

# Pacific Journal of Mathematics

**DETERMINATIONS OF JACOBSTHAL SUMS**

RONALD JAMES EVANS

## DETERMINATIONS OF JACOBSTHAL SUMS

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**The sign ambiguities are resolved in evaluations of Jacobsthal sums  $\sum_{m=1}^p (m(m^k + a)/p)$  for  $k = 2, 3, 4, 6, 10,$  and  $12,$  where  $( \ /p)$  denotes the Legendre symbol.**

**1. Introduction.** For a positive even integer  $e = 2n$ , a prime  $p = ef + 1$ , and an integer  $a$  prime to  $p$ , define the Jacobsthal sum of order  $e$  by

$$\varphi_n(a) = \sum_{m=1}^p \left( \frac{m(m^n + a)}{p} \right),$$

where  $( \ /p)$  denotes the Legendre symbol. In [1, §4], the values of Jacobsthal sums  $\varphi_n(a)$  of orders  $e = 4, 6, 8, 12, 20, 24$  are given up to some sign ambiguities. The purpose of this paper is to show how the precise values of  $\varphi_n(a)$  can be found.

In §3, we give congruence conditions (mod  $p$ ) which determine the correct choices of  $\pm$  signs. The computational complexity of these determinations for large  $p$  is much less than that of computing  $\varphi_n(a)$  directly from the definition.

In §4, we describe a method for determining the correct choices of  $\pm$  signs by congruence conditions (mod  $a$ ), when  $a$  is prime. If  $a$  is small compared with  $p$ , then the determinations in §4 (mod  $a$ ) turn out to be computationally simpler than those in §3 (mod  $p$ ).

The cases  $e = 4, 6$  and  $e = 8$  have already been treated by Hudson and Williams in [2] and [3], respectively. We employ different techniques based on Jacobi sums which work for all values  $e = 4, 6, 8, 12, 20, 24$ . Each of these values of  $e$  is considered in §3, but in §4, only the case  $e = 12$  is treated, for brevity.

It will be convenient to introduce the notation  $F_e(a)$  for the sum

$$(1) \quad F_e(a) = \sum_{m=1}^p \left( \frac{m(m^{e/2} - a)}{p} \right) = \varphi_n(-a).$$

An evaluation of  $F_e(a)$  immediately yields one for  $\varphi_n(a)$ , since [4, (7)]

$$F_e(a) = \varphi_n(-a) = \varphi_n(a)(-1)^{fn+f}.$$

In the sequel, attention will be focused on  $F_e(a)$ .

**2. Notation and Jacobi sums.** For a character  $\lambda \pmod{p}$ , define the Jacobi sums

$$J(\lambda) = \sum_{m=1}^p \lambda(m)\lambda(1-m), \quad K(\lambda) = \lambda(4)J(\lambda).$$

Write  $p = ef + 1$ . For each value of  $e = 4, 6, 8, 12, 20, 24$ , fix a character  $\chi = \chi_e \pmod{p}$  of order  $e$ . Let  $P$  be the prime ideal divisor of  $p$  in  $\mathbf{Z}[\exp(2\pi i/e)]$  chosen such that

$$(2) \quad \chi(\alpha) \equiv \alpha^{(p-1)/e} = \alpha^f \pmod{P}$$

for all  $\alpha \in \mathbf{Z}[\exp(2\pi i/e)]$ . It is easily seen that

$$(3) \quad K(\chi) \equiv 0 \pmod{P}.$$

In [1, §3] one finds the following evaluations of Jacobi sums  $K(\chi)$  of orders  $e = 4, 6, 8, 12, 20, 24$  in terms of parameters in quadratic partitions of  $p$ .

