution $u_0$ say, then for any $y$ in $\Omega$ there exists $\bar{y}$ in $T$ such that $y = \varphi^l(\bar{y})$; setting $\lambda(y) = lu_0$ we obtain the coloring $\lambda$.  \[\square\]
**Lemma II.2.** Let \( \langle \phi, \alpha \rangle \) be abelian and \( \lambda \) be a coloring orthogonal to \( \phi \). Then \( D_a \lambda \) is constant on the orbits of \( \phi \).

**Proof.** We have to show that \( D_a \lambda(\phi x) = D_a \lambda(x) \). But as \( D_a \lambda(\phi(x)) = \lambda \alpha \phi x - \lambda \phi x \) and since \( \alpha \) and \( \phi \) commute:

\[
D_a \phi \lambda(x) = \lambda \alpha \phi x - \lambda \phi x = D_\phi \lambda(\alpha x) + D_a \lambda(x) - D_\phi \lambda(x).
\]

As \( D_\phi \lambda \) is constant, also the result follows. Remark that \( D_a \lambda \) defines a coloring on \( \Omega/\phi \). For \( A \) in \( \Omega/\phi \), \( D_a \lambda(A) \) denotes the common value of \( D_a \lambda(x) \) for \( x \) in \( A \).

**Lemma II.3.** Let \( \langle \phi, \alpha \rangle \) be abelian and transitive on \( \Omega \). Then there exists a coloring \( \lambda \) orthogonal to \( \phi \), such that

\[
2 D_a \lambda(A) = z(\alpha) \quad \text{in } \mathbb{Z}/m\mathbb{Z}.
\]

**Proof.** Let \( |\Omega| = n, \alpha \) have order \( a \), and let \( r \) be the cardinality of the intersection of an orbit of \( \alpha \) with one of \( \phi \). As \( \alpha \) is transitive on \( \Omega/\phi \) there exists \( x \) such that \( T = \{ x, \alpha x, \ldots, \alpha^{n/m-1}x \} \) is a transversal of \( \Omega/\phi \). Let \( y = \alpha^{n/m}x \); we claim that there exists \( \lambda \) vanishing on \( T \) and such that \( \lambda(y) = z(\alpha) = n/a \).

By Lemma I.4 there exists \( k \) such that \( \phi^{n/a - k} = \alpha^{n/m} \); then \( y = \phi^{n/a - k}(x) \). By II.1 such a \( \lambda \) exists if the equation

\[
nku/a \equiv n/a
\]

has a solution in \( \mathbb{Z}/m\mathbb{Z} \).

But since \( (k, r) = 1 \) there exist \( u, v \), such that \( uk + vr = 1 \). Then

\[
nku/a + nvr/a = n/a
\]

and as \( nr/a = m \) (I.3), we are done.

**Lemma II.4.** Let \( G = \langle \phi, \alpha \rangle \) be abelian. Then there exists a coloring \( \lambda \) such that \( D_\phi \lambda \) is constant on \( G \)-orbits and such that

\[
\sum_{A \in \Omega/\phi} D_a \lambda(A) = z(\alpha).
\]

Moreover if \( \phi \) is of prime order and fixes only one orbit of \( \alpha \) then \( \lambda \) can be found orthogonal to \( \phi \).
Proof. By Lemma II.3 for any $G$-orbit $\Omega_h$ there exists a coloring $\lambda_h$ such that $D_\phi \lambda_h$ is constant on $\Omega_h$ and
\[
\sum_{A \in \Omega_h/\phi} D_\alpha \lambda_h(A) = z(\alpha_h)
\]
where $\alpha_h$ is the restriction of $\alpha$ to $\Omega_h$. Taking for $\lambda$ the union of the $\lambda_h$ we have the result, since $z(\alpha) = \Sigma z(\alpha_h)$. If $\phi$ is of prime order, then by I.3 $|\Omega_h/\alpha| = 1$ or $m$. In the first case, $\Omega_h$ is an $\alpha$ orbit fixed by $\phi$. This occurs only once, for $h_0$ say; the equation to solve in $\Omega_{h_0}$ is $ku \equiv 1 \pmod{m}$ which gives $u = k^{-1}$ in $\mathbb{Z}/m\mathbb{Z}$. In the second case $|\Omega_h/\alpha| = m$ and the equation to solve is $mk'u \equiv m \pmod{m}$ which is satisfied by any $u$, in particular for $u = k^{-1}$. We thus can choose $\lambda$ such that $D_\alpha \lambda = u$ on any $\Omega_h$. $D_\alpha \lambda$ is thus constant.

