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ON STRONGLY DECOMPOSABLE OPERATORS

I. ERDÉLYI AND SHENG-WANG WANG

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A strongly decomposable operator, as defined by C. Apostol, is a bounded linear operator T which, for every spectral maximal space Y , induces two decomposable operators: the restriction $T|Y$ and the coinduced T/Y on the quotient space X/Y . In this paper, we give some necessary and sufficient conditions for an operator to be strongly decomposable.

Throughout the paper, T is a bounded linear operator acting on an abstract Banach space X over the field \mathbf{C} of complex numbers. T^* denotes the conjugate of T on the dual space X^* . For a set S , S^c is the complement, \bar{S} is the closure, \bar{S}^w is the weak*-closure in X^* , S^\perp is the annihilator of $S \subset X$ in X^* , ${}^\perp S$ is the annihilator of $S \subset X^*$ in X and $\text{Int } S$ represents the interior of S . We write $\sigma(T)$ for the spectrum, $\rho(T)$ for the resolvent set of T and $R(\cdot; T)$ for the resolvent operator. If T is endowed with the single valued extension property (SVEP), then for any $x \in X$, $\sigma_T(x)$ denotes the local spectrum. For $S \subset \mathbf{C}$, we shall extensively use the spectral manifold

$$X_T(S) = \{x \in X: \sigma_T(x) \subset S\}.$$

We say that T satisfies condition α , if

(a) T has the SVEP, and (b) $X_T(F)$ is closed for every closed $F \subset \mathbf{C}$.

Two special types of subspaces, invariant under the given operator T , enter the theory of decomposable operators: (1) spectral maximal spaces [7]; (2) analytically invariant subspaces [9].

1. PROPOSITION. *Let Y be a spectral maximal space of T .*

(i) [9, Proposition 1] *If T has the SVEP then, for any $x \in X$,*

$$(1) \quad \sigma_T(x) = [\sigma_T(x) \cap \sigma(T|Y)] \cup \sigma_{\hat{T}}(\hat{x}), \quad \hat{x} = x + Y, \hat{T} = T/Y.$$

(ii) [2, Lemma 1.4]. *If T is decomposable, then*

$$(2) \quad \sigma(T/Y) = \overline{\sigma(T) - \sigma(T|Y)}.$$

(iii) [7, Theorem 2.3]. *If T satisfies condition α , then $Y = X_T[\sigma(T|Y)]$.*

(iv) [3, Proposition I.3.2]. *If $Z \subset Y$ is a spectral maximal space of T , then Y/Z is a spectral maximal space of T/Z .*

(v) [7, Lemma 2.1]. If T is decomposable and $G \subset \mathbf{C}$ is open, then $\sigma(T) \cap G \neq \emptyset$ implies that $X_T(\overline{G}) \neq \{0\}$.

(vi) [7, Theorem 2.3]. If T satisfies condition α , then for every closed $F \subset \mathbf{C}$, $X_T(F)$ is a spectral maximal space of T and

$$(3) \quad \sigma[T|X_T(F)] \subset F.$$

(vii) [12, Corollary 1(c)]. For T decomposable and for any closed $F \subset \mathbf{C}$,

$$\sigma[T/X_T(F)] \subset (\text{Int } F)^c.$$

(viii) [8, Theorem 1]. If T is decomposable then, for every closed $F \subset \mathbf{C}$, $X_T(F^c)^\perp$ is a spectral maximal space of T^* and $X_T(F^c)^\perp = X_{T^*}^*(F)$.

(ix) [9, Theorem 2]. If T has the SVEP, then Y is analytically invariant under T .

REMARK. More generally than in the original versions, properties (iii) and (vi) hold without the restriction of T being decomposable.

2. PROPOSITION. Let Y be an analytically invariant subspace under T . Then

(i) [9, Theorem 1]. T/Y has the SVEP (the converse property is also true).

(ii) [4, Lemma 3.4]. If T has the SVEP then, for every $y \in Y$,

$$\sigma_{T|Y}(y) = \sigma_T(y).$$

(iii) [9, Theorem 3]. If T is decomposable then, for every open $G \subset \mathbf{C}$, $X_T(\overline{G})$ is analytically invariant under T .