$$(4) \quad K(\chi_4) = a_4 + ib_4, \quad \text{where } p = a_4^2 + b_4^2, a_4 \equiv -(2/p) \pmod{4};$$

$$(5) \quad \left(\frac{-1}{p}\right)K(\chi_6) = K(\chi_6^2) = a_3 + ib_3\sqrt{3},$$

$$\text{where } p = a_3^2 + 3b_3^2, a_3 \equiv -1 \pmod{3};$$

$$(6) \quad K(\chi_8) = a_8 + ib_8\sqrt{2}, \quad \text{where } p = a_8^2 + 2b_8^2, a_8 \equiv -1 \pmod{4};$$

$$(7) \quad K(\chi_{12}) = \begin{cases} -a_4 - ib_4, & \text{if } 3 \mid a_4, \\ a_4 + ib_4, & \text{if } 3 \nmid a_4, \end{cases}$$

where

$$K(\chi_{12}^3) = a_4 + ib_4 \quad \text{as in (4);}$$

$$(8) \quad K(\chi_{24}) = a_{24} + ib_{24}\sqrt{6}, \quad \text{where } p = a_{24}^2 + 6b_{24}^2,$$

$$a_{24} \equiv a_8 \pmod{3}, \quad \text{with } K(\chi_{24}^3) = a_8 + ib_8\sqrt{2} \quad \text{as in (6);}$$

$$(9) \quad K(\chi_{20}) = \begin{cases} a_{20} + ib_{20}\sqrt{5}, & \text{if } 5 \nmid a_4, \\ ia_{20} - b_{20}\sqrt{5}, & \text{if } 5 \mid a_4, \end{cases}$$

where

$$p = a_{20}^2 + 5b_{20}^2 \quad \text{and} \quad a_{20} \equiv \begin{cases} a_4 \pmod{5}, & \text{if } 5 \nmid a_4, \\ b_4 \pmod{5}, & \text{if } 5 \mid a_4, \end{cases}$$

with  $K(\chi_{20}^5) = a_4 + ib_4$  as in (4).

**3. Congruence conditions (mod  $p$ ).** This section is to be read in conjunction with [1, §4]. We consider only those values of  $a$  for which the evaluations of  $F_e(a)$  in [1, §4] have sign ambiguities, and we resolve these ambiguities with congruence conditions (mod  $p$ ), for  $e = 4, 6, 8, 12, 20, 24$ .

*Case 1.  $e = 4, (a/p) = -1$ .*

The proof in [1, Theorem 4.4] shows that

$$(10) \quad F_4(a) = 2 \operatorname{Re}(\bar{\chi}(a)K(\chi)) = -2b_4i\chi(a) = \pm 2b_4.$$

To determine the correct sign, it remains to find  $F_4(a) \pmod{p}$ . By (3) and (4),  $-ib_4 \equiv a_4 \pmod{P}$ . Thus by (10) and (2),  $F_4(a) \equiv 2a_4a^f \pmod{P}$ , so

$$(11) \quad F_4(a) \equiv 2a_4a^f \pmod{p}.$$

**REMARK.** While it takes the computer  $O(p)$  operations to compute  $F_4(a)$  directly from the definition (1), it requires at most  $O(\sqrt{p})$  operations to compute  $F_4(a)$  from (10) and (11), since  $a^f \pmod{p}$  can be computed in  $O(\log p)$  steps.

*Case 2.  $e = 6, a$  is noncubic (mod  $p$ ).*

Write  $\lambda = \chi_6^2$ . Note that  $\lambda(a) = (-1 \pm i\sqrt{3})/2$ . The proof in [1, Theorem 4.2] shows that

$$(12) \quad \begin{aligned} F_6(a) &= -1 + 2 \operatorname{Re}(\bar{\lambda}(a)K(\lambda)) \\ &= -1 - a_3 + 2b_3\sqrt{3} \operatorname{Im} \lambda(a) = -1 - a_3 \pm 3b_3. \end{aligned}$$

It remains to determine  $F_6(a) \pmod{p}$ . By (3) and (5),  $a_3 \equiv -ib_3\sqrt{3} \pmod{P}$ , so by (12) and (2),

$$F_6(a) \equiv a_3(a^{2f} - a^{4f}) - 1 - a_3 \equiv 2a_3a^{2f} - 1 \pmod{p}.$$

*Case 3.  $e = 8, (a/p) = -1$ .*

From the proof in [1, Theorem 4.6],

$$(13) \quad \begin{aligned} F_8(a) &= -2 \operatorname{Re}(K(\chi)(\chi(a) + \chi^3(a))) \\ &= -2ib_8\sqrt{2}(\chi(a) + \chi^3(a)) = \pm 4b_8. \end{aligned}$$

Thus,

$$F_8(a) \equiv 2a_8(a^f + a^{3f}) \pmod{p}.$$

Case 4.  $e = 12$ ,  $(a/p) = -1$ .

