

# Pacific Journal of Mathematics

**HEREDITARY CONES, ORDER IDEALS AND HALF-NORMS**

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Let  $\mathfrak{B}$  be an ordered normed space with positive cone  $\mathfrak{B}_+$  and let  $N$  be the canonical half-norm of  $\mathfrak{B}_+$ , i.e.,

$$N(a) = \inf \{ \|a + b\|; b \in \mathfrak{B}_+ \}.$$

Then, for any hereditary subcone  $\mathcal{C}$  of  $\mathfrak{B}_+$ , the positive bipolar  $\mathcal{C}^{\perp\perp}$  coincides with the  $N$ -closure  $\mathcal{C}^N$  of  $\mathcal{C}$ , i.e.,

$$\mathcal{C}^N = \{ a \in \mathfrak{B}_+; N(a - a_n) \rightarrow 0 \text{ for some } a_n \in \mathcal{C} \}.$$

Similar facts are proved for order ideals and these results are used to derive a result of Størmer on archimedean order ideals.

**0. Introduction.** Let  $\mathfrak{B}$  be a normed space. A half-norm on  $\mathfrak{B}$  is a real-valued function  $N$  satisfying the following conditions:

- (1)  $0 \leq N(x) \leq k\|x\|$  for some  $k > 0$ ;
- (2)  $N(x + y) \leq N(x) + N(y)$ ;
- (3)  $N(\lambda x) = \lambda N(x)$  for  $\lambda \geq 0$ ;
- (4)  $N(x) \vee N(-x) = 0$  if, and only if,  $x = 0$ .

For a half-norm  $N$  on  $\mathfrak{B}$ , the set  $\{x \in \mathfrak{B}; N(-x) = 0\}$  is a closed proper cone which defines an order relation on  $\mathfrak{B}$ . Conversely, when  $\mathfrak{B}$  is an ordered normed space with positive cone  $\mathfrak{B}_+$ , which is closed and proper,

$$N(x) = \inf \{ \|x + y\|; y \in \mathfrak{B}_+ \}$$

is a half-norm on  $\mathfrak{B}$ . This half-norm is called the *canonical half-norm* [1] associated with the positive cone  $\mathfrak{B}_+$ . It is also called the *order half-norm* [6] associated with  $\mathfrak{B}_+$  because it can also be characterized by

$$N(x) = \inf \{ \lambda \geq 0; x \leq \lambda u \text{ for some } u \in \mathfrak{B}_1 \}$$

where  $\mathfrak{B}_1$  is the unit ball of  $\mathfrak{B}$ .

Half-norms were explicitly introduced by Arendt, Chernoff and Kato [1], although particular examples occurred in the works of Grosberg and Krein [4], Kadison [5], and many subsequent authors on order-unit and base-norm spaces. Half-norms are particularly useful for studying positive semigroups [1], [2], [9]. For further developments, see [6], [7].

The aim of this paper is to clarify the positive bipolar property considered in [2]. It turns out that the positive bipolar  $\mathcal{C}^{\perp\perp}$  of a hereditary subcone  $\mathcal{C}$  of  $\mathfrak{B}_+$  coincides with the  $N$ -closure  $\mathcal{C}^N$  of  $\mathcal{C}$ . A similar fact is valid also for order ideals. Hereditary subcones and order ideals are crucial notions in the study of the hermitian part of a  $C^*$ -algebra.

**1. Hereditary cones and order ideals.** Let  $\mathfrak{B}$  be a normed space ordered by a closed proper cone  $\mathfrak{B}_+$ , which we call the positive cone of  $\mathfrak{B}$ . A subcone  $\mathcal{C}$  of  $\mathfrak{B}_+$  is said to be *hereditary* if  $0 \leq a \leq c$  and  $c \in \mathcal{C}$  imply  $a \in \mathcal{C}$ . The following lemma is easily proved.

