

Pacific Journal of Mathematics

**WEAK COMPACTNESS IN SPACES OF BOCHNER
INTEGRABLE FUNCTIONS AND THE RADON-NIKODÝM
PROPERTY**

NASSIF A. GHOUSSOUB AND PAULETTE SAAB

WEAK COMPACTNESS IN SPACES OF BOCHNER INTEGRABLE FUNCTIONS AND THE RADON-NIKODYM PROPERTY

N. GHOUSSOUB AND P. SAAB

We characterize Banach spaces E such that E and E^* have the Radon-Nikodym property in terms of relatively weakly compact sets of $L^1[\lambda, E]$.

Introduction. It is well known [1] that if $(\Omega, \Sigma, \lambda)$ is a finite measure space and E is a Banach space, then a relatively weakly compact subset K of $L^1[\lambda, E]$ is *bounded, uniformly integrable* and for every $B \in \Sigma$, the set $\{\int_B f d\lambda, f \in K\}$ is *relatively weakly compact* in E . Moreover, it was shown in [1] that if the Banach space E and its dual E^* have the Radon-Nikodym property, then relatively weakly compact subsets of $L^1[\lambda, E]$ are completely characterized by the above three conditions. A question that arises naturally is the following: Are the conditions on E and E^* to have the Radon-Nikodym property necessary in order that relatively weakly compact subsets of $L^1[\lambda, E]$ be exactly those bounded, uniformly integrable subsets K such that for any $B \in \Sigma$, the set $\{\int_B f d\lambda, f \in K\}$ is relatively weakly compact in E ? In [1], it was shown that the condition on E to have the Radon-Nikodym property is indeed necessary. The object of this paper is to show that the condition on E^* to have the Radon-Nikodym property is also necessary. This gives a new characterization of Banach spaces E such that E and E^* have the Radon-Nikodym property. We also study bounded linear operators T between Banach spaces such that T and its adjoint T^* are strong Radon-Nikodym operators.

Definitions and Preliminaries.

DEFINITION 1. A closed bounded convex subset C of a Banach space E is a *Radon-Nikodym (R.N.P) set* if for every finite measure space $(\Omega, \Sigma, \lambda)$ and any vector measure $G: \Sigma \rightarrow E$ such that the set $\{G(B)/\lambda(B), B \in \Sigma, \lambda(B) > 0\}$ is contained in C , there exists a Bochner integrable Radon-Nikodym derivative $f: \Omega \rightarrow C$ such that $G(B) = \int_B f d\lambda$, for every $B \in \Sigma$.

For more on (R.N.P) sets see [3] and [4].

DEFINITION 2. A bounded linear operator T from a Banach E into a Banach space F is called a *strong Radon-Nikodym operator* if the closure of $\{Tx, x \in E, \|x\| \leq 1\}$ is an (R.N.P) set in E .

Accordingly, a Banach space E has the Radon-Nikodym property (R.N.P) iff its closed unit ball is an (R.N.P) set or equivalently if the identity operator on E is a strong Radon-Nikodym operator.

If $T: E \rightarrow F$ is a strong Radon-Nikodym operator then T is an (R.N.P) operator see [2] i.e., for every vector measure $G: \Sigma \rightarrow E$ with $\|G(B)\| \leq \lambda(B)$ for all $B \in \Sigma$, there exists a Bochner integrable function $f: \Omega \rightarrow F$ such that $TG(B) = \int_B f d\lambda$ for all $B \in \Sigma$. The converse is not true as any quotient map Q from l^1 onto c_0 is an (R.N.P) operator but is not a strong Radon-Nikodym operator. But it follows from [4] if $T: E \rightarrow F$ is a bounded linear operator, then its adjoint T^* is a strong Radon-Nikodym operator if and only if T^* is an (R.N.P) operator.

Finally, given a finite measure space $(\Omega, \Sigma, \lambda)$ E and F two Banach spaces and $T: E \rightarrow F$ a bounded linear operator, we shall denote by \tilde{T} the natural extension of T to a bounded linear operator from $L^1[\lambda, E]$ to $L^1[\lambda, F]$.

For all undefined statements and notations we refer the reader to [1].

The following theorem extends the result of [1, p. 101] to operators $T: E \rightarrow F$ such that T and T^* are strong Radon-Nikodym operators.