Subcase 4A.  $3 \mid a_4$ ,  $a$  is cubic (mod  $p$ ).

By [1, (4.3)],

$$(14) \quad F_{12}(a) = 6 \operatorname{Re}(\chi(a)(a_4 + ib_4)) = 6\chi(a)ib_4 = \pm 6b_4.$$

By (3) and (7),  $a_4 \equiv -ib_4 \pmod{P}$ , so

$$F_{12}(a) \equiv -6a_4a^f \pmod{p}.$$

Subcase 4B.  $3 \nmid a_4$ .

By [1, (4.5)],

$$(15) \quad F_{12}(a) = 2b_4/\operatorname{Im} \chi(a)$$

$$= 4ib_4/(\chi(a) + \chi^5(a)) = \begin{cases} \pm 4b_4, & \text{if } a \text{ is noncubic (mod } p) \\ \pm 2b_4, & \text{if } a \text{ is cubic (mod } p). \end{cases}$$

Thus,

$$F_{12}(a) \equiv -4a_4/(a^f + a^{5f}) \pmod{p}.$$

Case 5.  $e = 24$ ,  $(a/p) = -1$ .

This case is slightly different than those above in that *two* congruence conditions are required to determine  $F_{24}(a)$ . From the proof in [1, Theorem 4.10],

$$F_{24}(a) = A_{24} + B_{24},$$

where

$$\begin{aligned} A_{24} &= -2 \operatorname{Re}((a_8 + ib_8\sqrt{2})(\chi^3(a) + \chi^9(a))) \\ &= -2ib_8\sqrt{2}(\chi^3(a) + \chi^9(a)) = \pm 4b_8 \end{aligned}$$

and

$$\begin{aligned} B_{24} &= -2 \operatorname{Re}((a_{24} + ib_{24}\sqrt{6})(\chi(a) + \chi^5(a) + \chi^7(a) + \chi^{11}(a))) \\ &= -2ib_{24}\sqrt{6}(\chi(a) + \chi^5(a) + \chi^7(a) + \chi^{11}(a)) \\ &= \begin{cases} \pm 12b_{24}, & \text{if } a \text{ is noncubic (mod } p) \\ 0, & \text{if } a \text{ is cubic (mod } p). \end{cases} \end{aligned}$$

It remains to determine  $A_{24}$  and  $B_{24} \pmod{p}$ . Since  $a_8 \equiv -ib_8\sqrt{2}$  and  $a_{24} \equiv -ib_{24}\sqrt{6} \pmod{P}$ , we have

$$A_{24} \equiv 2a_8(a^{3f} + a^{9f}) \pmod{p}$$

and

$$B_{24} \equiv 2a_{24}(a^f + a^{5f} + a^{7f} + a^{11f}) \pmod{p}.$$

*Case 6.  $e = 20$ .*

This case is similar to Case 5, so we omit some details. From the proof in [1, Theorem 4.13],

$$F_{20}(a) = A_{20} + B_{20},$$

where

$$A_{20} = 2 \operatorname{Re}\{\chi^5(a)(a_4 - ib_4)\}$$

and

$$B_{20} = \begin{cases} 2 \operatorname{Re}\{(\chi(a) - \chi^3(a) - \chi^7(a) + \chi^9(a))(-ia_{20} - b_{20}\sqrt{5})\}, & \text{if } 5 \mid a_4, \\ 2 \operatorname{Re}\{(\chi(a) + \chi^3(a) + \chi^7(a) + \chi^9(a))(a_{20} - ib_{20}\sqrt{5})\}, & \text{if } 5 \nmid a_4. \end{cases}$$

It remains to determine  $A_{20}$  and  $B_{20}$  in each of the subcases below.

*Subcase 6A.  $5 \mid a_4, (a/p) = 1$ , a nonquintic (mod  $p$ ).*

Here  $A_{20} = \pm 2a_4$  and  $B_{20} = \pm 10b_{20}$ , with

$$(16) \quad A_{20} \equiv 2a_4 a^{5f} \pmod{p}$$

and

$$(17) \quad B_{20} \equiv 2(a^f - a^{3f} - a^{7f} + a^{9f})a_4 a_{20}/b_4 \pmod{p}.$$

Observe that there is no sign ambiguity in the right member of (17), since  $a_{20}/b_4 \equiv 1 \pmod{5}$ , as is noted after (9).

