ON STRONGLY DECOMPOSABLE OPERATORS

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A strongly decomposable operator, as defined by C. Apostol, is a bounded linear operator $T$ which, for every spectral maximal space $Y$, induces two decomposable operators: the restriction $T|Y$ and the coinduced $T/Y$ on the quotient space $X/Y$. In this paper, we give some necessary and sufficient conditions for an operator to be strongly decomposable.

Throughout the paper, $T$ is a bounded linear operator acting on an abstract Banach space $X$ over the field $\mathbb{C}$ of complex numbers. $T^*$ denotes the conjugate of $T$ on the dual space $X^*$. For a set $S$, $S^c$ is the complement, $\overline{S}$ is the closure, $S^w$ is the weak*-closure in $X^*$, $S^\perp$ is the annihilator of $S \subset X$ in $X^*$, $\overline{\perp}S$ is the annihilator of $S \subset X^*$ in $X$ and $\text{Int } S$ represents the interior of $S$. We write $\sigma(T)$ for the spectrum, $\rho(T)$ for the resolvent set of $T$ and $R(\cdot; T)$ for the resolvent operator. If $T$ is endowed with the single valued extension property (SVEP), then for any $x \in X$, $\sigma_T(x)$ denotes the local spectrum. For $S \subset \mathbb{C}$, we shall extensively use the spectral manifold

$$X_T(S) = \{x \in X : \sigma_T(x) \subset S\}.$$

We say that $T$ satisfies condition $\alpha$, if

(a) $T$ has the SVEP, and (b) $X_T(F)$ is closed for every closed $F \subset \mathbb{C}$.

Two special types of subspaces, invariant under the given operator $T$, enter the theory of decomposable operators: (1) spectral maximal spaces [7]; (2) analytically invariant subspaces [9].

1. Proposition. Let $Y$ be a spectral maximal space of $T$.

(i) [9, Proposition 1] If $T$ has the SVEP then, for any $x \in X$,

$$\sigma_T(x) = [\sigma_T(x) \cap \sigma(T|Y)] \cup \sigma_T(\hat{x}), \quad \hat{x} = x + Y, \hat{T} = T/Y.$$  

(ii) [2, Lemma 1.4]. If $T$ is decomposable, then

$$\sigma(T/Y) = \overline{\sigma(T) - \sigma(T|Y)}.$$  

(iii) [7, Theorem 2.3]. If $T$ satisfies condition $\alpha$, then $Y = X_T[\sigma(T|Y)]$.

(iv) [3, Proposition 1.3.2]. If $Z \subset Y$ is a spectral maximal space of $T$, then $Y/Z$ is a spectral maximal space of $T/Z$.
(v) [7, Lemma 2.1]. If \( T \) is decomposable and \( G \subset \mathbb{C} \) is open, then \( \sigma(T) \cap G \neq \emptyset \) implies that \( X_T(G) \neq \{0\} \).

(vi) [7, Theorem 2.3]. If \( T \) satisfies condition \( \alpha \), then for every closed \( F \subset \mathbb{C} \), \( X_T(F) \) is a spectral maximal space of \( T \) and

\[
\sigma[T|X_T(F)] \subset F.
\]

(vii) [12, Corollary 1(c)]. For \( T \) decomposable and for any closed \( F \subset \mathbb{C} \),

\[
\sigma[T/X_T(F)] \subset (\text{Int } F)^c.
\]

(viii) [8, Theorem 1]. If \( T \) is decomposable then, for every closed \( F \subset \mathbb{C} \), \( X_T(F^c)^\perp \) is a spectral maximal space of \( T^* \) and \( X_T(F^c)^\perp = X_{T^*}(F) \).

(ix) [9, Theorem 2]. If \( T \) has the SVEP, then \( Y \) is analytically invariant under \( T \).

**Remark.** More generally than in the original versions, properties (iii) and (vi) hold without the restriction of \( T \) being decomposable.

