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CHARACTERIZATIONS OF \mathfrak{S} -SPACES

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CHARACTERIZATIONS OF \aleph -SPACES

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Two simultaneous generalizations of metric spaces and \aleph_0 -spaces, the \aleph -spaces introduced by O'Meara and the cs - σ -spaces of Guthrie, are shown to be the same.

It was shown by Guthrie [2] that a regular space is an \aleph_0 -space if and only if it has a countable cs -network (see definitions below). We show here that, in parallel manner, O'Meara's \aleph -spaces may be characterized as the regular spaces admitting σ -locally finite cs -networks; that is, the classes of \aleph -spaces and cs - σ -spaces coincide. While this equivalence has been proved by Guthrie [3] for paracompact spaces, the fact that these classes contain non-paracompact examples [6] makes our result an honest improvement.

DEFINITION 1. A collection \mathcal{P} of subsets of a topological space X is a k -network for X if, given any compact subset C of X and any neighborhood U of C , there is a finite subcollection \mathcal{P}^* of \mathcal{P} so that $C \subset \bigcup \mathcal{P}^* \subset U$. A collection \mathcal{P} is a cs -network for X if, given any sequence σ converging to $x \in X$ and any neighborhood U of x , there is a $P \in \mathcal{P}$ so that $P \subset U$ and σ is eventually in P . A regular space is an \aleph_0 -space [5] (\aleph -space [6], [7], cs - σ -space [3]) if it has a countable k -network (σ -locally finite k -network, σ -locally finite cs -network); because of regularity, these collections can be chosen to consist of closed sets.

We say that a subset W of a topological space X is a *sequential neighborhood* of a subset F of W if every sequence converging to a member of F is eventually in W .

LEMMA 2. *A discrete family $\{F_\alpha: \alpha \in A\}$ of subsets of an \aleph -space X admits a pairwise disjoint family $\{W_\alpha: \alpha \in A\}$ of sequential neighborhoods.*

Proof. For every $n < \omega$, let \mathcal{P}_n be a locally finite collection of closed sets so that $\bigcup_{n < \omega} \mathcal{P}_n$ is a k -network for X . For $n < \omega$ and $B \subset A$, let

$$T(n, B) = \bigcup \{P \in \mathcal{P}_n: P \cap \bigcup \{F_\alpha: \alpha \in B\} = \emptyset\}.$$

For every $\alpha \in A$, let

$$W_\alpha = \bigcup_{n < \omega} [T(n, A \setminus \{\alpha\}) \setminus T(n, \{\alpha\})].$$

It is simple to verify that the W_α 's are pairwise disjoint. To see that W_α is a sequential neighborhood of F_α , note that for a sequence σ converging to a member of F_α there is an $n < \omega$ so that σ is eventually in $T(n, A \setminus \{\alpha\})$; hence σ is eventually in $T(n, A \setminus \{\alpha\}) \setminus T(n, \{\alpha\}) \subset W_\alpha$.

LEMMA 3. *Assume X has a point-countable k -network \mathcal{P} of closed sets so that \mathcal{P} is closed under finite intersections. If $x \in X$, if W is a sequential neighborhood of x , and if σ is a sequence converging to x , then there is a finite subset \mathcal{P}^* of \mathcal{P} so that $\bigcup \mathcal{P}^* \subset W$ and $\bigcup \mathcal{P}^*$ contains a tail of σ .*

Proof. Let $\{\mathcal{P}_n: n < \omega\}$ be the family of all finite subsets \mathcal{P}^* of \mathcal{P} such that $x \in \bigcap \mathcal{P}^*$ and σ is eventually in $\bigcup \mathcal{P}^*$. If no finite subset of \mathcal{P} satisfies the conclusion of the lemma, then we could find a $y_n \in \bigcap_{i \leq n} (\bigcup \mathcal{P}_i) \setminus W$ for every $n < \omega$. This sequence $\{y_n: n < \omega\}$ converges to x ; indeed, if U is a neighborhood of x , we could find a \mathcal{P}_m so that $\{y_n: n \geq m\} \subset \bigcup \mathcal{P}_m \subset U$. The convergence of $\{y_n: n < \omega\}$ contradicts that W is a sequential neighborhood of x .

