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Let \mathfrak{B} be an ordered normed space with positive cone \mathfrak{B}_+ and let N be the canonical half-norm of \mathfrak{B}_+ , i.e.,

$$N(a) = \inf \{ \|a + b\|; b \in \mathfrak{B}_+ \}.$$

Then, for any hereditary subcone \mathcal{C} of \mathfrak{B}_+ , the positive bipolar $\mathcal{C}^{\perp\perp}$ coincides with the N -closure \mathcal{C}^N of \mathcal{C} , i.e.,

$$\mathcal{C}^N = \{ a \in \mathfrak{B}_+; N(a - a_n) \rightarrow 0 \text{ for some } a_n \in \mathcal{C} \}.$$

Similar facts are proved for order ideals and these results are used to derive a result of Størmer on archimedean order ideals.

0. Introduction. Let \mathfrak{B} be a normed space. A half-norm on \mathfrak{B} is a real-valued function N satisfying the following conditions:

- (1) $0 \leq N(x) \leq k\|x\|$ for some $k > 0$;
- (2) $N(x + y) \leq N(x) + N(y)$;
- (3) $N(\lambda x) = \lambda N(x)$ for $\lambda \geq 0$;
- (4) $N(x) \vee N(-x) = 0$ if, and only if, $x = 0$.

For a half-norm N on \mathfrak{B} , the set $\{x \in \mathfrak{B}; N(-x) = 0\}$ is a closed proper cone which defines an order relation on \mathfrak{B} . Conversely, when \mathfrak{B} is an ordered normed space with positive cone \mathfrak{B}_+ , which is closed and proper,

$$N(x) = \inf \{ \|x + y\|; y \in \mathfrak{B}_+ \}$$

is a half-norm on \mathfrak{B} . This half-norm is called the *canonical half-norm* [1] associated with the positive cone \mathfrak{B}_+ . It is also called the *order half-norm* [6] associated with \mathfrak{B}_+ because it can also be characterized by

$$N(x) = \inf \{ \lambda \geq 0; x \leq \lambda u \text{ for some } u \in \mathfrak{B}_1 \}$$

where \mathfrak{B}_1 is the unit ball of \mathfrak{B} .

Half-norms were explicitly introduced by Arendt, Chernoff and Kato [1], although particular examples occurred in the works of Grosberg and Krein [4], Kadison [5], and many subsequent authors on order-unit and base-norm spaces. Half-norms are particularly useful for studying positive semigroups [1], [2], [9]. For further developments, see [6], [7].

The aim of this paper is to clarify the positive bipolar property considered in [2]. It turns out that the positive bipolar $\mathcal{C}^{\perp\perp}$ of a hereditary subcone \mathcal{C} of \mathfrak{B}_+ coincides with the N -closure \mathcal{C}^N of \mathcal{C} . A similar fact is valid also for order ideals. Hereditary subcones and order ideals are crucial notions in the study of the hermitian part of a C^* -algebra.

1. Hereditary cones and order ideals. Let \mathfrak{B} be a normed space ordered by a closed proper cone \mathfrak{B}_+ , which we call the positive cone of \mathfrak{B} . A subcone \mathcal{C} of \mathfrak{B}_+ is said to be *hereditary* if $0 \leq a \leq c$ and $c \in \mathcal{C}$ imply $a \in \mathcal{C}$. The following lemma is easily proved.

LEMMA 1. *A subcone \mathcal{C} of \mathfrak{B}_+ is hereditary if, and only if $\mathcal{C} = \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)$.*

Hereditary cones are closely related to order ideals. An *order ideal* in \mathfrak{B} is a linear subspace \mathcal{J} of \mathfrak{B} such that $a \leq x \leq b$ and $a, b \in \mathcal{J}$ imply $x \in \mathcal{J}$. In other words, a linear subspace \mathcal{J} of \mathfrak{B} is an order ideal if, and only if,

$$\mathcal{J} = (\mathcal{J} + \mathfrak{B}_+) \cap (\mathcal{J} - \mathfrak{B}_+).$$

An order ideal is said to be *positively generated* if $\mathcal{J} = \mathcal{J}_+ - \mathcal{J}_+$ for $\mathcal{J}_+ = \mathcal{J} \cap \mathfrak{B}_+$.