**LEMMA 1.** *A subcone  $\mathcal{C}$  of  $\mathfrak{B}_+$  is hereditary if, and only if  $\mathcal{C} = \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)$ .*

Hereditary cones are closely related to order ideals. An *order ideal* in  $\mathfrak{B}$  is a linear subspace  $\mathcal{J}$  of  $\mathfrak{B}$  such that  $a \leq x \leq b$  and  $a, b \in \mathcal{J}$  imply  $x \in \mathcal{J}$ . In other words, a linear subspace  $\mathcal{J}$  of  $\mathfrak{B}$  is an order ideal if, and only if,

$$\mathcal{J} = (\mathcal{J} + \mathfrak{B}_+) \cap (\mathcal{J} - \mathfrak{B}_+).$$

An order ideal is said to be *positively generated* if  $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_+$  for  $\mathcal{J}_+ = \mathcal{J} \cap \mathfrak{B}_+$ .

**LEMMA 2.** *If  $\mathcal{J}$  is a positively generated order ideal, then  $\mathcal{J}_+$  is an hereditary subcone of  $\mathfrak{B}_+$ . If  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$ , its linear span  $L(\mathcal{C})$  is a positively generated order ideal.*

*Proof.* Let  $\mathcal{J}$  be a positively generated order ideal. Then,

$$\begin{aligned} \mathcal{J}_+ &= \mathcal{J} \cap \mathfrak{B}_+ = (\mathcal{J} + \mathfrak{B}_+) \cap (\mathcal{J} - \mathfrak{B}_+) \cap \mathfrak{B}_+ \\ &\supset (\mathcal{J}_+ + \mathfrak{B}_+) \cap (\mathcal{J}_+ - \mathfrak{B}_+) \cap \mathfrak{B}_+ \\ &= (\mathcal{J}_+ - \mathfrak{B}_+) \cap \mathfrak{B}_+ \supset \mathcal{J}_+. \end{aligned}$$

Hence, by Lemma 1,  $\mathcal{J}_+$  is hereditary. Conversely, if  $\mathcal{C}$  is hereditary, we have

$$\mathcal{C} = L(\mathcal{C}) \cap \mathfrak{B}_+.$$

It is obvious from this equality that  $L(\mathcal{C})$  is positively generated with  $L(\mathcal{C})_+ = \mathcal{C}$ . Furthermore, since  $\mathcal{C} + \mathfrak{B}_+ = \mathfrak{B}_+$ ,

$$\begin{aligned} & (L(\mathcal{C}) + \mathfrak{B}_+) \cap (L(\mathcal{C}) - \mathfrak{B}_+) \\ &= (\mathcal{C} - \mathcal{C} + \mathfrak{B}_+) \cap (\mathcal{C} - \mathcal{C} - \mathfrak{B}_+) = (\mathfrak{B}_+ - \mathcal{C}) \cap (\mathcal{C} - \mathfrak{B}_+) \\ &\subset \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+) - \mathcal{C} = L(\mathcal{C}). \end{aligned}$$

Therefore,  $L(\mathcal{C})$  is also an order ideal.

**2.  $N$ -closures.** Let  $\mathfrak{B}$  be an ordered normed space with positive cone  $\mathfrak{B}_+$  and let  $N$  be the canonical half-norm associated with  $\mathfrak{B}_+$ . For a subset  $\mathfrak{M}$  of  $\mathfrak{B}$ , we define the  $N$ -closure  $\mathfrak{M}^N$  of  $\mathfrak{M}$  by

$$\mathfrak{M}^N = \{a \in \mathfrak{B}_+ ; N(a - a_n) \rightarrow 0 \text{ for some } a_n \in \mathfrak{M}\}.$$

By definition,  $\mathfrak{M}^N$  is a subset of  $\mathfrak{B}_+$ .