2. **Proposition.** Let \( Y \) be an analytically invariant subspace under \( T \). Then

(i) [9, Theorem 1]. \( T/Y \) has the SVEP (the converse property is also true).

(ii) [4, Lemma 3.4]. If \( T \) has the SVEP then, for every \( y \in Y \),

\[
\sigma_{T/Y}(y) = \sigma_T(y).
\]

(iii) [9, Theorem 3]. If \( T \) is decomposable then, for every open \( G \subset \mathbb{C} \), \( X_T(G) \) is analytically invariant under \( T \).

3. **Theorem.** The following assertions are equivalent:

(i) \( T \) is strongly decomposable;

(ii) (a) \( T \) satisfies condition \( \alpha \);

(b) for every spectral maximal space \( Y \) of \( T \) and any \( x \in X \),

\[
\sigma_{\hat{T}}(\hat{x}) = \sigma_T(x) - \sigma(T|Y), \quad \hat{T} = T/Y, \quad \hat{x} = x + Y;
\]

(c) for every special maximal space \( Y \) of \( T \) and any open \( G \subset \mathbb{C} \), \( G \cap \sigma(T|Y) \neq \emptyset \) implies that \( X_T[G \cap \sigma(T|Y)] \neq \{0\} \).

**Proof.** (i) \( \Rightarrow \) (ii). (a) is evident. (b). (1) implies

\[
\sigma_{\hat{T}}(\hat{x}) \supset \sigma_T(x) - [\sigma_T(x) \cap \sigma(T|Y)] = \sigma_T(x) - \sigma(T|Y)
\]
and hence
\[ \sigma_f(x) \supset \sigma(x) - \sigma(T|Y). \]

To obtain the opposite inclusion, for \( x \in X \), put
\[ F = \sigma(x) \cup (T|Y) \]
and for the decomposable \( T|X \), use (2) and (3) as follows:
\[ \sigma(\bar{T}|X) = \sigma(T|X) \subset F - \sigma(T|Y) \]
\[ = \sigma(x) - \sigma(T|Y). \]

By (5), \( x \in X \) and hence \( \hat{x} = x + Y \in X \). Consequently,
\[ \sigma(\hat{x}) \subset \sigma(\bar{T}|X) \subset \sigma(x) - \sigma(T|Y) \]
and this establishes (4).

Since \( T|Y \) is decomposable, (c) is a consequence of Proposition 1 (v).

(ii) \Rightarrow (i): Let \( Y \) be a spectral maximal space of \( T \). By (a) and Proposition 1 (iii), \( Y \) has a representation \( Y = X \sigma(T|Y) \).

Let \( G \subset C \) be open and put \( Z = X \sigma(T|Y) \). We shall prove inclusion
\[ G \cap \sigma(T|Y) \subset \sigma(T|Y \cap Z). \]

If \( G \cap \sigma(T|Y) = \emptyset \), then (6) is evident. Therefore, assume
\[ G \cap \sigma(T|Y) \neq \emptyset. \]

Let \( \lambda_0 \in G \cap \sigma(T|Y) \) and let \( \delta_0 \subset G \) be a neighborhood of \( \lambda_0 \). Then, since \( \delta_0 \cap (T|Y) \neq \emptyset \), (c) implies that \( X[\delta_0 \cap \sigma(T|Y)] \neq \emptyset \) and hence
\[ \sigma(T|X) \neq \emptyset. \]