*Subcase 6B.  $5 \mid a_4, (a/p) = -1$ .*

Here,

$$A_{20} = \pm 2b_4 \quad \text{and} \quad B_{20} = \begin{cases} \pm 8a_{20}, & \text{if } a \text{ is quintic (mod } p) \\ \pm 2a_{20}, & \text{if } a \text{ is nonquintic (mod } p), \end{cases}$$

with the congruences (16) and (17) again holding.

Subcase 6C.  $5 \nmid a_4, (a/p) = -1$ .

Here

$$A_{20} = \pm 2b_4 \quad \text{and} \quad B_{20} = \begin{cases} \pm 10b_{20}, & \text{if } a \text{ is nonquintic (mod } p) \\ 0, & \text{if } a \text{ is quintic (mod } p), \end{cases}$$

with (16) holding and

$$B_{20} \equiv 2a_{20}(a^f + a^{3f} + a^{7f} + a^{9f}) \pmod{p}.$$

**4. Congruence conditions (mod  $a$ ).** Throughout this section,  $e = 12$ ,  $p = 12f + 1$ ,  $\chi$  is a character (mod  $p$ ) of order 12,  $(a/p) = -1$ , and  $a$  is prime. From (14) and (15),

$$(18) \quad F_{12}(a) = t \operatorname{Im} K(\chi^3) / \operatorname{Im} \chi(a) = tb_4 / \operatorname{Im} \chi(a) = \pm hb_4$$

where

$$(19) \quad K(\chi^3) = a_4 + ib_4$$

and

$$\begin{aligned} h = t = -6, & \quad \text{if } 3 \mid a_4 \text{ and } a \text{ is cubic (mod } p), \\ h = t = 2, & \quad \text{if } 3 \nmid a_4 \text{ and } a \text{ is cubic (mod } p), \\ h = 4, t = 2, & \quad \text{if } 3 \nmid a_4 \text{ and } a \text{ is noncubic (mod } p). \end{aligned}$$

If the prime  $a$  is odd, then  $a \nmid b_4$ , otherwise we would have

$$p = a_4^2 + b_4^2 \equiv a_4^2 \pmod{a},$$

which contradicts  $(a/p) = -1$ . Thus we can resolve the ambiguity in (18) by determining  $F_{12}(a) \pmod{a}$ , if  $a > 3$ . (Note  $a \neq 3$ , as  $(a/p) = -1$ .) For  $a = 2$ , we will resolve the ambiguity by determining  $F_{12}(2)$  modulo an appropriate power of 2, in (20) and (21) below.

Case 1.  $a = 2$ .

It is classical [4, p. 107] that

$$b_4 \equiv -2i\chi^3(2) \pmod{8}.$$

If 2 is a cubic residue (mod  $p$ ), then

$$\frac{b_4}{\operatorname{Im} \chi(2)} = \frac{ib_4}{\chi(2)} \equiv \frac{2\chi^3(2)}{\chi(2)} = -2 \pmod{8},$$

so by (18),

$$(20) \quad F_{12}(2) \equiv -2t \equiv -4 \pmod{16}, \quad \text{if } 2 \text{ is cubic (mod } p).$$

If  $3 \nmid a_4$  and 2 is noncubic (mod  $p$ ), then

$$\begin{aligned} F_{12}(2) &= \frac{2b_4}{\operatorname{Im} \chi(2)} = \frac{4ib_4}{\chi(2) - \bar{\chi}(2)} \equiv \frac{8\chi^3(2)}{\chi(2) - \bar{\chi}(2)} \\ &= \frac{8}{\chi^{10}(2) - \chi^8(2)} \pmod{32}. \end{aligned}$$

Since  $\chi^8(2) = (-1 \pm i\sqrt{3})/2$  and  $\chi^{10}(2) = (1 \pm i\sqrt{3})/2$ ,

$$(21) \quad F_{12}(2) \equiv 8 \pmod{32}, \quad \text{if } 3 \nmid a_4 \text{ and 2 is noncubic (mod } p).$$