3. THEOREM. The following assertions are equivalent:

(i) T is strongly decomposable;

(ii) (a) T satisfies condition α ;

(b) for every spectral maximal space Y of T and any $x \in X$,

$$(4) \quad \sigma_{\hat{T}}(\hat{x}) = \overline{\sigma_T(x) - \sigma(T|Y)}, \quad \hat{T} = T/Y, \hat{x} = x + Y;$$

(c) for every special maximal space Y of T and any open $G \subset \mathbf{C}$, $G \cap \sigma(T|Y) \neq \emptyset$ implies that $X_T[\overline{G} \cap \sigma(T|Y)] \neq \{0\}$.

Proof. (i) \Rightarrow (ii). (a) is evident. (b). (1) implies

$$\sigma_{\hat{T}}(\hat{x}) \supset \sigma_T(x) - [\sigma_T(x) \cap \sigma(T|Y)] = \sigma_T(x) - \sigma(T|Y)$$

and hence

$$\sigma_{\hat{T}}(\hat{x}) \supset \overline{\sigma_T(x) - \sigma(T|Y)}.$$

To obtain the opposite inclusion, for $x \in X$, put

$$(5) \quad F = \sigma_T(x) \cup \sigma(T|Y)$$

and for the decomposable $T|X_T(F)$ use (2) and (3) as follows:

$$\begin{aligned} \sigma[\hat{T}|X_T(F)/Y] &= \overline{\sigma[T|X_T(F)] - \sigma(T|Y)} \subset \overline{F - \sigma(T|Y)} \\ &= \overline{\sigma_T(x) - \sigma(T|Y)}. \end{aligned}$$

By (5), $x \in X_T(F)$ and hence $\hat{x} = x + Y \in X_T(F)/Y$. Consequently,

$$\sigma_{\hat{T}}(\hat{x}) \subset \sigma[\hat{T}|X_T(F)/Y] \subset \overline{\sigma_T(x) - \sigma(T|Y)}$$

and this establishes (4).

Since $T|Y$ is decomposable, (c) is a consequence of Proposition 1 (v).

(ii) \Rightarrow (i): Let Y be a spectral maximal space of T . By (a) and Proposition 1 (iii), Y has a representation $Y = X_T[\sigma(T|Y)]$.

Let $G \subset \mathbf{C}$ be open and put $Z = X_T(\bar{G})$. We shall prove inclusion

$$(6) \quad \overline{G \cap \sigma(T|Y)} \subset \sigma(T|Y \cap Z).$$

If $G \cap \sigma(T|Y) = \emptyset$, then (6) is evident. Therefore, assume

$$G \cap \sigma(T|Y) \neq \emptyset.$$

Let $\lambda_0 \in G \cap \sigma(T|Y)$ and let $\delta_0 \subset G$ be a neighborhood of λ_0 . Then, since $\delta_0 \cap (T|Y) \neq \emptyset$, (c) implies that $X_T[\bar{\delta}_0 \cap \sigma(T|Y)] \neq \{0\}$ and hence

$$\sigma(T|X_T[\bar{\delta}_0 \cap \sigma(T|Y)]) \neq \emptyset.$$

Let $\lambda_1 \in \sigma(T|X_T[\bar{\delta}_0 \cap \sigma(T|Y)])$. Then $\lambda_1 \in \bar{\delta}_0$ and it follows from

$$X_T[\bar{\delta}_0 \cap \sigma(T|Y)] \subset X_T[\bar{G} \cap \sigma(T|Y)] = X_T[\sigma(T|Y)] \cap Z = Y \cap Z$$

that $\lambda_1 \in \bar{\delta}_0 \cap \sigma(T|Y \cap Z)$. Thus,

$$\bar{\delta}_0 \cap \sigma(T|Y \cap Z) \neq \emptyset$$

and since δ_0 is an arbitrary neighborhood of λ_0 , we must have $\lambda_0 \in \sigma(T|Y \cap Z)$. By the definition of λ_0 , inclusion (6) holds. Finally, we shall conclude the proof by showing that $T|Y$ is decomposable. The subspace $W = Y \cap Z$ is a spectral maximal space of T . By denoting $\tilde{T} = T/W$ and for $x \in Y$, $\tilde{x} = x + W$, with the help of condition (b) and inclusion (6),

we obtain successively

$$(7) \quad \sigma_{\hat{T}}(\tilde{x}) = \overline{\sigma_T(x) - \sigma(T|W)} \subset \overline{\sigma_T(x) - [G \cap \sigma(T|Y)]} \\ \subset \overline{\sigma(T|Y) - [G \cap \sigma(T|Y)]} = \overline{\sigma(T|Y) - G} \subset G^c.$$