Let \( \lambda_1 \in \sigma(T|X[\delta_0 \cap \sigma(T|Y)]) \). Then \( \lambda_1 \in \bar{\delta}_0 \) and it follows from
\[ X[\bar{\delta}_0 \cap \sigma(T|Y)] \subset X[\bar{G} \cap \sigma(T|Y)] = X[\sigma(T|Y)] \cap Z = Y \cap Z \]
that \( \lambda_1 \in \bar{\delta}_0 \cap \sigma(T|Y \cap Z) \). Thus,
\[ \bar{\delta}_0 \cap \sigma(T|Y \cap Z) \neq \emptyset \]
and since \( \delta_0 \) is an arbitrary neighborhood of \( \lambda_0 \), we must have \( \lambda_0 \in \sigma(T|Y \cap Z) \). By the definition of \( \lambda_0 \), inclusion (6) holds. Finally, we shall conclude the proof by showing that \( T|Y \) is decomposable. The subspace \( W = Y \cap Z \) is a spectral maximal space of \( T \). By denoting \( \bar{T} = T/W \) and for \( x \in Y \), \( \bar{x} = x + W \), with the help of condition (b) and inclusion (6),
we obtain successively

\begin{equation}
\sigma_{\hat{T}}(\hat{x}) = \sigma_T(x) - \rho(T|W) \subseteq \sigma_T(x) - [G \cap \sigma(T|Y)] \\
\subseteq \sigma(T|Y) - [G \cap \sigma(T|Y)] = \sigma(T|Y) - G \subseteq G^c.
\end{equation}

Since \( Y \) is a spectral maximal space of \( T \) and \( W \) is a spectral maximal space of \( T|Y \), Proposition 1 (iv) implies \( Y/W \) is a spectral maximal space of \( T/W \). Then, with the help of (7) and [13, Theorem 1.1 (g)], we obtain

\[
\sigma[\hat{T}|(Y/W)] = \bigcup_{\hat{x} \in Y/W} \sigma_{\hat{T}}(\hat{x}) \subseteq G^c.
\]

Consequently, \( T|Y \) is decomposable by [5, Theorem 12] and [1] (or [11]), (see also [10]).

If one slightly strengthens condition (b) in Theorem 3, then (c) becomes redundant.

4. THEOREM. The following assertions are equivalent:

(I) \( T \) is strongly decomposable;

(II) (A) \( T \) satisfies condition \( \alpha \);

(B) for every closed \( F \subseteq \mathcal{C} \), and each \( x \in X \),

\begin{equation}
\sigma_{\hat{T}}(\hat{x}) = \sigma_T(x) - F
\end{equation}

where \( \hat{T} = T/X_T(F) \), \( \hat{x} = x + X_T(F) \).

(III) (A) \( T \) satisfies condition \( \alpha \);

(C) For every pair \( F_1, F_2 \) of closed sets in \( \mathcal{C} \),

\begin{equation}
\sigma[(T/Y_2)|X_T(F_1 \cup F_2)/Y_2] \subseteq F_1, \quad \text{where } Y_2 = X_T(F_2).
\end{equation}

Proof. (I) \( \Rightarrow \) (III). Let \( F_1, F_2 \) be closed in \( \mathcal{C} \). Since \( T \) is strongly decomposable, \( T|X_T(F_1 \cup F_2) \) is decomposable. Let \( G_1, G_2 \) be open sets in \( \mathcal{C} \) such that \( F_1 \cup F_2 \subseteq G_1 \cup G_2 \), \( F_1 \subseteq G_1 \) and \( G_2 \cap F_1 = \emptyset \). For \( x \in X_T(F_1 \cup F_2) \), we have a representation

\[ x = x_1 + x_2 \quad \text{with } x_i \in X_T(F_1 \cup F_2) \cap X_T(\overline{G_i}), \ i = 1, 2. \]

It follows from

\[ \sigma_T(x_2) \subseteq (F_1 \cup F_2) \cap G_2 = F_2 \cap \overline{G_2} \subseteq F_2 \]

that \( x_2 \in X_T(F_2) = Y_2. \)