*Case 2.  $a$  is a prime  $> 3$ .*

To determine  $F_{12}(a) \pmod{a}$ , it suffices, by (18), to determine

$$S(\chi) = \operatorname{Im} \chi(a)/b_4$$

modulo  $a$ . To do this, we need some formulas for Gauss sums  $G(\psi)$ , defined for characters  $\psi \pmod{p}$  by

$$G(\psi) = \sum_{n=1}^p \psi(n) \exp(2\pi in/p).$$

From [1, Theorems 2.2 and 3.1],

$$G(\chi)^{12} = pJ^4(\chi^4)K^6(\chi)$$

so by [1, Theorem 3.19],

$$(22) \quad G(\chi)^{12} = pJ^4(\chi^4)K^6(\chi^3).$$

From [1, (3.28) and Theorems 2.2 and 3.1],

$$G^5(\chi)/G(\chi^5) = J^2(\chi^4)K^2(\chi),$$

so by [1, Theorem 3.19],

$$(23) \quad G^5(\chi)/G(\chi^5) = J^2(\chi^4)K^2(\chi^3).$$

Here, as in [1, Theorem 3.4],

$$(24) \quad 2J(\chi^4) = r_3 + 3it_3\sqrt{3}, \quad \text{where } 4p = r_3^2 + 27t_3^2, r_3 \equiv 1 \pmod{3}.$$

It is clear from the definition of  $G(\chi)$  that, in the ring of algebraic integers,

$$(25) \quad G^a(\chi) \equiv \bar{\chi}^a(a)G(\chi^a) \pmod{a}.$$

We will complete the proof by determining  $S(\chi) \pmod{a}$  in (27)–(30) in terms of the parameters  $p$ ,  $r_3$ , and  $a_4$  unambiguously defined in (4) and (24).



*Subcase 2A.*  $a \equiv 5 \pmod{12}$ .

By (25) and (23),

$$\chi^7(a) \equiv G^{a-5}(\chi)G^5(\chi)/G(\chi^5) = G^{a-5}(\chi)J^2(\chi^4)K^2(\chi^3) \pmod{a}.$$

Thus, by (22),

$$\chi^7(a) \equiv p^{(a-5)/12}J^{(a+1)/3}(\chi^4)K^{(a-1)/2}(\chi^3) \pmod{a}.$$

Replacing  $\chi$  by  $\chi^7$ , we obtain

$$(26) \quad \chi(a) \equiv p^{(a-5)/12}J^{(a+1)/3}(\chi^4)K^{(a-1)/2}(\bar{\chi}^3) \pmod{a}.$$

Each member of (26) is a rational linear combination of  $1, i, \sqrt{3}, i\sqrt{3}$  by (19) and (24). The respective coefficients of  $i$  must be congruent  $\pmod{a}$ . Since  $\text{Im } \chi(a)$  is rational, it follows that

$$\text{Im } \chi(a) \equiv -p^{(a-5)/12} \text{Re } J^{(a+1)/3}(\chi^4) \text{Im } K^{(a-1)/2}(\chi^3) \pmod{a}$$

so

$$(27) \quad S(\chi) \equiv -p^{(a-5)/12}b_4^{-1} \text{Re } J^{(a+1)/3}(\chi^4) \text{Im } K^{(a-1)/2}(\chi^3) \pmod{a}.$$

For example, when  $a = 5$ , (27) yields

$$\begin{aligned} S(\chi) &\equiv (-4b_4)^{-1} \text{Re}(r_3 + 3it_3\sqrt{3})^2 \text{Im}(a_4 + ib_4)^2 \\ &\equiv 2a_4(r_3^2 - 27t_3^2) \pmod{5}. \end{aligned}$$

*Subcase 2B.*  $a \equiv 7 \pmod{12}$ .