THEOREM 1. *Let E and F be two Banach spaces and let $T: E \rightarrow F$ be a bounded linear operator such that T and T^* are strong Radon-Nikodym operators. Then for any finite measure space $(\Omega, \Sigma, \lambda)$, the operator $\tilde{T}: L^1[\lambda, E] \rightarrow L^1[\lambda, F]$ sends into relatively weakly compact subsets of $L^1[\lambda, F]$ any bounded, uniformly integrable subsets K of $L^1[\lambda, E]$ such that for every $B \in \Sigma$ the set $\{\int_B f d\lambda, f \in K\}$ is relatively weakly compact in E .*

Proof. Let $T: E \rightarrow F$ be a bounded linear operator such that T and T^* are strong Radon-Nikodym operators. Let $(\Omega, \Sigma, \lambda)$ be a finite measure space and let $K \subseteq L^1[\lambda, E]$ be a bounded and uniformly integrable subset of $L^1[\lambda, E]$ such that for any $B \in \Sigma$ the set $\{\int_B f d\lambda, f \in K\}$ is relatively weakly compact in E . Let $(f_n)_n$ be a sequence in K . Proceed now as in [1, p. 101] to get a countably generated σ -field Σ_1 , such that each f_n is measurable with respect to Σ_1 , find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and define a countably additive vector measure $G: \Sigma_1 \rightarrow E$ of bounded variation by

$$G(B) = \text{weak limit}_k \int_B f_{n_k} d\lambda, \quad \text{for every } B \in \Sigma_1.$$

Since T^* is a Radon-Nikodym operator, it follows from [4] that there exist a Banach space Z , such that Z^* has RNP, and bounded linear operators $T_1: E \rightarrow Z$ and $T_2: Z \rightarrow F$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ T_1 \searrow & & \nearrow T_2 \\ & Z & \end{array}$$

Case 1. Assume that for some $\alpha > 0$ $\|G(B)\| \leq \alpha\lambda(B)$, for all $B \in \Sigma_1$. It follows that the set $\{T_1G(B)/\lambda(B), \lambda(B) > 0, B \in \Sigma_1\}$ is contained in the closure C in Z of the set $\{T_1x, x \in E, \|x\| \leq \alpha\}$. But a glance at the construction of [4] reveals that the set C is affinely homeomorphic to the closure in F of the set $\{Tx, x \in E, \|x\| \leq \alpha\}$, and one can show that the set C is an R.N.P. set. Therefore there exists a Bochner integrable function $h: \Omega \rightarrow C$ such that

$$T_1G(B) = \int_B h \, d\lambda, \quad \text{for all } B \in \Sigma_1.$$

Moreover since Z^* has R.N.P and since $(\int_B T_1 f_{n_k} \, d\lambda)_k$ converges weakly to $\int_B h \, d\lambda$ in Z for every $B \in \Sigma_1$, it follows that the sequence $(\tilde{T}_1 f_{n_k})_k$ converges weakly to h in $L^1[\Sigma_1, \lambda, Z]$, thus $(\tilde{T}f_{n_k})_k$ converges weakly to $\tilde{T}_2 h$ in $L^1[\Sigma_1, \lambda, F]$, and hence in $L^1[\lambda, F]$. An appeal to Eberlein's theorem shows that $\{\tilde{T}f, f \in K\}$ is relatively weakly compact in $L^1[\lambda, F]$ and completes the proof of Case 1.

General case. Let $(\Omega_m)_m$ be a partition of Ω of elements of Σ_1 and such that

$$\|G(B)\| \leq m\lambda(B)$$

for all elements B of Σ_1 contained in Ω_m . By restricting the sequence $(f_{n_k})_k$ to each of the sets Ω_m , by Case 1, and by an appropriate diagonal process, one can produce a subsequence $(h_j)_j$ of $(f_{n_k})_k$ and a sequence $(g_m)_m$ of Bochner integrable functions $g_m: \Omega_m \rightarrow F$ such that:

- (i) the sequence $(\tilde{T}h_{j\Omega_m})_m$ converges weakly to g_m in $L^1[\Omega_m, \lambda, F]$,
- (ii) $TG(B \cap \Omega_m) = \int_{B \cap \Omega_m} g_m \, d\lambda$, for $B \in \Sigma_1$.

Let $g: \Omega \rightarrow F$ be defined as follows:

$$g(w) = g_m(w) \quad \text{if } w \in \Omega_m.$$

It is clear that $g \in L^1[\lambda, F]$ for g is obviously measurable and it follows from (ii) that

$$\int_{\Omega} \|g(w)\| d\lambda \leq \sum_m |TG|(\Omega_m) = |TG|(\Omega) < \infty.$$

The proof will be complete when we show that the sequence $(\tilde{T}h_j)_j$ converges weakly to g in $L^1[\lambda, F]$. For this let $L \in (L^1[\lambda, F])^*$. For each $m \geq 1$ let L_m be the restriction of L to $L^1[\Omega_m, \lambda, F]$. For every $m \geq 1$ we have

$$|L(\tilde{T}h_j - g)| \leq \left| \sum_{i=1}^m L_i(\tilde{T}h_{j\Omega_i} - g_i) \right| + \|L\| \int_{\cup \Omega_i; i > m} \|\tilde{T}h_i - g\| d\lambda.$$