Let \( \lambda_0 \notin \overline{G_1} \). Then \( \lambda_0 \in \rho(T|X_T[(F_1 \cup F_2) \cap \overline{G_1}]) \) and hence there is \( y \in X_T[(F_1 \cup F_2) \cap \overline{G_1}] \) verifying

\[ (\lambda_0 - T)y = x_1. \]
By the natural homomorphism $X \to X/Y_2$, we obtain
\[(\lambda_0 - T/Y_2)\hat{\varphi} = \hat{x}_1 = \hat{x},\]
and hence $\lambda_0 - (T/Y_2)|_{X_T(F_1 \cup F_2)/Y_2}$ is surjective. Since $T/Y_2$ has the SVEP by Proposition 1 (vi), (ix) and Proposition 2 (i), we have $\lambda_0 \in \rho[(T/Y_2)|_{X_T(F_1 \cup F_2)/Y_2}]$ by [6, Theorem 2]. By the definition of $\lambda_0$, we have
\[\sigma[(T/Y_2)|_{X_T(F_1 \cup F_2)/Y_2}] \subset \overline{G_1},\]
and since $G_1 \supset F_1$ is arbitrary, inclusion (9) holds.

(III) $\Rightarrow$ (II): Let $x \in X$ and $F \subset \mathbb{C}$ be closed. For $F_1 = \overline{\sigma_T(x) - F}$ and $Y = X_T(F)$, (9) implies
\[\sigma[(T/Y)|_{X_T(F_1 \cup F)/Y}] \subset F_1 = \overline{\sigma_T(x) - F}.\]
It follows from the definition of $F_1$ that $x \in X_T(F_1 \cup F)$. Consequently, for $\hat{x} = x + Y$ and $\hat{T} = T/Y$, we have
\[\sigma_{\hat{T}}(\hat{x}) \subset \sigma[\hat{T}|_{X_T(F_1 \cup F)/Y}] \subset \overline{\sigma_T(x) - F}.\]
On the other hand, it follows from Proposition 1 (i) that
\[\sigma_{\hat{T}}(\hat{x}) \supset \overline{\sigma_T(x) - \sigma(T|Y)} \supset \sigma_T(x) - F\]
and hence (8) holds.

(II) $\Rightarrow$ (I). In view of Theorem 3, we only have to prove that, for every open $G$ and spectral maximal space $Y = X_T[\sigma(T|Y)]$,
\[(10) \quad G \cap \sigma(T|Y) \neq \emptyset\]
implies that $X_T[\overline{G} \cap \sigma(T|Y)] \neq \{0\}$. Choose an open $G$ verifying (10), denote $Z = X_T[\overline{G} \cap \sigma(T|Y)]$ and for $x \in X$, let $\hat{x} = x + Z$. If $Z = \{0\}$, then
\[(11) \quad \sigma_{\hat{T}}(\hat{x}) = \sigma_T(x), \quad \hat{T} = T/Z.\]
In view of (11), by hypothesis, we have
\[\sigma_T(x) = \sigma_{\hat{T}}(\hat{x}) = \overline{\sigma_T(x) - \overline{G} \cap \sigma(T|Y)}}\]
\[= \left[\sigma_T(x) - \overline{G}\right] \cup \left[\sigma_T(x) - \sigma(T|Y)\right].\]
Let $x \in Y$. Since $\sigma_T(x) \subset \sigma(T|Y)$, we have
\[\sigma_T(x) = \overline{\sigma_T(x) - G}\]
and hence
\[\sigma_T(x) \cap G = \emptyset.\]
Now, with the help of [13, Theorem 1.1 (g)], Proposition 1 (v), (ix) and Proposition 2 (ii), we obtain

\[
\sigma(T|_Y) \cap G = \bigcup_{x \in Y} \sigma_{\eta_Y}(x) \bigcap G = \bigcup_{x \in Y} \sigma_T(x) \bigcap G
\]

\[
= \bigcup_{x \in Y} \left[ \sigma_T(x) \cap G \right] = \emptyset.
\]

But this contradicts hypothesis (10). Therefore, \( Z = X_T[\overline{G} \cap \sigma(T|_Y)] \neq \{0\} \).

Next, we shall obtain a characterization of a strongly decomposable operator in terms of the conjugate operator. First, we need some preparation.