By (25) and (23),

$$\begin{aligned} \chi^5(a) &\equiv G^{a+5}(\chi)\chi(-1)p^{-1}G(\chi^5)/G^5(\chi) \\ &\equiv G^{a+5}(\chi)\chi(-1)p^{-1}/(J^2(\chi^4)K^2(\chi^3)) \pmod{a}. \end{aligned}$$

Thus, by (22),

$$\chi^5(a) \equiv p^{(a-7)/12}\chi(-1)J^{(a-1)/3}(\chi^4)K^{(a+1)/2}(\chi^3) \pmod{a}.$$

Replacing  $\chi$  by  $\chi^5$ , we obtain

$$\chi(a) \equiv p^{(a-7)/12}(-1)^f J^{(a-1)/3}(\bar{\chi}^4)K^{(a+1)/2}(\chi^3) \pmod{a},$$

so

$$(28) \quad \begin{aligned} S(\chi) &\equiv p^{(a-7)/12}(-1)^f \text{Re } J^{(a-1)/3}(\chi^4) \\ &\quad \times \text{Im } K^{(a+1)/2}(\chi^3)/b_4 \pmod{a}. \end{aligned}$$

For example, when  $a = 7$ , (28) yields

$$\begin{aligned} S(\chi) &\equiv (-1)^f (4b_4)^{-1} \operatorname{Re}(r_3 + 3it_3\sqrt{3})^2 \operatorname{Im}(a_4 + ib_4)^4 \\ &\equiv (-1)^f a_4 (r_3^2 - 27t_3^2) (2a_4^2 - p) \pmod{7}. \end{aligned}$$

*Subcase 2C.  $a \equiv 11 \pmod{12}$ .*

By (25) and (22),

$$\begin{aligned} \chi(a) &\equiv p^{-1} \chi(-1) G^{a+1}(\chi) \\ &\equiv p^{(a-11)/12} \chi(-1) J^{(a+1)/3}(\chi^4) K^{(a+1)/2}(\chi^3) \pmod{a}. \end{aligned}$$

Thus,

$$(29) \quad \begin{aligned} S(\chi) &\equiv p^{(a-11)/12} (-1)^f \operatorname{Re} J^{(a+1)/3}(\chi^4) \\ &\quad \times \operatorname{Im} K^{(a+1)/2}(\chi^3) / b_4 \pmod{a}. \end{aligned}$$

For example, when  $a = 11$ , (29) yields

$$\begin{aligned} S(\chi) &= (-1)^f (16b_4)^{-1} \operatorname{Re}(r_3 + 3it_3\sqrt{3})^4 \operatorname{Im}(a_4 + ib_4)^6 \\ &\equiv (-1)^f a_4 (3b_4^4 - 10a_4^2 b_4^2 + 3a_4^4) (r_3^4 - 162r_3^2 t_3^2 + 729t_3^4) / 8 \\ &\equiv 7a_4 (-1)^f (3b_4^4 + a_4^2 b_4^2 + 3a_4^4) (r_3^4 + 3r_3^2 t_3^2 + 3t_3^4) \pmod{11}. \end{aligned}$$

*Subcase 2D.  $a \equiv 1 \pmod{12}$ .*

By (25) and (22),

$$\chi(a) \equiv G^{a-1}(\bar{\chi}) \equiv p^{(a-1)/12} J^{(a-1)/3}(\bar{\chi}^4) K^{(a-1)/2}(\bar{\chi}^3) \pmod{a}.$$

Thus,

$$(30) \quad S(\chi) \equiv -p^{(a-1)/12} \operatorname{Re} J^{(a-1)/3}(\chi^4) \operatorname{Im} K^{(a-1)/2}(\chi^3) / b_4 \pmod{a}.$$

For example, when  $a = 13$ , (30) yields

$$\begin{aligned} S(\chi) &\equiv -p (16b_4)^{-1} \operatorname{Re}(r_3 + 3it_3\sqrt{3})^4 \operatorname{Im}(a_4 + ib_4)^6 \\ &\equiv -pa_4 (3b_4^4 - 10a_4^2 b_4^2 + 3a_4^4) (r_3^4 - 162r_3^2 t_3^2 + 729t_3^4) / 8 \\ &\equiv -2pa_4 (b_4^4 + a_4^2 b_4^2 + a_4^4) (r_3^4 + 7r_3^2 t_3^2 + t_3^4) \pmod{13}. \end{aligned}$$

*Numerical examples.*

$a$	5	5	5	7	7	7	11	11	11	13	13	13
$p$	13	37	157	61	73	157	61	193	337	37	193	229
$F_{12}(a)$	12	24	-24	-24	48	-12	-12	24	-96	24	-24	12

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Received March 3, 1982 and in revised form June 7, 1982. Supported by NSF grant MCS 81-01860.

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