**LEMMA 3.** (1)  $\mathfrak{M} \cap \mathfrak{B}_+ \subset \mathfrak{M}^N$ ;

(2)  $\mathfrak{M}^N$  is closed;

(3) If  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$ ,  $\mathcal{C}^N$  is also an hereditary subcone of  $\mathfrak{B}_+$ ,  $\mathcal{C}^N = L(\mathcal{C})^N$ , and

$$\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$$

(4) If  $\mathcal{I}$  is an order ideal in  $\mathfrak{B}$ ,  $\mathcal{I}^N$  is also an order ideal and

$$\mathcal{I}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{I} - \mathfrak{B}_+)}$$

*Proof.* (1) is obvious. To prove (2), assume that  $a_n \in \mathfrak{M}^N$  and  $\|a_n - a\| \rightarrow 0$  for some  $a \in \mathfrak{B}$ . Then, one can choose  $b_n \in \mathfrak{M}$  such that  $N(a_n - b_n) < 1/n$ . Since  $N$  is canonical, one always has  $0 \leq N(x) \leq \|x\|$  for every  $x \in \mathfrak{B}$ . Hence,

$$N(a - b_n) \leq N(a - a_n) + N(a_n - b_n) \leq \|a - a_n\| + N(a_n - b_n) \rightarrow 0.$$

Therefore,  $a \in \mathfrak{M}^N$ .

To prove (3), let  $\mathcal{C}$  be an hereditary subcone of  $\mathfrak{B}_+$ . If  $0 \leq a \leq c$  and  $c \in \mathcal{C}^N$ , we can choose  $c_n \in \mathcal{C}$  such that  $N(c - c_n) \rightarrow 0$ . Since  $N$  is canonical, we always have  $N(x) \leq N(y)$  if  $x \leq y$ . Hence,

$$0 \leq N(a - c_n) \leq N(c - c_n) \rightarrow 0.$$

Therefore,  $a \in \mathcal{C}^N$ , i.e.,  $\mathcal{C}^N$  is hereditary. Next, let  $a \in L(\mathcal{C})^N$  and choose  $a_n \in L(\mathcal{C})$  such that  $N(a - a_n) \rightarrow 0$ . Since  $L(\mathcal{C}) = \mathcal{C} - \mathcal{C}$  by Lemma 2, there exist  $c_n, d_n \in \mathcal{C}$  such that  $a_n = c_n - d_n$ . Then,  $a_n \leq c_n$  and, hence,  $N(a - c_n) \rightarrow 0$ . This shows  $a \in \mathcal{C}^N$ . Therefore,  $\mathcal{C}^N = L(\mathcal{C})^N$ . To prove

the equality  $\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$ , let  $a \in \mathcal{C}^N$  and choose  $c_n \in \mathcal{C}$  such that  $N(a - c_n) \rightarrow 0$ . By the definition of the canonical half-norm, there exist  $b_n \in \mathfrak{B}_+$  such that  $\|a - c_n + b_n\| \rightarrow 0$ . Therefore,  $a \in \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$ . Conversely, if  $a \in \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$ , then  $a \in \mathfrak{B}_+$  and there exist  $c_n \in \mathcal{C}$  and  $b_n \in \mathfrak{B}$  such that  $\|a - c_n + b_n\| \rightarrow 0$ . Then,  $N(a - c_n) \rightarrow 0$  and, hence,  $a \in \mathcal{C}^N$ .

Finally, to prove (4), let  $\mathcal{F}$  be an order ideal and  $a \leq x \leq b$  for  $a, b \in \mathcal{F}^N$ . Then, for  $b_n \in \mathcal{F}$  such that  $N(b - b_n) \rightarrow 0$ , we have  $0 \leq N(x - b_n) \leq N(b - b_n) \rightarrow 0$ . Hence  $x \in \mathcal{F}^N$ . The remaining equality can be proved in the same manner as in (3).

When  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$ , we have

$$\mathcal{C} = \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+),$$

by Lemma 1. Then

$$\bar{\mathcal{C}} = \overline{\mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)}.$$

If  $\bar{\mathcal{C}}$  is hereditary, again by Lemma 1,

$$\bar{\mathcal{C}} = \mathfrak{B}_+ \cap (\bar{\mathcal{C}} - \mathfrak{B}_+),$$

whereas by Lemma 3,

$$\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)},$$

and, in general, we only have

$$\overline{\mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)} \subset \mathfrak{B}_+ \cap (\bar{\mathcal{C}} - \mathfrak{B}_+) \subset \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}.$$

Similar relations hold for order ideals. To obtain relations  $\bar{\mathcal{C}} = \mathcal{C}^N$  and  $\bar{\mathcal{F}} \cap \mathfrak{B}_+ = \mathcal{F}^N$ , we introduce new types of hereditary cones and order ideals.

