ON LOCAL CONVEXITY OF BOUNDED WEAK TOPOLOGIES ON BANACH SPACES

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In this paper we prove that the bw topology on a Banach space $E$, i.e. the finest topology which agrees with the weak topology on bounded sets of $E$, is a locally convex topology if and only if the Banach space $E$ is reflexive.

1. Introduction. If $E$ is a Banach space, the bounded weak (bw) topology is the finest topology which agrees with the weak topology on bounded sets. Wheeler in [7, p. 251] proves that the bw topology on $c_0$ is not locally convex. This result gives a counterexample to a remark of Day [2, p. 42] which said that the bw topology is locally convex always. This fact suggests a question: Under what conditions on $E$ is it true that bw is a locally convex topology?. The theorem of Banach and Dieudonné (2.2) shows that reflexivity is a sufficient condition. In this paper we obtain that reflexivity is also a necessary condition.

2. Notations, definitions and preliminary results. The notations for topological vector spaces are taken primarily from [6], but we employ the definition of polarity found in [4].

DEFINITION 2.1. If $E$ is a locally convex space (lcs), the equicontinuous weak* (ew*) topology on $E'$ is the finest topology on $E'$ which coincides with the weak* topology $\sigma(E',E)$ on equicontinuous sets of $E'$.

The following result characterizes this topology when $E$ is a metrizable lcs.

THEOREM 2.2. (Banach-Dieudonné.) Let $E$ be a metrizable locally convex space and $E'$ its dual. The ew* topology on $E'$ is the topology of the uniform convergence on precompact subsets of $E$.

For a demonstration of this theorem we refer the reader to [4] or [6].
As an immediate consequence of this theorem we have that if $E$ is a metrizable lcs, the ew* topology on $E'$ is locally convex.
Other results about $\text{ew}^*$ may be found in [1] and [2].

**Definition 2.3.** If $E$ is a locally convex space, the bounded weak (bw) topology on $E$ is the finest topology on $E$ which agrees with the weak topology $\sigma(E, E')$ on bounded sets.

This definition is equivalent to Day's [2, p. 41]:

"The bw topology is the collection of all subsets $U$ of $E$ satisfying: for each bounded set $B$ of $E$, there is a $\sigma(E, E')$-open $V$ with $U \cap B = V \cap B$.”

Obviously the last definition is not changed if we choose the bounded sets on a fundamental family of bounded sets in $E$.

As follows from [1, p. 265], the bw topology is semi-linear, i.e. addition and scalar multiplication functions are separately continuous. Moreover, if $E$ is a Banach space, it can be shown [3, p. 21] that bw is a vectorial topology if and only if it is a locally convex one.

A general result of Collins [1, p. 266], which can be extended to the complex case, makes the following definition valid.

**Definition 2.4.** The convex bw (cbw) topology on a locally convex space $E$ is the unique locally convex topology with a base of all convex neighborhoods of 0 in the bw topology.

It is easy to see that the cbw topology is the finest locally convex topology which agrees with the weak topology on bounded sets. In [7, p. 251] may be found the following result which characterizes the cbw topology:

**Theorem 2.5.** If $E$ is a lcs, the cbw topology on $E$ is that of uniform convergence on compact subsets of the completion of $(E', \beta(E', E'))$.

As consequence of this result and Theorem 2.2 we obtain:

**Corollary 2.6.** If $E$ is a Banach space, the cbw topology is the restriction to $E$ of the ew* topology on $E''$.

In particular if $E$ is reflexive we have:

**Corollary 2.7.** If $E$ is a reflexive Banach space, the bw topology on $E$ is a locally convex topology.
A different introduction and other results about this topology may be seen in [3].

If $E$ is a Banach space, we denote by $B_1^E$, $B_2^E$, $S$ the closed unit ball of $E$ and $E''$ and the unit sphere of $E$, respectively, and we will write, for each $n \in \mathbb{N}$, $B_n = nB$, $B_n'' = nB''$ and $S_n = nS$.

3. The bw topology and local convexity.

**Lemma 3.1.** Let $E$ be a separable, non-reflexive Banach space. If $E$ contains no subspace isomorphic to $l^1$, there exists a subset $A$ of $E$ which is bw-closed but is not closed in the restriction to $E$ of the ew* topology on $E''$.