Since Y is a spectral maximal space of T and W is a spectral maximal space of $T|Y$, Proposition 1 (iv) implies Y/W is a spectral maximal space of T/W . Then, with the help of (7) and [13, Theorem 1.1 (g)], we obtain

$$\sigma[\hat{T}|(Y/W)] = \bigcup_{\tilde{x} \in Y/W} \sigma_{\hat{T}}(\tilde{x}) \subset G^c.$$

Consequently, $T|Y$ is decomposable by [5, Theorem 12] and [1] (or [11]), (see also [10]). □

If one slightly strengthens condition (b) in Theorem 3, then (c) becomes redundant.

4. THEOREM. *The following assertions are equivalent:*

- (I) T is strongly decomposable;
- (II) (A) T satisfies condition α ;
- (B) for every closed $F \subset \mathbf{C}$, and each $x \in X$,

$$(8) \quad \sigma_{\hat{T}}(\hat{x}) = \overline{\sigma_T(x) - F}$$

where $\hat{T} = T/X_T(F)$, $\hat{x} = x + X_T(F)$.

- (III) (A) T satisfies condition α ;
- (C) For every pair F_1, F_2 of closed sets in \mathbf{C} ,

$$(9) \quad \sigma[(T/Y_2)|X_T(F_1 \cup F_2)/Y_2] \subset F_1, \quad \text{where } Y_2 = X_T(F_2).$$

Proof. (I) \Rightarrow (III). Let F_1, F_2 be closed in \mathbf{C} . Since T is strongly decomposable, $T|X_T(F_1 \cup F_2)$ is decomposable. Let G_1, G_2 be open sets in \mathbf{C} such that $F_1 \cup F_2 \subset G_1 \cup G_2$, $F_1 \subset G_1$ and $\overline{G_2} \cap F_1 = \emptyset$. For $x \in X_T(F_1 \cup F_2)$, we have a representation

$$x = x_1 + x_2 \quad \text{with } x_i \in X_T(F_1 \cup F_2) \cap X_T(\overline{G_i}), i = 1, 2.$$

It follows from

$$\sigma_T(x_2) \subset (F_1 \cup F_2) \cap \overline{G_2} = F_2 \cap \overline{G_2} \subset F_2$$

that $x_2 \in X_T(F_2) = Y_2$.

Let $\lambda_0 \notin \overline{G_1}$. Then $\lambda_0 \in \rho(T|X_T[(F_1 \cup F_2) \cap \overline{G_1}])$ and hence there is $y \in X_T[(F_1 \cup F_2) \cap \overline{G_1}]$ verifying

$$(\lambda_0 - T)y = x_1.$$

By the natural homomorphism $X \rightarrow X/Y_2$, we obtain

$$(\lambda_0 - T/Y_2)\hat{y} = \hat{x}_1 = \hat{x},$$

and hence $\lambda_0 - (T/Y_2) | X_T(F_1 \cup F_2)/Y_2$ is surjective. Since T/Y_2 has the SVEP by Proposition 1 (vi), (ix) and Proposition 2 (i), we have $\lambda_0 \in \rho[(T/Y_2) | X_T(F_1 \cup F_2)/Y_2]$ by [6, Theorem 2]. By the definition of λ_0 , we have

$$\sigma[(T/Y_2) | X_T(F_1 \cup F_2)/Y_2] \subset \overline{G}_1$$

and since $G_1 \supset F_1$ is arbitrary, inclusion (9) holds.