A subcone  $\mathcal{C}$  of  $\mathfrak{B}_+$  is said to be *topologically hereditary* if the following condition is satisfied: if  $a_n \leq c_n$ ,  $c_n \in \mathcal{C}$ ,  $a \in \mathfrak{B}_+$  and  $\|a - a_n\| \rightarrow 0$ , then  $a \in \bar{\mathcal{C}}$ . An order ideal  $\mathcal{F}$  is called a *topological order ideal* if the following condition is satisfied: if  $x_n \leq a_n$ ,  $a_n \in \mathcal{F}$ ,  $a \in \mathfrak{B}_+$  and  $\|a - x_n\| \rightarrow 0$ , then  $a \in \bar{\mathcal{F}}$ . An example of a topologically hereditary cone is  $\mathcal{C}^N$  for a hereditary cone  $\mathcal{C}$ . It will be shown in Lemma 10, that archimedean order ideals are topological order ideals.

**LEMMA 4.** (1) *Let  $\mathcal{C}$  be an hereditary subcone of  $\mathfrak{B}_+$ . Then  $\mathcal{C}$  is topologically hereditary if, and only if,  $\bar{\mathcal{C}} = \mathcal{C}^N$ .*

(2) *Let  $\mathcal{F}$  be an order ideal. Then  $\mathcal{F}$  is a topological order ideal if, and only if,  $\bar{\mathcal{F}} \cap \mathfrak{B}_+ = \mathcal{F}^N$ .*

*Proof.* To prove (1), let  $\mathcal{C}$  be an hereditary subcone of  $\mathfrak{B}_+$ . If  $\mathcal{C}$  is topologically hereditary and  $a \in \mathcal{C}^N$ , by Lemma 3, there exist  $c_n \in \mathcal{C}$  and  $b_n \in \mathfrak{B}_+$  such that  $\|a - c_n + b_n\| \rightarrow 0$ . Then, for  $a_n = c_n - b_n$ , we have  $a_n \leq c_n$  and  $\|a - a_n\| \rightarrow 0$ . Hence,  $a \in \overline{\mathcal{C}}$ . Therefore,  $\mathcal{C}^N \subset \overline{\mathcal{C}}$ . But  $\overline{\mathcal{C}} \subset \mathcal{C}^N$  follows from (1) and (2) of Lemma 3. Conversely, assume  $\overline{\mathcal{C}} = \mathcal{C}^N$ ,  $a_n \leq c_n$ ,  $c_n \in \mathcal{C}$ ,  $a \in \mathfrak{B}_+$  and  $\|a - a_n\| \rightarrow 0$ . Then, since

$$0 \leq N(a - c_n) \leq N(a - a_n) \leq \|a - a_n\| \rightarrow 0,$$

we have  $a \in \mathcal{C}^N = \overline{\mathcal{C}}$ .

Property (2) can be proved in the same manner, using (4) in Lemma 3.

There are two cases where some points of  $\mathcal{C}^N$  automatically belong to  $\overline{\mathcal{C}}$ .

**PROPOSITION 5.** *Let  $\mathfrak{M}$  be an hereditary subcone of  $\mathfrak{B}_+$ , or an order ideal in  $\mathfrak{B}$ . If  $\mathfrak{B}_+$  contains an interior point  $u$ ,  $u \in \mathfrak{M}^N$  implies  $u \in \overline{\mathfrak{M}}$ .*

*Proof.* If  $u \in \mathfrak{M}^N$ , there exist  $a_n \in \mathfrak{M}$  and  $x_n \in \mathfrak{B}$  such that  $x_n \leq a_n$  and  $\|u - x_n\| \rightarrow 0$ . Since  $u$  is an interior point of  $\mathfrak{B}_+$ , we may suppose  $x_n \in \mathfrak{B}_+$ . Then, by assumption,  $x_n \in \mathfrak{M}$  and, hence,  $u \in \overline{\mathfrak{M}}$ .