THEOREM 4. *The following are equivalent for a regular space X .*

- (a) X has a σ -discrete cs-network.
- (b) X has a σ -discrete k -network.
- (c) X has a σ -locally finite cs-network.
- (d) X has a σ -locally finite k -network.

Proof. It is clear that (a) implies (c) and (b) implies (d). As Guthrie observed in [3], his proof of the countable case in [2] can be adapted to show (c) implies (d), and the same is true for (a) implies (b). It therefore suffices to show (d) implies (a).

For every $m < \omega$ let \mathcal{P}_m be a locally finite collection of closed sets (our only use of regularity) which is closed under finite intersections, so that $\mathcal{P}_m \subset \mathcal{P}_{m+1}$ and $\mathcal{P} = \bigcup_{m < \omega} \mathcal{P}_m = \{P_\alpha: \alpha \in A\}$ is a k -network for X .

For each m let \mathcal{U}_m be an open cover of X that witnesses the local finiteness of \mathcal{P}_m . Since a space X satisfying (d) is clearly subparacompact [1], it follows from [1] that \mathcal{U}_m has a σ -discrete closed refinement $\bigcup_{n < \omega} \{F_\beta: \beta \in B_{m,n}\}$, where $\{F_\beta: \beta \in B_{m,n}\}$ is discrete for each n . It follows that, if $\beta \in \bigcup_{n < \omega} B_{m,n}$, then $F_\beta \cap P_\alpha \neq \emptyset$ for only finitely many $P_\alpha \in \mathcal{P}_m$.

By Lemma 2 we can find, for every $\langle m, n \rangle \in \omega^2$, a pairwise disjoint family $\{W_\beta: \beta \in B_{m,n}\}$ of sequential neighborhoods for $\{F_\beta: \beta \in B_{m,n}\}$.

For every pair $\langle m, n \rangle \in \omega^2$ let

$$C_{m,n} = \{ \langle \alpha, \beta \rangle : P_\alpha \in \mathcal{P}_m, \beta \in B_{m,n}, P_\alpha \cap F_\beta \neq \emptyset \}.$$

Let us check that the collection $\{P_\alpha \cap W_\beta: \langle \alpha, \beta \rangle \in C_{m,n}\}$ is star-finite. Indeed, if $\langle \alpha, \beta \rangle \in C_{m,n}$ and $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \emptyset$ (where $\langle \gamma, \delta \rangle \in C_{m,n}$), the fact that β and δ are in $B_{m,n}$ with $W_\beta \cap W_\delta \neq \emptyset$ forces $\beta = \delta$. Consequently, $\langle \gamma, \beta \rangle \in C_{m,n}$; it follows that $P_\gamma \cap F_\beta \neq \emptyset$. So P_γ is one of the finitely many members of \mathcal{P}_m which meets F_β . So there are only finitely many pairs $\langle \gamma, \delta \rangle \in C_{m,n}$ for which $(P_\alpha \cap W_\beta) \cap (P_\gamma \cap W_\delta) \neq \emptyset$.

Fix $\langle m, n \rangle \in \omega^2$. Now if $\langle \alpha, \beta \rangle \in C_{m,n}$ and $r < \omega$, let

$$S(\alpha, \beta, r) = \cup \{P_\alpha \cap P_\gamma: P_\gamma \in \mathcal{P}_r \text{ and } P_\gamma \subset W_\beta\}$$

and

$$\mathfrak{S}(m, n, r) = \{S(\alpha, \beta, r): \langle \alpha, \beta \rangle \in C_{m,n}\}.$$

Since $S(\alpha, \beta, r) \subset P_\alpha \cap W_\beta$ for every $r < \omega$, the collections $\mathfrak{S}(m, n, r)$ inherit the star-finite property from $\{P_\alpha \cap W_\beta: \langle \alpha, \beta \rangle \in C_{m,n}\}$. Note too that every member of $\mathfrak{S}(m, n, r)$ is the union of a subcollection of the locally finite collection $\{P_\alpha \cap P_\gamma: P_\alpha \in \mathcal{P}_m, P_\gamma \in \mathcal{P}_r\}$ and thus $\mathfrak{S}(m, n, r)$ is closure-preserving. Because a star-finite collection of sets is σ -disjoint and because a disjoint and closure-preserving collection of closed sets is discrete, we have that $\mathfrak{S}(m, n, r)$ is σ -discrete.