(III) \Rightarrow (II): Let $x \in X$ and $F \subset \mathbb{C}$ be closed. For $F_1 = \overline{\sigma_T(x) - F}$ and $Y = X_T(F)$, (9) implies

$$\sigma[(T/Y) | X_T(F_1 \cup F)/Y] \subset F_1 = \overline{\sigma_T(x) - F}.$$

It follows from the definition of F_1 that $x \in X_T(F_1 \cup F)$. Consequently, for $\hat{x} = x + Y$ and $\hat{T} = T/Y$, we have

$$\sigma_{\hat{T}}(\hat{x}) \subset \sigma[\hat{T} | X_T(F_1 \cup F)/Y] \subset \overline{\sigma_T(x) - F}.$$

On the other hand, it follows from Proposition 1 (i) that

$$\sigma_{\hat{T}}(\hat{x}) \supset \overline{\sigma_T(x) - \sigma(T|Y)} \supset \overline{\sigma_T(x) - F}$$

and hence (8) holds.

(II) \Rightarrow (I). In view of Theorem 3, we only have to prove that, for every open G and spectral maximal space $Y = X_T[\sigma(T|Y)]$,

$$(10) \quad G \cap \sigma(T|Y) \neq \emptyset$$

implies that $X_T[\overline{G} \cap \sigma(T|Y)] \neq \{0\}$. Choose an open G verifying (10), denote $Z = X_T[\overline{G} \cap \sigma(T|Y)]$ and for $x \in X$, let $\tilde{x} = x + Z$. If $Z = \{0\}$, then

$$(11) \quad \sigma_{\tilde{T}}(\tilde{x}) = \sigma_T(x), \quad \tilde{T} = T/Z.$$

In view of (11), by hypothesis, we have

$$\begin{aligned} \sigma_T(x) &= \sigma_{\tilde{T}}(\tilde{x}) = \overline{\sigma_T(x) - [\overline{G} \cap \sigma(T|Y)]} \\ &= \overline{[\sigma_T(x) - \overline{G}] \cup [\sigma_T(x) - \sigma(T|Y)]}. \end{aligned}$$

Let $x \in Y$. Since $\sigma_T(x) \subset \sigma(T|Y)$, we have

$$\sigma_T(x) = \overline{\sigma_T(x) - \overline{G}}$$

and hence

$$\sigma_T(x) \cap G = \emptyset.$$

Now, with the help of [13, Theorem 1.1 (g)], Proposition 1 (v), (ix) and Proposition 2 (ii), we obtain

$$\begin{aligned}\sigma(T|Y) \cap G &= \left[\bigcup_{x \in Y} \sigma_{\eta_Y}(x) \right] \cap G = \left[\bigcup_{x \in Y} \sigma_T(x) \right] \cap G \\ &= \bigcup_{x \in Y} [\sigma_T(x) \cap G] = \emptyset.\end{aligned}$$

But this contradicts hypothesis (10). Therefore, $Z = X_T[\overline{G} \cap \sigma(T|Y)] \neq \{0\}$. \square

Next, we shall obtain a characterization of a strongly decomposable operator in terms of the conjugate operator. First, we need some preparation.

5. LEMMA. *Given T , let Y and Z be invariant subspaces of X with $Z \subset Y$. Then*

$$(12) \quad (T/Z)^*|(Y/Z)^\perp \cong T^*|Y^\perp.$$

Proof. The mapping $X/Z \rightarrow X/Y$ is a continuous surjective homomorphism with kernel Y/Z . Therefore, the quotient spaces $(X/Z)/(Y/Z)$ and X/Y are isomorphic. Given $x \in X$, we use the following notations for the equivalent classes containing x in the corresponding quotient spaces: $\hat{x} \in X/Y$, $\tilde{x} \in X/Z$, $\tilde{\tilde{x}} \in (X/Z)/(Y/Z)$. Note that $u \in \hat{x}$ iff $u - x \in Y$ iff $(u - x) \sim \tilde{u} \in Y/Z$ iff $\tilde{u} \in \tilde{\tilde{x}}$. Since

$$\inf_{v \in \tilde{u}} \|v\| \leq \|u\|,$$

we have

$$(13) \quad \|\tilde{\tilde{x}}\| = \inf_{\tilde{u} \in \tilde{\tilde{x}}} \|\tilde{u}\| = \inf_{\tilde{u} \in \tilde{\tilde{x}}} \inf_{v \in \tilde{u}} \|v\| \leq \inf_{u \in \hat{x}} \|u\| = \|\hat{x}\|.$$