**PROPOSITION 6.** *Let  $\mathfrak{B}$  be a normed lattice. Then, every hereditary subcone of  $\mathfrak{B}_+$  is topologically hereditary, and every positively generated order ideal is a topological order ideal.*

*Proof.* Let  $\mathfrak{M}$  be an hereditary subcone, or a positively generated order ideal, and suppose  $x_n \leq a_n$ ,  $a_n \in \mathfrak{M}$ ,  $a \in \mathfrak{B}_+$ , and  $\|a - x_n\| \rightarrow 0$ . Then,  $0 \leq x_n^+ \leq a_n^+$  and  $\|a - x_n^+\| \rightarrow 0$ , where  $x_n^+$  and  $a_n^+$  denote the positive parts of  $x_n$  and  $a_n$ . In both cases,  $a_n^+ \in \mathfrak{M}$  and, hence  $x_n^+ \in \mathfrak{M}$ . Then we have  $a \in \overline{\mathfrak{M}}$ .

**REMARK.** When  $\mathcal{J}$  is a positively generated order ideal in a vector lattice  $\mathfrak{B}$ , every  $a \in \mathcal{J}$  has a decomposition  $a = a_1 - a_2$  with  $a_i \in \mathcal{J} \cap \mathfrak{B}_+$ . Then  $a \leq a^+ \leq a_1$ , which implies  $a^+ \in \mathcal{J}$ . This fact was used in the above proof and it also means that  $\mathcal{J}$  is a sublattice.

**3. Positive bipolars.** Let  $\mathfrak{B}$  be an ordered normed space with positive cone  $\mathfrak{B}_+$  and let  $N$  be the canonical half-norm associated with  $\mathfrak{B}_+$ . Let  $\mathfrak{B}^*$  be the dual of  $\mathfrak{B}$  and denote by  $f(x)$  the value of  $f \in \mathfrak{B}^*$  at  $x \in \mathfrak{B}$ . Then  $f \in \mathfrak{B}^*$  is defined to be positive if  $f(x) \geq 0$  for all  $x \in \mathfrak{B}_+$ . The *dual cone*  $\mathfrak{B}_+^*$  consists of all positive elements of  $\mathfrak{B}^*$ .

We recall a result, proved in [7], Lemma 2, which states that  $f \in \mathfrak{B}^*$  is positive if, and only if,

$$f(x) \leq \|f\|N(x) \quad \text{for all } x \in \mathfrak{B}.$$

We start with a proposition which illustrates the relevance of the  $N$ -closure.

**PROPOSITION 7.** *Let  $\mathfrak{M}$  be a linear subspace of  $\mathfrak{B}$  and let  $a \in \mathfrak{B}_+$ . Then, there exists  $f \in \mathfrak{B}_+^*$  such that*

$$f(a) > 0 \quad \text{and} \quad f(x) = 0 \quad \text{for all } x \in \mathfrak{M}$$

*if, and only if,  $a \notin \mathfrak{M}^N$ .*

*Proof.* If such an  $f \in \mathfrak{B}_+^*$  exists, then, for any  $b \in \mathfrak{M}$ ,

$$0 < f(a) = f(a - b) \leq \|f\|N(a - b),$$

which implies  $a \notin \mathfrak{M}^N$ . Conversely, assume  $a \notin \mathfrak{M}^N$  and set

$$\alpha = \inf_{b \in \mathfrak{M}} N(a - b) > 0.$$

Let  $\mathfrak{M}_a$  be the linear subspace spanned by  $a$  and  $\mathfrak{M}$ , and define a linear functional  $g$  on  $\mathfrak{M}_a$  by