5. Lemma. Given \( T \), let \( Y \) and \( Z \) be invariant subspaces of \( X \) with \( Z \subset Y \). Then

\[ (T/Z)^*|_{(Y/Z)^\perp} \cong T^*|_{Y^\perp}. \]

Proof. The mapping \( X/Z \to X/Y \) is a continuous surjective homomorphism with kernel \( Y/Z \). Therefore, the quotient spaces \( (X/Z)/(Y/Z) \) and \( X/Y \) are isomorphic. Given \( x \in X \), we use the following notations for the equivalent classes containing \( x \) in the corresponding quotient spaces:

\( \hat{x} \in X/Y, \tilde{x} \in X/Z, \check{x} \in (X/Z)/(Y/Z) \). Note that \( u \in \hat{x} \) iff \( u - x \in Y \) iff \( (u - x)^\perp \in Y/Z \) iff \( \check{u} \in \check{x} \). Since

\[
\inf_{v \in \check{u}} \|v\| \leq \|u\|,
\]

we have

\[ \|\hat{x}\| = \inf_{\check{u} \in \check{x}} \|\check{u}\| = \inf_{\check{u} \in \check{x}} \inf_{v \in \check{u}} \|v\| \leq \inf_{u \in \hat{x}} \|u\| = \|\hat{x}\|. \]

On the other hand, for every \( u \in \hat{x} \), \( \check{u} = u + Z \subset u + Y = \hat{x} \) and hence \( \check{u} \subset \check{x} \). Thus,

\[
\inf_{v \in \check{u}} \|v\| \geq \|\hat{x}\|
\]

and hence

\[ \|\hat{x}\| \cong \inf_{\check{u} \in \check{x}} \inf_{v \in \check{u}} \|v\| \geq \|\hat{x}\|.
\]

Then, by (13) and (14), \( \|\check{x}\| \cong \|\hat{x}\| \). Thus, it follows from the isometrical isomorphisms

\[ (X/Y)^* \cong Y^\perp, \quad [(X/Z)/(Y/Z)^* \cong (Y/Z)]^\perp \]

that the unitary equivalence (12) holds.
6. **Lemma.** If $T$ is decomposable then, for every open $G \subset C$,

\[(15) \quad X_T(G^c)^\perp = \overline{X_{T^*}(G)^w}.\]

**Proof.** Let $T$ be decomposable. By [14], for every closed $F \subset C$,

\[(16) \quad JX_T(F) = JX \cap X_{T^*}^*(F)\]

where $J$ is the natural embedding of $X$ into $X^{**}$. By Proposition 1 (viii) and the fact that $T$ decomposable implies $T^*$ decomposable,

\[(17) \quad X_{T^*}^*(F) = X_{T^*}^*(F^c)^\perp.\]

Relations (16) and (17) imply

\[X_T(F) = \perp X_{T^*}^*(F^c)\]

and hence, for $F = G^c$, (15) follows. \qed

7. **Lemma.** If $T^*$ is decomposable then, for every open $G \subset C$, $\overline{X_{T^*}^*(G)^w}$ (i.e. the weak*-closure of $X_{T^*}^*(G)$) is analytically invariant under $T^*$.

**Proof.** Let $f^*: D \to X^*$ be analytic on an open $D \subset C$ and verify condition

\[(\lambda - T^*)f^*(\lambda) \in \overline{X_{T^*}^*(G)^w}\] on $D$.

We may assume $D$ is connected. Put $F = G^c$, $Y = X_T(F)$, use Lemma 6, Proposition 1 (vii) and obtain successively

\[\sigma[T^*| \overline{X_{T^*}^*(G)^w}] = \sigma(T| Y^\perp) = \sigma[(T/Y)^*] = \sigma(T/Y) \subset (\text{Int } F)^c = \overline{G}.\]

First, assume $D \subset \overline{G}$. Then $D \subset G \subset \rho(T| Y)$ and, for every $x \in Y$, $\lambda \in D$, we have

\[
\langle x, f^*(\lambda) \rangle = \langle (\lambda - T)R(\lambda; T| Y)x, f^*(\lambda) \rangle = \langle R(\lambda; T| Y)x, (\lambda - T^*)f^*(\lambda) \rangle = 0.
\]

Since $x \in Y$ is arbitrary, $f^*(\lambda) \in Y^\perp = \overline{X_{T^*}^*(G)^w}$ on $D$.