Since the sequence $(h_j)_j$ is uniformly integrable, there exists $m \geq 1$ such that $\int_{\cup \Omega_i; i \geq m} \|\tilde{T}h_j - g\| d\lambda$ is arbitrary small for all $j \geq 1$. Since $\tilde{T}h_{j\Omega_i}$ converges weakly to g_i , it follows that $|\sum_{i=1}^m L(\tilde{T}h_{j\Omega_i} - g_i)|$ is arbitrary small as $j \rightarrow \infty$. Hence $L(\tilde{T}h_j - g) \rightarrow 0$ as $j \rightarrow \infty$. This completes the proof.

The following proposition establishes the fact that if T^* fails to be a strong Radon-Nikodym operator, then the conclusion of Theorem 1 is no more valid.

PROPOSITION. *If T is a bounded linear operator from a Banach space E into a Banach space F such that T^* fails to be a strong Radon-Nikodym operator, then there exists a finite measure space $(\Omega, \Sigma, \lambda)$, a bounded uniformly integrable subset K of $L^1[\lambda, E]$ such that the set $\{\int_B f d\lambda, f \in K\}$ is relatively weakly compact in E for any $B \in \Sigma$, but the set $\{\tilde{T}f, f \in K\}$ is not relatively weakly compact in $L^1[\lambda, F]$.*

Proof. Suppose that T^* fails to be a strong Radon-Nikodym operator. Let $\Delta = \{-1, 1\}^{\mathbb{N}}$ denote the Cantor group with Haar measure m and let $\{\Delta_{n,i}, 1 \leq i \leq 2^n\}$ denote the standard n th partition of Δ with $\Delta_{0,1} = \Delta$, $\Delta_{n,i} = \Delta_{n+1,2i-1} \cup \Delta_{n+1,2i}$, $\Delta_{n,i}$ is clopen, and $m(\Delta_{n,i}) = 1/2^n$. It follows from the dichotomy theorem of Stegall [4] that the operator T must factor the Haar operator $H: l^1 \rightarrow L_{\infty}(m)$ which takes the basis of l^1 into the usual Haar basis of $C(\Delta)$ considered as a subspace of $L_{\infty}(m)$. Indeed the Haar operator is defined as follows:

$$\text{if } h_{ni} = \chi_{\Delta_{n+1,2i-1}} - \chi_{\Delta_{n+1,2i}}, \quad n \geq 0, 1 \leq i \leq 2^n \quad \text{then} \quad He_{ni} = h_{ni},$$

here $\{e_{ni}, n \geq 0, 1 \leq i \leq 2^n\}$ is an enumeration of the usual l^1 basis. Let $U: l^1 \rightarrow E$ and $V: F \rightarrow L_{\infty}(m)$ be bounded linear operators such that $H = V \circ T \circ U$ as illustrated in the following diagram.

$$\begin{array}{ccc}
 E & \xrightarrow{T} & F \\
 U \uparrow & & \downarrow V \\
 l^1 & \xrightarrow{H} & L_\infty(m)
 \end{array}$$

Consider the following sequence $(f_n)_n$ in $L^1[m, l^1]$ with

$$f_n(t) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{2^j} h_{ji}(t) e_{ji}, \quad \text{for } t \in \Delta.$$

The sequence $(f_n)_n$ is easily seen [2] to have the following properties

(i) $\sup_n \|f_n(t)\| = 1$ m.a.e.

(ii) $\sup_{\|x^*\| \leq 1} \int |x^* \circ f_n| dm$ approaches zero as $n \rightarrow \infty$.

It follows that for every Borel B set in Δ the sequence $(\|\int_B f_n dm\|)_n$ approaches zero as $n \rightarrow \infty$. The sequence $(f_n)_n$ is bounded and uniformly integrable in $L^1[m, l^1]$ and $(\int_B f_n dm)_n$ is a null sequence in l^1 . We claim that the sequence $(\tilde{H}f_n)$ is not relatively weakly compact in $L^1[m, L_\infty(m)]$. For this note that for each $n \geq 1$

$$\tilde{H}f_n(t) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{2^j} h_{ji}(t) h_{ji}, \quad t \in \Delta;$$

therefore $\tilde{H}f_n(t)$ takes its values in $C(\Delta)$, to prove the claim all we need to show is that $(\tilde{H}f_n)_n$ is not relatively weakly compact in $L^1[m, C(\Delta)]$. To this end note that since for every Borel set B the sequence $(\int_B \tilde{H}f_n dm)_n$ converges to zero in $C(\Delta)$, it follows that every weakly convergent subsequence of $(\tilde{H}f_n)_n$ in $L^1[m, C(\Delta)]$ must converge to zero. Let $L \in (L^1[m, C(\Delta)])^*$ be defined as follows: for $\psi \in L^1[m, C(\Delta)]$

$$L(\psi) = \int_\Delta \psi(t)(t) dm$$

then

$$L(\tilde{H}f_n) = \frac{1}{n} \int_\Delta \sum_{j=1}^n \sum_{i=1}^{2^j} h_{ji}(t) h_{ji}(t) dm = 1.$$