$$g(\xi a + b) = \alpha \xi$$

for all  $\xi \in \mathbf{R}$  and  $b \in \mathfrak{M}$ . Then one has

$$g(c) \leq N(c) \quad \text{for all } c \in \mathfrak{M}_a,$$

because, if  $c = \xi a + b$  with  $\xi \leq 0$ ,

$$g(c) = \alpha \xi \leq 0 \leq N(\xi a + b),$$

and, if  $c = \xi a + b$  with  $\xi > 0$ ,

$$g(c) = \alpha \xi \leq \xi N(a + b/\xi) = N(\xi a + b).$$

Hence  $g$  can be extended to an  $f \in \mathfrak{B}^*$  such that

$$f(c) \leq N(c) \quad \text{for all } c \in \mathfrak{B}.$$

Therefore,  $f$  is positive and, by construction,  $f(a) > 0$  and  $f(c) = 0$  for all  $c \in \mathfrak{M}$ .

An immediate consequence is a characterization of the positive bipolars  $\mathcal{C}^{\perp\perp}$  of an hereditary subcone  $\mathcal{C}$ , and  $\mathcal{I}^{\perp\perp}$  of an order ideal  $\mathcal{I}$ . The

definitions of these polars are as follows:

$$\begin{aligned} \mathcal{C}^\perp &= \{f \in \mathfrak{B}_+^* ; f(c) = 0 \text{ for every } c \in \mathcal{C}\}, \\ \mathcal{C}^{\perp\perp} &= \{a \in \mathfrak{B}_+ ; f(a) = 0 \text{ for every } f \in \mathcal{C}^\perp\}, \\ \mathcal{F}^\perp &= \{f \in \mathfrak{B}_+^* ; f(x) = 0 \text{ for every } x \in \mathcal{F}\}, \\ \mathcal{F}^{\perp\perp} &= \{a \in \mathfrak{B} ; f(a) = 0 \text{ for every } f \in \mathcal{F}^\perp\}. \end{aligned}$$

**THEOREM 8.** (1) *If  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$ , then  $\mathcal{C}^{\perp\perp} = \mathcal{C}^N$ .*  
 (2) *If  $\mathcal{F}$  is an order ideal,  $\mathcal{F}^{\perp\perp} \cap \mathfrak{B}_+ = \mathcal{F}^N$ .*

*Proof.* When  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$ ,  $\mathcal{C}^{\perp\perp} \subset \mathcal{C}^N$  follows from Proposition 7. Conversely, if  $a \in \mathcal{C}^N$  and  $f \in \mathcal{C}^\perp$ , there exist  $c_n \in \mathcal{C}$  such that  $N(a - c_n) \rightarrow 0$  and

$$0 \leq f(a) = f(a - c_n) \leq \|f\|N(a - c_n) \rightarrow 0.$$

Therefore,  $\mathcal{C}^N \subset \mathcal{C}^{\perp\perp}$ . The second part can be proved in the same manner.

Now we can characterize the subcones  $\mathcal{C}$  which satisfy the equality  $\mathcal{C} = \mathcal{C}^{\perp\perp}$ .

**COROLLARY 9.** *Let  $\mathcal{C}$  be a subcone of  $\mathcal{C}_+$ . Then  $\mathcal{C} = \mathcal{C}^{\perp\perp}$  if, and only if,  $\mathcal{C}$  is closed and topologically hereditary.*

*Proof.* By Lemma 4, if  $\mathcal{C}$  is closed and topologically hereditary, we have  $\mathcal{C} = \mathcal{C}^N$ . Conversely, if  $\mathcal{C} = \mathcal{C}^{\perp\perp}$ , then  $\mathcal{C}$  is hereditary and  $\mathcal{C} = \mathcal{C}^N$ . Hence, by Lemma 4,  $\mathcal{C} = \bar{\mathcal{C}}$  and  $\mathcal{C}$  is topologically hereditary.

In [2] it was proved that if  $\mathfrak{B}$  is the hermitian part of a  $C^*$ -algebra,  $\mathfrak{B}_+$  the positive elements of the algebra, and  $\mathcal{C}$  an hereditary subcone of  $\mathfrak{B}_+$ , then one has the bipolar property  $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ . A similar property was established for separable, countably order-complete, Banach lattices. But this latter result can be improved by the foregoing.