Next, assume $D \not\subset \overline{G}$. Then, for $\lambda \in D - \overline{G}$, the resolvent operator $R[\lambda; T^*| \overline{X_{T^*}^*(G)^w}]$ is defined, and for $h^*(\lambda) = (\lambda - T^*)f^*(\lambda)$ we have

\[(\lambda - T^*)\{f^*(\lambda) - R[\lambda; T^*| \overline{X_{T^*}^*(G)^w}]h^*(\lambda)\} = 0.\]

Since $T^*$ has the SVEP,

\[f^*(\lambda) = R[\lambda; T^*| \overline{X_{T^*}^*(G)^w}]h^*(\lambda) \in \overline{X_{T^*}^*(G)^w}\]

on $D - \overline{G}$, and $f^*(\lambda) \in \overline{X_{T^*}^*(G)^w}$ on $D$, by analytic continuation. \qed
8. THEOREM. The bounded operator \( T \) (resp. \( T^* \)) is strongly decomposable iff:

(i) \( T \) (resp. \( T^* \)) has the SVEP and for open \( G \subset \mathbb{C} \), \( T^* \mid X_T^*(G) \) (resp. \( T \mid X_T(G) \)) is decomposable;

(ii) for every pair \( G, H \) of open sets in \( \mathbb{C} \),

\[
X^*_T(G \cap H)^w = Y^*_T \cap \gamma^*(H)^w \quad \text{(resp. } X_T(G \cap H) = Y_T \cap \gamma(H))\]

Proof. We confine the proof to the operator \( T \), the proof concerning \( T^* \) being similar.

(only if): Assume \( T \) is strongly decomposable. Let \( G \subset \mathbb{C} \) be open, \( F = G^c \) and \( Z = X_T(F) \). The operator \( (T/Z) \mid (X/Z) \) is decomposable. Then, by Lemma 6, \( X_T(F)^+ = X_T^*(G)^w \) and hence

\[
(X/Z)^* \cong X_T^*(G)^w.
\]

By [8, Theorem 2] and [12], \( T^* \mid X_T^*(G)^w \) is decomposable. Apply Lemma 5 to a closed \( F_1 \supset F \), and obtain

\[
\left[ X_T(F_1)/Z \right]^+ \cong X_T(F_1)^+.
\]

Denote \( \tilde{T} = T/Z \), \( \tilde{X} = X/Z \). Before embarking on the proof of (ii), we shall show that

\[
\tilde{X}_T(F_1 - F) = X_T(F_1)/Z.
\]

In fact, if \( \tilde{x} \in \tilde{X}_T(F_1 - F) \), then \( \sigma_{\tilde{T}}(\tilde{x}) \subset \overline{F_1 - F} \) and hence, for every \( x \in \tilde{x} \),

\[
\sigma_T(x) \subset (\overline{F_1 - F}) \cup F = F_1.
\]

Therefore, \( \tilde{x} \in \tilde{X}_T(\overline{F_1 - F}) \) implies \( x \in X_T(F_1) \) and hence \( \tilde{x} \in X_T(F_1)/Z \). Conversely, if \( \tilde{x} \in X_T(F_1)/Z = X_T(F_1 - F \cup F)/Z \), then Theorem 4 (III, C) implies

\[
\sigma_{\tilde{T}}(\tilde{x}) \subset \sigma[\tilde{T} \mid X_T(\overline{F_1 - F} \cup F)/Z] \subset \overline{F_1 - F}
\]

and hence \( \tilde{x} \in \tilde{X}_T(\overline{F_1 - F}) \). Thus (21) is proved.