On the other hand, for every $u \in \hat{x}$, $\tilde{u} = u + Z \subset u + Y = \hat{x}$ and hence $\tilde{u} \subset \hat{x}$. Thus,

$$\inf_{v \in \tilde{u}} \|v\| \geq \|\hat{x}\|$$

and hence

$$(14) \quad \|\tilde{\tilde{x}}\| = \inf_{\tilde{u} \in \tilde{\tilde{x}}} \inf_{v \in \tilde{u}} \|v\| \geq \|\hat{x}\|.$$

Then, by (13) and (14), $\|\tilde{\tilde{x}}\| = \|\hat{x}\|$. Thus, it follows from the isometrical isomorphisms

$$(X/Y)^* \cong Y^\perp, \quad [(X/Z)/(Y/Z)]^* \cong (Y/Z)^\perp$$

that the unitary equivalence (12) holds. \square

6. LEMMA. *If T is decomposable then, for every open $G \subset \mathbb{C}$,*

$$(15) \quad X_T(G^c)^\perp = \overline{X_{T^*}^*(G)}^w.$$

Proof. Let T be decomposable. By [14], for every closed $F \subset \mathbb{C}$,

$$(16) \quad JX_T(F) = JX \cap X_{T^{**}}(F)$$

where J is the natural imbedding of X into X^{**} . By Proposition 1 (viii) and the fact that T decomposable implies T^* decomposable,

$$(17) \quad X_{T^{**}}(F) = X_{T^*}^*(F^c)^\perp.$$

Relations (16) and (17) imply

$$X_T(F) = {}^\perp X_{T^*}^*(F^c)$$

and hence, for $F = G^c$, (15) follows. □

7. LEMMA. *If T^* is decomposable then, for every open $G \subset \mathbb{C}$, $\overline{X_{T^*}^*(G)}^w$ (i.e. the weak*-closure of $X_{T^*}^*(G)$) is analytically invariant under T^* .*

Proof. Let $f^*: D \rightarrow X^*$ be analytic on an open $D \subset \mathbb{C}$ and verify condition

$$(\lambda - T^*)f^*(\lambda) \in \overline{X_{T^*}^*(G)}^w \quad \text{on } D.$$

We may assume D is connected. Put $F = G^c$, $Y = X_T(F)$, use Lemma 6, Proposition 1 (vii) and obtain successively

$$\sigma[T^* | \overline{X_{T^*}^*(G)}^w] = \sigma(T | Y^\perp) = \sigma[(T/Y)^*] = \sigma(T/Y) \subset (\text{Int } F)^c = \overline{G}.$$

First, assume $D \subset \overline{G}$. Then $D \subset G \subset \rho(T | Y)$ and, for every $x \in Y$, $\lambda \in D$, we have

$$\begin{aligned} \langle x, f^*(\lambda) \rangle &= \langle (\lambda - T)R(\lambda; T | Y)x, f^*(\lambda) \rangle \\ &= \langle R(\lambda; T | Y)x, (\lambda - T^*)f^*(\lambda) \rangle = 0. \end{aligned}$$

Since $x \in Y$ is arbitrary, $f^*(\lambda) \in Y^\perp = \overline{X_{T^*}^*(G)}^w$ on D .

Next, assume $D \not\subset \overline{G}$. Then, for $\lambda \in D - \overline{G}$, the resolvent operator $R[\lambda; T^* | \overline{X_{T^*}^*(G)}^w]$ is defined, and for $h^*(\lambda) = (\lambda - T^*)f^*(\lambda)$ we have

$$(\lambda - T^*)\{f^*(\lambda) - R[\lambda; T^* | \overline{X_{T^*}^*(G)}^w]h^*(\lambda)\} = 0.$$

Since T^* has the SVEP,

$$f^*(\lambda) = R[\lambda; T^* | \overline{X_{T^*}^*(G)}^w]h^*(\lambda) \in \overline{X_{T^*}^*(G)}^w$$

on $D - \overline{G}$, and $f^*(\lambda) \in \overline{X_{T^*}^*(G)}^w$ on D , by analytic continuation. □

8. THEOREM. *The bounded operator T (resp. T^*) is strongly decomposable iff:*

- (i) *T (resp. T^*) has the SVEP and for open $G \subset \mathbf{C}$, $T^* | \overline{X_{T^*}^*(G)}^w$ (resp. $T | \overline{X_T(G)}$) is decomposable;*
- (ii) *for every pair G, H of open sets in \mathbf{C} ,*

$$(18) \quad \overline{X_{T^*}^*(G \cap H)}^w = \overline{Y_{T^*Y^*}^*(H)}^w \quad (\text{resp. } \overline{X_T(G \cap H)} = \overline{Y_{TY}(H)}),$$

where $Y^* = \overline{X_{T^*}^*(G)}^w$ (resp. $Y = \overline{X_T(G)}$).