This shows that the sequence $(\tilde{H}f_n)_n$ has no weakly convergent subsequence in $L^1[m, C(\Delta)]$. The sequence $(\tilde{U}f_n)_n$ is bounded and uniformly integrable in $L^1[m, E]$ and the set $\{\int_B \tilde{U}f_n d\lambda, n \geq 1\}$ is relatively weakly compact in E for all Borel sets B of Δ , yet since T factors the Haar operator H , the sequence $(\tilde{T}\tilde{U}f_n)_n$ cannot have a weakly convergent subsequence in $L^1[m, F]$. This completes the proof.

COROLLARY 3. *A Banach space E and its dual E^* have (R.N.P) if and only if for every finite measure space $(\Omega, \Sigma, \lambda)$, any bounded and uniformly integrable subset K of $L^1[\lambda, E]$ is relatively weakly compact whenever for every $B \in \Sigma$, the set $\{\int_B f d\lambda, f \in K\}$ is relatively weakly compact in E .*

REFERENCES

- [1] J. Diestel and J. J. Uhl, Jr. *Vector measures*, Mathematical Survey No. 15, A.M.S., Providence, 1977.
- [2] G. A. Edgar, *Asplund operators and A. E. convergences*, J. Multivariate Anal., **10** (1980), 460–466.
- [3] E. Saab, *A characterization of w^* -compact convex sets having the Radon-Nikodym property*, Bull. Sci. Math., (2), **104** (1980), 79–88.
- [4] C. Stegall, *The Radon-Nikodym property in conjugate Banach spaces, II*, Trans. Amer. Math. Soc., **264** (1981), 507–519.

Received February 22, 1982.

THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C., CANADA V6T 1Y4

AND

THE UNIVERSITY OF MISSOURI-COLUMBIA
COLUMBIA, MO 65211

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR OGUS
University of California
Berkeley, CA 94720

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113

R. FINN and H. SAMELSON
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906–1982)

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

| | |
|--|-----|
| Wojciech Abramczuk , A class of surjective convolution operators | 1 |
| K. Adachi , Extending bounded holomorphic functions from certain subvarieties of a weakly pseudoconvex domain | 9 |
| Malvina Florica Baica , An algorithm in a complex field and its application to the calculation of units | 21 |
| Giuliana Bianchi and Robert Cori , Colorings of hypermaps and a conjecture of Brenner and Lyndon | 41 |
| Ronald James Evans , Determinations of Jacobsthal sums | 49 |
| Leslie Foged , Characterizations of \aleph -spaces | 59 |
| Nassif A. Ghoussoub and Paulette Saab , Weak compactness in spaces of Bochner integrable functions and the Radon-Nikodým property | 65 |
| J. Gómez Gil , On local convexity of bounded weak topologies on Banach spaces | 71 |
| Masaru Hara , On Gamelin constants | 77 |
| Wilfried Hauenschild, Eberhard Kaniuth and Ajay Kumar , Harmonic analysis on central hypergroups and induced representations | 83 |
| Eugenio Hernandez , An interpolation theorem for analytic families of operators acting on certain H^p spaces | 113 |
| Thomas Alan Keagy , On “Tauberian theorems via block-dominated matrices” | 119 |
| Thomas Landes , Permanence properties of normal structure | 125 |
| Daniel Henry Luecking , Closed ranged restriction operators on weighted Bergman spaces | 145 |
| Albert Thomas Lundell , The p -equivalence of $SO(2n + 1)$ and $Sp(n)$ | 161 |
| Mark D. Meyerson , Remarks on Fenn’s “the table theorem” and Zaks’ “the chair theorem” | 167 |
| Marvin Victor Mielke , Homotopically trivial toposes | 171 |
| Gerard J. Murphy , Hyperinvariant subspaces and the topology on $\text{Lat } A$... | 183 |
| Subhashis Nag , On the holomorphy of maps from a complex to a real manifold | 191 |
| Edgar Milan Palmer and Robert William Robinson , Enumeration of self-dual configurations | 203 |
| John J. Walsh and David Clifford Wilson , Continuous decompositions into cells of different dimensions | 223 |
| Walter John Whiteley , Infinitesimal motions of a bipartite framework | 233 |