**Proof.** It is well known that for each $n \in \mathbb{N}$ the sphere $S_n$ is $\sigma(E'', E')$-dense in the closed ball $B_n''$. As $E$ is a separable Banach space that contains no subspace isomorphic to $l^1$ it follows from Rosenthal ([5], Theorem 3) that $S_n$ is $\sigma(E'', E')$-sequentially dense in the ball $B_n''$, i.e. each $z \in B_n''$ can be approximated in $\sigma(E'', E')$ by a sequence contained in $S_n$. Hence if $\phi \in E'' \setminus E$ and $\|\phi\| = 1$, there exists for each $n \in \mathbb{N}$ a sequence $(x_{k,n})_{k \in \mathbb{N}}$ contained in $S_n$ and converging to $n^{-1}\phi$ in $\sigma(E'', E')$.

We define $A = \{x_{k,n}: k, n \in \mathbb{N}\}$. For each $m \in \mathbb{N}$ we have

$$A \cap B_m = \{x_{k,n}: k \in \mathbb{N}, n \leq m\} = (\{x_{k,n}: k \in \mathbb{N}, n \leq m\} \cup \{n^{-1}\phi: n \leq m\}) \cap B_m,$$

and since the set $\{x_{k,n}: k \in \mathbb{N}, n \leq m\} \cup \{n^{-1}\phi: n \leq m\}$ is $\sigma(E'', E')$-compact, it is $\sigma(E'', E')$-closed; then the set $A \cap B_m$ is closed in the restriction of $\sigma(E'', E')$ to $B_m$, but this topology is the same as the restriction of $\sigma(E, E')$ to $B_m$. This proves that $A$ is bw-closed.

On the other hand, let $U$ be a neighborhood of $0$ in the ew* topology; there exists $W$, ew*-neighborhood of $0$ such that $W + W \subseteq U$, and as $W$ is absorbent, there exists $n_0 \in \mathbb{N}$ such that $n_0^{-1}\phi \in W$. By the definition of ew* topology we know there exists a $V$, $\sigma(E'', E')$-neighborhood of $0$ satisfying

$$W \cap B_1^E = V \cap B_1^E.$$

As $(x_{k,n})_{k \in \mathbb{N}}$ converges to $n^{-1}\phi$ in the $\sigma(E'', E')$-topology there exists $k_0 \in \mathbb{N}$ such that $x_{k_0,n_0} - n_0^{-1}\phi \in V$ and then

$$x_{k_0,n_0} = x_{k_0,n_0} - n_0^{-1}\phi + n_0^{-1}\phi \in (V \cap B_1^E) + W \subseteq W + W \subseteq U.$$

This proves that $0$ belongs to the closure of $A$ in the ew* topology, and since $0 \in E$, $0$ is in the closure of $A$ in the restriction of ew* to $E$ (we denote this topology rew*). Thus $A$ is not closed in rew*. □
**Proposition 3.2.** Let $E$ be a separable Banach space that contains no subspace isomorphic to $l^1$. The bw topology on $E$ is locally convex if and only if $E$ is reflexive.

*Proof.* If $E$ is reflexive we saw in (2.7) that bw is a locally convex topology. Conversely, if $E$ is not reflexive, (3.1) and (2.6) prove that the bw topology is not locally convex. □

**Lemma 3.3.** Let $E$ be a Banach space and $F$ a closed linear subspace of $E$. The bw topology of $F$ is the restriction to $F$ of the bw topology on $E$.

*Proof.* We denote bw$(E)$ and bw$(F)$ the bw topology on $E$ and $F$ respectively. It is clear that the restriction of bw$(E)$ to $F$ is coarser than bw$(F)$.

On the other hand, if $U$ is bw$(F)$-open, let $V$ be the union of $U$ and $E \setminus F$. It is sufficient to prove that $V$ is bw$(E)$-open. If $B$ is a bounded subset of $E$, as $\sigma(F, F')$ coincides with the restriction of $\sigma(E, E')$ to $F$, there exists $W, \sigma(E, E')$-open, such that

$$U \cap B = U \cap (B \cap F) = (W \cap F) \cap (B \cap F) = (W \cap F) \cap B,$$

and then

$$V \cap B = (W \cup (E \setminus F)) \cap B,$$

and since $W \cup (E \setminus F)$ is $\sigma(E, E')$-open, $V$ is bw$(E)$-open. □

**Proposition 3.4.** Let $E$ be a Banach space that contains no subspace isomorphic to $l^1$. The bw topology on $E$ is locally convex if and only if $E$ is reflexive.