Lemma II.5. Let $G = \langle \phi, \alpha \rangle$ be abelian and such that the intersection $A_i \cap B_j$ of an orbit of $\phi$ with one of $\alpha$ contains at most one element. Then for any coloring $\lambda$ orthogonal to $\phi$ we have
\[
\sum_{A \in \Omega/\phi} D_\alpha \lambda(A) = 0.
\]
Proof. It suffices to show that the sum vanishes on each $\alpha$ orbit in $\Omega/\phi$. Let $C$ be such an orbit; under the hypothesis of the lemma, there exists an orbit $\Gamma$ in $\Omega$ of length $|C|$ and $\Sigma_{c \in C} D_\alpha \lambda(c) = \Sigma_{x \in \Gamma} D_\alpha \lambda(x)$. But $\Gamma = \{x, \alpha x, \ldots, \alpha^k x\}$ and the last sum is $\Sigma_{i=0}^k (\lambda \alpha^{i+1} x - \lambda \alpha^i x)$ which vanishes as $\alpha^{k+1} x = x$.

Lemma II.6. Let $\alpha$ and $\beta$ be any two permutations commuting with $\phi$, then for any coloring $\lambda$ orthogonal to $\phi$ one has
\[
\sum_{A \in \Omega/\phi} D_{\alpha \beta} \lambda(A) = \sum_{A \in \Omega/\phi} D_\alpha \lambda(A) + \sum_{A \in \Omega/\phi} D_\beta \lambda(A).
\]
Let $\Gamma$ be any subset of $\Omega$ having exactly one element in each cycle of $\phi$. Then
\[
\sum_{A \in \Omega/\phi} D_{\alpha \beta} \lambda(A) = \sum_{x \in \Gamma} D_{\alpha \beta} \lambda(x).
\]
Since $D_{\alpha \beta} \lambda(a) = \lambda(\alpha \beta(a)) - \lambda(\beta(a)) + \lambda(\beta(a)) - \lambda(\alpha) \lambda(a)$ we have
\[
\sum_{A \in \Omega/\phi} D_{\alpha \beta} \lambda(A) = \sum_{x' \in \beta(\Gamma)} D_\alpha \lambda(x') + \sum_{x \in \Gamma} D_\beta \lambda(x).
\]
But $\beta(\Gamma)$ is also a subset of $\Omega$ having one element in each cycle of $\phi$ and the result follows.
III. The main theorems.

**Theorem 1.** Let $H = (\alpha, \beta)$ be a hypermap $\varphi$ an automorphism of $H$ of prime order $p$. Then the number of cells fixed by $\varphi$ is necessarily different from one.

**Proof.** Let us show that the assumption that $\varphi$ fixes exactly one cell leads to a contradiction. Suppose that this cell is an orbit of $\alpha$ (a similar proof holds for an orbit of $\beta$ or $\alpha \beta$). If $\varphi$ fixes no other cycle of $\alpha$ then $z(\alpha) - 1$ is clearly divisible by $p$. Then by Lemma II.4 there exists a coloring orthogonal to $\varphi$ such that $\sum_{A \in \Omega/\varphi} D_{\alpha \beta} \lambda(A) \equiv 1 \pmod{p}$, but by Lemma II.6.

$$
\sum_{A \in \Omega/\varphi} D_{\alpha \beta} \lambda(A) = \sum_{A \in \Omega/\varphi} D_{\alpha} \lambda(A) + \sum_{A \in \Omega/\varphi} D_{\beta} \lambda(A)
$$

and Lemma II.5 insures the nullity of $\sum_{A \in \Omega/\varphi} D_{\alpha \beta} \lambda(A)$ and $\sum_{A \in \Omega/\varphi} D_{\beta} \lambda(A)$. As no cycle of either $\beta$ or $\alpha \beta$ is fixed by $\varphi$, we have thus found a contradiction and Theorem II.1 is proved.