Proof. We confine the proof to the operator T , the proof concerning T^* being similar.

(only if): Assume T is strongly decomposable. Let $G \subset \mathbf{C}$ be open, $F = G^c$ and $Z = X_T(F)$. The operator $(T/Z) | (X/Z)$ is decomposable. Then, by Lemma 6, $X_T(F)^\perp = \overline{X_{T^*}^*(G)}^w$ and hence

$$(19) \quad (X/Z)^* \cong \overline{X_{T^*}^*(G)}^w.$$

By [8, Theorem 2] and [12], $T^* | \overline{X_{T^*}^*(G)}^w$ is decomposable. Apply Lemma 5 to a closed $F_1 \supset F$, and obtain

$$(20) \quad [X_T(F_1)/Z]^\perp \cong X_T(F_1)^\perp.$$

Denote $\tilde{T} = T/Z$, $\tilde{X} = X/Z$. Before embarking on the proof of (ii), we shall show that

$$(21) \quad \tilde{X}_{\tilde{T}}(\overline{F_1 - F}) = X_T(F_1)/Z.$$

In fact, if $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$, then $\sigma_{\tilde{T}}(\tilde{x}) \subset \overline{F_1 - F}$ and hence, for every $x \in \tilde{x}$,

$$\sigma_T(x) \subset (\overline{F_1 - F}) \cup F = F_1.$$

Therefore, $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$ implies $x \in X_T(F_1)$ and hence $\tilde{x} \in X_T(F_1)/Z$. Conversely, if $\tilde{x} \in X_T(F_1)/Z = X_T(\overline{F_1 - F} \cup F)/Z$, then Theorem 4 (III, C) implies

$$\sigma_{\tilde{T}}(\tilde{x}) \subset \sigma[\tilde{T} | X_T(\overline{F_1 - F} \cup F)/Z] \subset \overline{F_1 - F}$$

and hence $\tilde{x} \in \tilde{X}_{\tilde{T}}(\overline{F_1 - F})$. Thus (21) is proved.

Now we are in a position to prove (ii). To simplify notation, put $X^* = (\tilde{X})^*$ and $T^* = (\tilde{T})^*$. Let H be open and let $F_1 = G^c \cup H^c$. Then $F_1 \supset F$ and $\overline{F_1 - F} \subset H^c$. By Lemma 6, Lemma 5, (20), (21) and (19), we obtain successively:

$$\begin{aligned} \overline{X_{T^*}^*(G \cap H)}^w &= X_T(F_1)^\perp \cong [X_T(F_1)/Z]^\perp = \tilde{X}_{\tilde{T}}(\overline{F_1 - F})^\perp \supset [\tilde{X}_{\tilde{T}}(H^c)]^\perp \\ &= \overline{X_{T^*}^*(H)}^w = \overline{Y_{T^*Y^*}^*(H)}^w. \end{aligned}$$

For the last equality, we used the equivalence

$$T^* = [T/X_T(F)]^* \cong T^* | \overline{X_{T^*}^*(G)}^w = T^* | Y^*.$$

To obtain the opposite inclusion, note that if $x^* \in X_{T^*}^*(G \cap H)$, then

$$\sigma_{T^*}(x^*) = \subset G \cap H \subset G$$

and hence $x^* \in X_{T^*}^*(G) \subset Y^*$. Since Y^* is analytically invariant under T^* (Lemma 7), in view of Proposition 2 (ii), we obtain