*Proof.* If $E$ is not reflexive, there exists a separable nonreflexive subspace $F$ of $E$. Obviously $F$ contains no subspace isomorphic to $l^1$. From (3.2) it follows that the bw topology on $F$ is not locally convex and then (3.3) shows that bw is not a locally convex topology on $E$. This fact and (2.7) prove the theorem. □

**Lemma 3.5.** There exists a subset $A$ of $l^1$ which is bw-closed but is not closed in the restriction to $l^1$ of the ew*-topology of $(l^1)''$.

*Proof.* For each $n \in \mathbb{N}$, we denote by $e_n$ the sequence of $l^1(0,0,\ldots,1,0,\ldots)$ where the one is in the $n$th place.
Let $A_0$ be the set $A_0 = \{e_n: n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, $e_n$ is a $\sigma(l^1, l^\infty)$-isolated point of $A_0$, and then $A_0$ is a $\sigma(l^1, l^\infty)$-closed set which is not $\sigma(l^1, l^\infty)$-compact. Consequently $A_0$ is not $\sigma((l^1)'', l^\infty)$-closed. Thus there exists a linear form $\phi$ that belongs to the $\sigma((l^1)'', l^\infty)$-closure of $A_0$ and $\phi \not\in A_0$. Obviously $\|\phi\| \leq 1$ and $\phi \in (l^1)'' \setminus l^1$.

Now, for each $n \in \mathbb{N}$, we define

$$A_n = \{ne_p - e_q + e_k/n: p, q, k \in \mathbb{N}, p \neq q, p < k, q < k\}.$$  

$A_n$ is contained in the sphere of radius $2n + 1/n$ of $l^1$ and if $n \in \mathbb{N}$, $n \geq 2$, it is not hard to check that $A_n$ is $\sigma(l^1, l^\infty)$-closed.

Let $V$ be a balanced, convex $\sigma((l^1)'', l^\infty)$-neighborhood of $0$. As $\phi$ is an accumulation point of $A_0$, $\phi + V/3n$ contains an infinite number of points of $A_0$. If $e_p, e_q, e_k \in (\phi + V/3n) \cap A_0$ with $p < q < k$, we have

$$\left[n(e_p - e_q) + \frac{1}{n}e_k\right] - \frac{1}{n}\phi = n(e_p - \phi) + n(\phi - e_q) + \frac{1}{n}(e_k - \phi) \in \frac{1}{3}V + \frac{1}{3}V + \frac{1}{3n^2}V \subseteq V.$$  

Thus $(\phi/n + V) \cap A_n$ is a nonempty set. This proves that $\phi/n$ belongs to the $\sigma((l^1)'', l^\infty)$-closure of $A_n$.

Now, we define $A = \bigcup_{n=2}^\infty A_n$. For each $m \in \mathbb{N}$, we have

$$A \cap B_m = \bigcup \{A_n: n \in \mathbb{N}, 5 \leq 2n^2 + 1 \leq mn\}. $$

This set is obviously $\sigma(l^1, l^\infty)$-closed, and thus $A$ is a bw-closed set.

On the other hand, as for each $n \in \mathbb{N}$, $\phi/n$ belongs to the $\sigma((l^1)'', l^\infty)$-closure of $A_n$; reasoning as in the last part of the proof of Lemma 3.1 proves that $0$ belongs to the closure on $\text{ew}^*$ of $A$, and as $0$ does not belong to $A$, we see that $A$ is not closed in the topology restriction to $l^1$ of $\text{ew}^*$ on $(l^1)''$.  

**Proposition 3.6.** Let $E$ be a Banach space that contains a subspace isomorphic to $l^1$. Then $\text{bw}$ is not a locally convex topology on $E$.

**Proof.** From (3.5) we get that the bw topology on $l^1$ is not locally convex. Hence if $E$ contains a subspace isomorphic to $l^1$, Lemma 3.3 and the conservation of bw topologies under isomorphisms prove that bw is not a locally convex topology on $E$.

**Theorem 3.7.** Let $E$ be a Banach space. $\text{bw}$ is a locally convex topology on $E$ if and only if $E$ is reflexive.
Proof. If $E$ is reflexive, (2.7) gives us the result. Conversely, if $bw$ is a locally convex topology on $E$, from (3.6) it follows that $E$ contains no subspace isomorphic to $l^1$ and (3.4) shows that $E$ must be reflexive.

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