Now we are in a position to prove (ii). To simplify notation, put \( X' = (\tilde{X})^* \) and \( T' = (\tilde{T})^* \). Let \( H \) be open and let \( F_1 = G^c \cup H^c \). Then \( F_1 \supset F \) and \( \overline{F_1 - F} \subset H^c \). By Lemma 6, Lemma 5, (20), (21) and (19), we obtain successively:

\[
X_T^*(G \cap H)^w = X_T(F_1)^+ \cong [X_T(F_1)/Z]^+ = \tilde{X}_T(\overline{F_1 - F}) \supset [\tilde{X}_T(H^c)] \supset \overline{X_T(H)^w} = Y_T \cap \gamma^*(H)^w.
\]
For the last equality, we used the equivalence

\[ T^* = \left[ T/X_T(F) \right]^* = T^* | X_T^*(G)^w = T^* | Y^*. \]

To obtain the opposite inclusion, note that if \( x^* \in X_T^*(G \cap H) \), then

\[ \sigma_{T^*}(x^*) = G \cap H \subset G \]

and hence \( x^* \in X_T^*(G) \subset Y^*. \) Since \( Y^* \) is analytically invariant under \( T^* \) (Lemma 7), in view of Proposition 2 (ii), we obtain

\[ \sigma_{T^*|Y^*}(x^*) = \sigma_{T^*}(x^*) \subset H \]

and hence

\[ x^* \in Y_{T^*|Y^*}(H) \subset Y_{T^*|Y^*}(H)^w. \]

Thus

\[ X_T^*(G \cap H)^w \subset Y_{T^*|Y^*}(H)^w. \]

(if): Assume conditions (i) and (ii) are satisfied. Let \( F, F_1 \subset C \) be closed. Since \( X_T^*(C) = X^* \), we conclude that \( T^* \) is decomposable and hence \( T \) is decomposable by [14, Corollary 2.8]. Therefore, \( Z = X_T(F) \) is closed. Also \( T^* | X_T^*(F^c)^w \) is decomposable. Then, by Lemma 6,

\[ T^* | X_T^*(F^c)^w = T^* | X_T(F) \perp \cong T^*, \]

where \( \tilde{T} = T/Z \) and \( T^* = (\tilde{T}^*)^* \). Thus \( T^* \) is decomposable and hence \( \tilde{T} \) is decomposable. Therefore, letting \( \tilde{X} = X/Z, \tilde{X}_T(F_1) \) is closed and

(22)

\[ \sigma[\tilde{T} | \tilde{X}_T(F_1)] \subset F_1. \]

Put \( G = F^c, H = F_1^c \) and \( Y^* = X_T^*(G)^w \). It follows from Lemma 6 that

\[ T^* | X_T(F \cup F_1) \perp = T^* | X_T^*(G \cap H)^w, \]

\[ T^* | \tilde{X}_T(F_1) \perp \cong T^* | \tilde{Y}_{T^*|Y^*}(H)^w. \]

Then (18) implies

(23)

\[ T^* | \tilde{X}_T(F_1) \perp \cong T^* | X_T(F \cup F_1) \perp. \]

By Lemma 5 we have

(24)

\[ T^* [ X_T(F \cup F_1)/Z ] \perp \cong T^* | X_T(F \cup F_1) \perp. \]

Consequently, with the help of (24), (23) and (22), we obtain

\[ \sigma[\tilde{T} | X_T(F \cup F_1)/Z] = \sigma[T^* [ X_T(F \cup F_1)/Z ] \perp] = \sigma[T^* | \tilde{X}_T(F_1) \perp] \]

\[ = \sigma[\tilde{T} | \tilde{X}_T(F_1)] \subset F_1. \]
Thus, conditions (III) of Theorem 4 are satisfied and hence $T$ is strongly decomposable.

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References


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