Thus $\mathfrak{S} = \cup \{\mathfrak{S}(m, n, r): \langle m, n, r \rangle \in \omega^3\}$ is σ -discrete; write $\mathfrak{S} = \cup_{k < \omega} \mathfrak{S}_k$ so that every \mathfrak{S}_k is a discrete collection of closed sets and $\mathfrak{S}_j \cap \mathfrak{S}_k = \emptyset$ if $j \neq k$. Let

$$\mathbf{F} = \{\mathcal{F}: \mathcal{F} \text{ is a finite subset of } \mathfrak{S}, \cap \mathcal{F} \neq \emptyset\},$$

and for every finite subset Φ of ω , let

$$\mathbf{F}_\Phi = \{\mathcal{F} \in \mathbf{F}: \{k < \omega: \mathcal{F} \cap \mathfrak{S}_k \neq \emptyset\} = \Phi\}.$$

Note that for a particular $k < \omega$, a collection $\mathcal{F} \in \mathbf{F}$ may contain at most one member of \mathfrak{S}_k , as \mathfrak{S}_k is pairwise disjoint.

Now for a given finite subset Φ of ω consider the collection $\{\cap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$. It is locally finite because it is comprised of finite intersections of the locally finite family $\cup_{k \in \Phi} \mathfrak{S}_k$. It is also pairwise disjoint: if $\mathcal{F}_1 \neq \mathcal{F}_2$ are members of \mathbf{F}_Φ , then $\mathcal{F}_1 \cap \mathfrak{S}_k \neq \mathcal{F}_2 \cap \mathfrak{S}_k$ for some $k \in \Phi$; i.e. if $\{S_1\} = \mathcal{F}_1 \cap \mathfrak{S}_k$ and $\{S_2\} = \mathcal{F}_2 \cap \mathfrak{S}_k$, then $S_1 \neq S_2$. Pairwise disjointness

of \mathcal{S}_k gives $S_1 \cap S_2 = \emptyset$, and thus $(\cap \mathcal{F}_1) \cap (\cap \mathcal{F}_2) = \emptyset$. So $\{\cap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$ is both pairwise disjoint and a locally finite collection of closed sets; therefore it is discrete.

Again we apply Lemma 2 to find, for every finite subset Φ of ω , a pairwise disjoint family $\{V(\mathcal{F}): \mathcal{F} \in \mathbf{F}_\Phi\}$ of sequential neighborhoods of $\{\cap \mathcal{F}: \mathcal{F} \in \mathbf{F}_\Phi\}$. For $j < \omega$ and $\mathcal{F} \in \mathbf{F}_\Phi$ let

$$V(\mathcal{F}, j) = \cup \{S \cap P_\delta: S \in \mathcal{F}, P_\delta \in \mathcal{P}_j, P_\delta \subset V(\mathcal{F})\}.$$

Now $V(\mathcal{F}, j) \subset V(\mathcal{F})$, so for a fixed $j < \omega$ the collection

$$\mathcal{V}(\Phi, j) = \{V(\mathcal{F}, j): \mathcal{F} \in \mathbf{F}_\Phi\}$$

is pairwise disjoint. Further, every $V(\mathcal{F}, j) \in \mathcal{V}(\Phi, j)$ is the union of a subcollection of the locally finite family of closed sets $\{S \cap P_\delta: S \in \cup_{k \in \Phi} \mathcal{S}_k, P_\delta \in \mathcal{P}_j\}$. Hence

$$\begin{aligned} \mathcal{V} &= \{V(\mathcal{F}, j): \mathcal{F} \in \mathbf{F}, j < \omega\} \\ &= \cup \{\mathcal{V}(\Phi, j): \Phi \text{ is a finite subset of } \omega, j < \omega\} \end{aligned}$$

is σ -discrete. We will now verify that \mathcal{V} is a cs-network for X .