**Lemma III.1** Let $\varphi$ be a permutation of order $p^2$ commuting with $\alpha$ of order $p$. Then for any $\lambda$ orthogonal to $\varphi$

$$
p \sum_{A \in \Omega/\varphi} D_{\alpha} \lambda(A) \equiv 0 \pmod{p^2}.
$$

**Proof.** We can assume that $\langle \varphi, \alpha \rangle$ acts transitively on $\Omega$; the general case is then obtained by summing over the orbits of $\langle \varphi, \alpha \rangle$.

Since $\alpha$ is of order $p$, by Lemma I.3 the cardinality of the intersection of a cycle of $\varphi$ and one of $\alpha$ is either 1 or $p$. If it is 1, then by Lemma II.4 we have

$$
\sum_{A \in \Omega/\varphi} D_{\alpha} \lambda(A) \equiv 0 \pmod{p^2}.
$$

If it is $p$, then $\Omega/\varphi$ has only one element. Let $\varphi = (b_1, b_2, \ldots, b_p)$. The sum $\sum_{A \in \Omega/\varphi} D_{\alpha} \lambda(A)$ equals $D_{\alpha} \lambda(b_1)$ and we find

$$
D_{\alpha} \lambda(b_1) = \lambda(\alpha(b_1)) - \lambda(b_1).
$$

But as $\varphi$ and $\alpha$ commute and $\varphi$ is a cycle, $\alpha$ is a power of $\varphi$ and $\alpha = \varphi^p$, $0 \leq i \leq p - 1$. Thus as $\lambda$ is orthogonal to $\varphi$, $\lambda(\alpha(b_1)) = \lambda(\varphi^p(b_1)) = \lambda(b_1) + i p u$, so that $D_{\alpha} \lambda(b_1) = i p u$, as required.

We are now able to prove our main theorem.
THEOREM 2. Let $p$ and $q$ be two distinct primes $\alpha$ and $\beta$ be two permutations such that

1. $\alpha \beta$ is a cycle,
2. $\alpha^q = \beta^p = 1$.

Then the automorphism group of $(\alpha, \beta)$ is either trivial or one of $C_p$, $C_q$, $C_{pq}$.

Proof. It is clear that $\text{Aut}(\alpha, \beta)$ is cyclic.

Let now $\phi$ be an automorphism of prime order, clearly $\phi$ fixes one cell of the hypermap $(\alpha, \beta)$: the unique cycle of $\alpha \beta$. By Theorem 1 it fixes one more cell, if this cell is of length one then $\phi$ is the identity, if it is of length $p$ or $q$ then clearly $\phi$ has orbits of length dividing $p$ or $q$ and $\phi$ is of order $p$, $q$ or 1. This proves that $\text{Aut}(\alpha, \beta)$ is of order $p^u q^v$. To obtain the complete result we will show that assuming the existence of an automorphism of order $m = p^2$ (or $m = q^2$ similarly) we have a contradiction. Let $\phi$ be such an automorphism, let $\lambda$ be the coloring constructed in Lemma II.3 for $\alpha \beta$, we have

$$\sum_{A \in \Omega/\phi} D_{\alpha \beta} \lambda(A) \equiv z(\alpha \beta) \pmod{p^2}.$$ 

But by Lemma II.6: 

$$1 \equiv \sum_{A \in \Omega/\phi} D_{\alpha \beta} \lambda(A) \equiv \sum_{A \in \Omega/\phi} D_{\alpha} \lambda(A) + \sum_{A \in \Omega/\phi} D_{\beta} \lambda(A) \pmod{p^2}$$

and

$$\sum_{A \in \Omega/\phi} D_{\beta} \lambda(A) \equiv 0 \pmod{p^2}$$

as the cardinality $r$ of the intersection of a cycle of $\phi$ and one of $\beta$ is 0 or 1 ($r$ dividing $p^2$ and $q$).

We thus have using Lemma III.1 and multiplying by $p$ the above equality:

$$p \equiv p \sum_{A \in \Omega/\phi} D_{\alpha} \lambda(A) \equiv 0 \pmod{p^2}.$$ 

Which is the contradiction we are looking for.

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REFERENCES


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