**COROLLARY 10.** *Let  $\mathfrak{B}$  be a normed lattice and  $\mathcal{C}$  an hereditary subcone of  $\mathfrak{B}_+$ . Then  $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$  and hence  $\bar{\mathcal{C}}$  is hereditary.*

*Proof.* It follows from Proposition 6 that  $\mathcal{C}$  is topologically hereditary and hence  $\bar{\mathcal{C}} = \mathcal{C}^N$  by Lemma 4. But  $\mathcal{C}^N = \mathcal{C}^{\perp\perp}$  by Theorem 8 and hence  $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ .

Of course in the  $\mathcal{C}^*$ -algebra case this argument could be reversed. If  $\mathcal{C}$  is an hereditary subcone of  $\mathfrak{B}_+$  then  $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$  by [2] and hence  $\bar{\mathcal{C}} = \mathcal{C}^N$ . Thus each hereditary subcone is topologically hereditary.

The case of order ideals is discussed in the next section.

**4. Archimedean order ideals.** As an application of the above results, we derive the positive bipolar property for archimedean order ideals (see [8]).

According to Kadison [5], an archimedean ordered vector space is an ordered vector space  $\mathfrak{B}$  with an order unit  $u$  whose order is archimedean, i.e., if  $x \leq \lambda u$  for all  $\lambda > 0$ , then  $x \leq 0$ .

When  $\mathfrak{B}$  is an archimedean ordered vector space, its positive cone  $\mathfrak{B}_+$  is a proper cone. We set

$$N(x) = \inf\{\lambda > 0; x \leq \lambda u\}$$

and

$$\|x\| = N(x) \vee N(-x).$$

Under the assumption that the order is archimedean, this defines a norm and  $\mathfrak{B}_+$  is closed with respect to the corresponding norm topology. Furthermore,  $N$  is the canonical half-norm of  $\mathfrak{B}_+$ . (For more details about the canonical half-norms defined by order units, see [6].) Therefore, an archimedean ordered vector space in the sense of Kadison is an ordered normed space equipped with the canonical half-norm defined by an order unit.

An *archimedean order ideal* is a closed, positively generated, order ideal  $\mathcal{I}$  such that the quotient  $\mathfrak{B}/\mathcal{I}$  is archimedean with respect to the order defined by the cone  $\theta(\mathfrak{B}_+)$ , where  $\theta: \mathfrak{B} \mapsto \mathfrak{B}/\mathcal{I}$  is the canonical map. When  $\mathcal{A}$  is a  $\mathcal{C}^*$ -algebra,  $\mathcal{I}$  is an archimedean ideal if, and only if,  $\mathcal{I}$  is the hermitian part of a two sided ideal of  $\mathcal{A}$ .

When  $\mathcal{I}$  is an archimedean order ideal in  $\mathfrak{B}$ , since  $\mathfrak{B}/\mathcal{I}$  is archimedean, the positive cone  $\theta(\mathfrak{B}_+)$  is closed with respect to the quotient norm:

$$\|\theta(x)\| = \inf\{\|y\|; y \in \theta(x)\}.$$

Evidently,  $\theta: \mathfrak{B} \mapsto \mathfrak{B}/\mathcal{I}$  is continuous with respect to this norm topology.

**LEMMA 11.** *Archimedean order ideals are topological order ideals.*

*Proof.* Let  $\mathcal{I}$  be an archimedean order ideal and suppose  $x_n \leq a_n$ ,  $a_n \in \mathcal{I}$ ,  $a \in \mathfrak{B}_+$  and  $\|a - x_n\| \rightarrow 0$ . Then,  $\theta(x_n) \leq \theta(a_n) = 0$  and  $\|\theta(a) - \theta(x_n)\| \rightarrow 0$ . Hence,  $\theta(a) \leq 0$ . On the other hand,  $\theta(a) \geq 0$  because  $a \geq 0$ . Therefore,  $\theta(a) = 0$ , i.e.,  $a \in \mathcal{I}$ .