Suppose U is open and σ is a sequence converging to $x \in U$. Because \mathcal{P} is a k -network for X , we can find an $m < \omega$ and a finite subset \mathcal{P}_m^* of \mathcal{P}_m so that $\cup \mathcal{P}_m^* \subset U$, σ is eventually in $\cup \mathcal{P}_m^*$, and, because the members of \mathcal{P} are closed, we may choose such a \mathcal{P}_m^* so that $x \in \cap \mathcal{P}_m^*$.

Since $X \subset \cup_{n < \omega} \{F_\beta: \beta \in B_{m,n}\}$, we can find an $n < \omega$ and a $\beta \in B_{m,n}$ so that $x \in F_\beta$. Now W_β is a sequential neighborhood of F_β , hence of x , so by applying Lemma 3 we can find an $r < \omega$ and a finite subset \mathcal{P}_r^{**} of \mathcal{P}_r so that $\cup \mathcal{P}_r^{**} \subset W_\beta$ and σ is eventually in $\cup \mathcal{P}_r^{**}$. Because the members of \mathcal{P} are closed, necessarily $x \in \cup \mathcal{P}_r^{**}$.

If $P_\alpha \in \mathcal{P}_m^*$, the fact that $x \in P_\alpha \cap F_\beta$ implies $\langle \alpha, \beta \rangle \in C_{m,n}$. If, in addition, $P_\gamma \in \mathcal{P}_r^{**}$, then $P_\gamma \in \mathcal{P}_r$ and $P_\gamma \subset W_\beta$, and thus $P_\alpha \cap P_\gamma \subset S(\alpha, \beta, r)$. From this we see that $(\cup \mathcal{P}_m^*) \cap (\cup \mathcal{P}_r^{**}) \subset \cup \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$ and, because there is a γ so that $x \in P_\gamma \in \mathcal{P}_r^{**}$, that $x \in \cap \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$. Let $\mathcal{F} = \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\}$ (a finite subset of \mathcal{S}). The previous sentence implies σ is eventually in $\cup \mathcal{F}$ (since σ is eventually in $\cup \mathcal{P}_m^* \cap \cup \mathcal{P}_r^{**}$) and $\cap \mathcal{F} \neq \emptyset$ (since $x \in \cap \mathcal{F}$). In particular, $\mathcal{F} \in \mathbf{F}$.

As $V(\mathcal{F})$ is a sequential neighborhood of $\cap \mathcal{F}$, hence of x , Lemma 3 enables us to find a $j < \omega$ and a finite subset \mathcal{P}_j^{***} of \mathcal{P}_j so that $\mathcal{P}_j^{***} \subset V(\mathcal{F})$ and σ is eventually in $\cup \mathcal{P}_j^{***}$.

Now if $P_\delta \in \mathcal{P}_j^{***}$, then $P_\delta \in \mathcal{P}_j$ and $P_\delta \subset V(\mathcal{F})$; therefore for any $S \in \mathcal{F}$ we have $S \cap P_\delta \subset V(\mathcal{F}, j)$. It follows that $(\cup \mathcal{F}) \cap (\cup \mathcal{P}_j^{***}) \subset V(\mathcal{F}, j)$. As a result, σ is eventually in $V(\mathcal{F}, j)$.

In addition,

$$V(\mathcal{F}, j) \subset \cup \mathcal{F} = \cup \{S(\alpha, \beta, r): P_\alpha \in \mathcal{P}_m^*\} \subset \cup \mathcal{P}_m^* \subset U.$$

So \mathcal{V} is a cs-network for X .

Our Theorem 4, taken with Theorem 2 of [3], gives the following answer to Michael's question in [4].

COROLLARY 5. *If X is an \aleph_0 -space and Y is an \aleph -space, then the space of continuous functions from X to Y equipped with the compact-open topology is an \aleph -space.*

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