$$\sigma_{T^*|Y^*}(x^*) = \sigma_{T^*}(x^*) \subset H$$

and hence

$$x^* \in Y_{T^*|Y^*}^*(H) \subset \overline{Y_{T^*|Y^*}^*(H)}^w.$$

Thus

$$\overline{X_{T^*}^*(G \cap H)}^w \subset \overline{Y_{T^*|Y^*}^*(H)}^w.$$

(if): Assume conditions (i) and (ii) are satisfied. Let $F, F_1 \subset \mathbb{C}$ be closed. Since $X_{T^*}^*(\mathbb{C}) = X^*$, we conclude that T^* is decomposable and hence T is decomposable by [14, Corollary 2.8]. Therefore, $Z = X_T(F)$ is closed. Also $T^* | \overline{X_{T^*}^*(F^c)}^w$ is decomposable. Then, by Lemma 6,

$$T^* | \overline{X_{T^*}^*(F^c)}^w = T^* | X_T(F)^\perp \cong T^*,$$

where $\tilde{T} = T/Z$ and $T^* = (\tilde{T})^*$. Thus T^* is decomposable and hence \tilde{T} is decomposable. Therefore, letting $\tilde{X} = X/Z$, $\tilde{X}_{\tilde{T}}(F_1)$ is closed and

$$(22) \quad \sigma[\tilde{T} | \tilde{X}_{\tilde{T}}(F_1)] \subset F_1.$$

Put $G = F^c$, $H = F_1^c$ and $Y^* = \overline{X_{T^*}^*(G)}^w$. It follows from Lemma 6 that

$$T^* | X_T(F \cup F_1)^\perp = T^* | \overline{X_{T^*}^*(G \cap H)}^w,$$

$$T^* | \tilde{X}_{\tilde{T}}(F_1)^\perp \cong T^* | \overline{Y_{T^*|Y^*}^*(H)}^w.$$

Then (18) implies

$$(23) \quad T^* | \tilde{X}_{\tilde{T}}(F_1)^\perp \cong T^* | X_T(F \cup F_1)^\perp.$$

By Lemma 5 we have

$$(24) \quad T^* | [X_T(F \cup F_1)/Z]^\perp \cong T^* | X_T(F \cup F_1)^\perp.$$

Consequently, with the help of (24), (23) and (22), we obtain

$$\begin{aligned} \sigma[T^* | X_T(F \cup F_1)/Z] &= \sigma\{T^* | [X_T(F \cup F_1)/Z]^\perp\} = \sigma[T^* | \tilde{X}_{\tilde{T}}(F_1)^\perp] \\ &= \sigma[\tilde{T} | \tilde{X}_{\tilde{T}}(F_1)] \subset F_1. \end{aligned}$$

Thus, conditions (III) of Theorem 4 are satisfied and hence T is strongly decomposable. \square

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Vol. 110, No. 2

October, 1984

Robert A. Bekes , The range of convolution operators	257
Dennis K. Burke and Sheldon Davis , Subsets of ${}^\omega\omega$ and generalized metric spaces	273
Giovanni Emmanuele , A remark on a paper: "Common fixed points of nonexpansive mappings by iteration"	283
I. Erdélyi and Sheng-Wang Wang , On strongly decomposable operators ...	287
Gerhard Gierz , Injective Banach lattices with strong order units	297
Maurizio Letizia , Quotients by complex conjugation of nonsingular quadrics and cubics in \mathbf{P}_C^3 defined over \mathbf{R}	307
P. H. Maserick and Franciszek Hugon Szafraniec , Equivalent definitions of positive definiteness	315
Costel Peligrad and S. Rubinstein , Maximal subalgebras of C^* -crossed products	325
Derek W. Robinson and Sadayuki Yamamuro , Hereditary cones, order ideals and half-norms	335
Derek W. Robinson and Sadayuki Yamamuro , The Jordan decomposition and half-norms	345
Richard Rochberg , Interpolation of Banach spaces and negatively curved vector bundles	355
Dale Rolfsen , Rational surgery calculus: extension of Kirby's theorem	377
Walter Iaan Seaman , Helicoids of constant mean curvature and their Gauss maps	387
Diana Shelstad , Endoscopic groups and base change \mathbf{C}/\mathbf{R}	397
Jerrold Norman Siegel and Frank Williams , Numerical invariants of homotopies into spheres	417
Alladi Sitaram , Some remarks on measures on noncompact semisimple Lie groups	429
Teruhiko Soma , Atoroidal, irreducible 3-manifolds and 3-fold branched coverings of S^3	435
Jan de Vries , On the G -compactification of products	447
Hans Weber , Topological Boolean rings. Decomposition of finitely additive set functions	471