We are now ready to reproduce a result in [8], Theorem 2.4 from the previous results.

PROPOSITION 12. For an archimedean order ideal  $\mathcal{F}$ ,  $\mathcal{F} = \mathcal{F}^{\perp\perp}$ .

*Proof.* By Lemma 11,  $\mathcal{F}$  is a closed topological order ideal. Hence, by Lemma 4,  $\mathcal{F} \cap \mathfrak{B}_+ = \mathcal{F}^N$ . Therefore, by Theorem 8,  $\mathcal{F}_+ = \mathcal{F} \cap \mathfrak{B}_+ = \mathcal{F}^{\perp\perp} \cap \mathfrak{B}_+$ . Let  $a \in \mathcal{F}^{\perp\perp}$ . Then, by a version of a result of Effros [3], Theorem 2.4, and Størmer [8], Lemma 2.3, there exists  $b \in \mathfrak{B}_+$  such that  $b \geq a$  and  $b \in \mathcal{F}^{\perp\perp}$ . Since  $\mathcal{F}$  is positively generated,  $\mathcal{F} = \mathcal{F}_+ - \mathcal{F}_+$  and  $\mathcal{F}^\perp = \mathcal{F}_+^\perp$ . Now, since  $a = b - (b - a)$  and  $b, b - a \in \mathcal{F}^{\perp\perp} \cap \mathfrak{B}_+ = \mathcal{F}^N = \mathcal{F} \cap \mathfrak{B}_+ = \mathcal{F}_+$  by Lemma 4 and Theorem 8, we conclude that  $a \in \mathcal{F}$ .

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Received May 13, 1982.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$132.00 a year (6 Vol., 12 issues). Special rate: \$66.00 a year to individual members of supporting institutions.

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The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

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# Pacific Journal of Mathematics

Vol. 110, No. 2

October, 1984

<b>Robert A. Bekes</b> , The range of convolution operators .....	257
<b>Dennis K. Burke and Sheldon Davis</b> , Subsets of ${}^\omega\omega$ and generalized metric spaces .....	273
<b>Giovanni Emmanuele</b> , A remark on a paper: "Common fixed points of nonexpansive mappings by iteration" .....	283
<b>I. Erdélyi and Sheng-Wang Wang</b> , On strongly decomposable operators ...	287
<b>Gerhard Gierz</b> , Injective Banach lattices with strong order units .....	297
<b>Maurizio Letizia</b> , Quotients by complex conjugation of nonsingular quadrics and cubics in $\mathbf{P}_C^3$ defined over $\mathbf{R}$ .....	307
<b>P. H. Maserick and Franciszek Hugon Szafraniec</b> , Equivalent definitions of positive definiteness .....	315
<b>Costel Peligrad and S. Rubinstein</b> , Maximal subalgebras of $C^*$ -crossed products .....	325
<b>Derek W. Robinson and Sadayuki Yamamuro</b> , Hereditary cones, order ideals and half-norms .....	335
<b>Derek W. Robinson and Sadayuki Yamamuro</b> , The Jordan decomposition and half-norms .....	345
<b>Richard Rochberg</b> , Interpolation of Banach spaces and negatively curved vector bundles .....	355
<b>Dale Rolfsen</b> , Rational surgery calculus: extension of Kirby's theorem .....	377
<b>Walter Iaan Seaman</b> , Helicoids of constant mean curvature and their Gauss maps .....	387
<b>Diana Shelstad</b> , Endoscopic groups and base change $\mathbf{C}/\mathbf{R}$ .....	397
<b>Jerrold Norman Siegel and Frank Williams</b> , Numerical invariants of homotopies into spheres .....	417
<b>Alladi Sitaram</b> , Some remarks on measures on noncompact semisimple Lie groups .....	429
<b>Teruhiko Soma</b> , Atoroidal, irreducible 3-manifolds and 3-fold branched coverings of $S^3$ .....	435
<b>Jan de Vries</b> , On the $G$ -compactification of products .....	447
<b>Hans Weber</b> , Topological Boolean rings. Decomposition of finitely additive set functions .....	471