LEMMA 2. *If \mathcal{J} is a positively generated order ideal, then \mathcal{J}_+ is an hereditary subcone of \mathfrak{B}_+ . If \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ , its linear span $L(\mathcal{C})$ is a positively generated order ideal.*

Proof. Let \mathcal{J} be a positively generated order ideal. Then,

$$\begin{aligned} \mathcal{J}_+ &= \mathcal{J} \cap \mathfrak{B}_+ = (\mathcal{J} + \mathfrak{B}_+) \cap (\mathcal{J} - \mathfrak{B}_+) \cap \mathfrak{B}_+ \\ &\supset (\mathcal{J}_+ + \mathfrak{B}_+) \cap (\mathcal{J}_+ - \mathfrak{B}_+) \cap \mathfrak{B}_+ \\ &= (\mathcal{J}_+ - \mathfrak{B}_+) \cap \mathfrak{B}_+ \supset \mathcal{J}_+. \end{aligned}$$

Hence, by Lemma 1, \mathcal{J}_+ is hereditary. Conversely, if \mathcal{C} is hereditary, we have

$$\mathcal{C} = L(\mathcal{C}) \cap \mathfrak{B}_+.$$

It is obvious from this equality that $L(\mathcal{C})$ is positively generated with $L(\mathcal{C})_+ = \mathcal{C}$. Furthermore, since $\mathcal{C} + \mathfrak{B}_+ = \mathfrak{B}_+$,

$$\begin{aligned} & (L(\mathcal{C}) + \mathfrak{B}_+) \cap (L(\mathcal{C}) - \mathfrak{B}_+) \\ &= (\mathcal{C} - \mathcal{C} + \mathfrak{B}_+) \cap (\mathcal{C} - \mathcal{C} - \mathfrak{B}_+) = (\mathfrak{B}_+ - \mathcal{C}) \cap (\mathcal{C} - \mathfrak{B}_+) \\ &\subset \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+) - \mathcal{C} = L(\mathcal{C}). \end{aligned}$$

Therefore, $L(\mathcal{C})$ is also an order ideal.

2. N -closures. Let \mathfrak{B} be an ordered normed space with positive cone \mathfrak{B}_+ and let N be the canonical half-norm associated with \mathfrak{B}_+ . For a subset \mathfrak{M} of \mathfrak{B} , we define the N -closure \mathfrak{M}^N of \mathfrak{M} by

$$\mathfrak{M}^N = \{a \in \mathfrak{B}_+ ; N(a - a_n) \rightarrow 0 \text{ for some } a_n \in \mathfrak{M}\}.$$

By definition, \mathfrak{M}^N is a subset of \mathfrak{B}_+ .

LEMMA 3. (1) $\mathfrak{M} \cap \mathfrak{B}_+ \subset \mathfrak{M}^N$;

(2) \mathfrak{M}^N is closed;

(3) If \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ , \mathcal{C}^N is also an hereditary subcone of \mathfrak{B}_+ , $\mathcal{C}^N = L(\mathcal{C})^N$, and

$$\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)};$$

(4) If \mathcal{I} is an order ideal in \mathfrak{B} , \mathcal{I}^N is also an order ideal and

$$\mathcal{I}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{I} - \mathfrak{B}_+)}.$$

Proof. (1) is obvious. To prove (2), assume that $a_n \in \mathfrak{M}^N$ and $\|a_n - a\| \rightarrow 0$ for some $a \in \mathfrak{B}$. Then, one can choose $b_n \in \mathfrak{M}$ such that $N(a_n - b_n) < 1/n$. Since N is canonical, one always has $0 \leq N(x) \leq \|x\|$ for every $x \in \mathfrak{B}$. Hence,

$$N(a - b_n) \leq N(a - a_n) + N(a_n - b_n) \leq \|a - a_n\| + N(a_n - b_n) \rightarrow 0.$$

Therefore, $a \in \mathfrak{M}^N$.

To prove (3), let \mathcal{C} be an hereditary subcone of \mathfrak{B}_+ . If $0 \leq a \leq c$ and $c \in \mathcal{C}^N$, we can choose $c_n \in \mathcal{C}$ such that $N(c - c_n) \rightarrow 0$. Since N is canonical, we always have $N(x) \leq N(y)$ if $x \leq y$. Hence,

$$0 \leq N(a - c_n) \leq N(c - c_n) \rightarrow 0.$$

Therefore, $a \in \mathcal{C}^N$, i.e., \mathcal{C}^N is hereditary. Next, let $a \in L(\mathcal{C})^N$ and choose $a_n \in L(\mathcal{C})$ such that $N(a - a_n) \rightarrow 0$. Since $L(\mathcal{C}) = \mathcal{C} - \mathcal{C}$ by Lemma 2, there exist $c_n, d_n \in \mathcal{C}$ such that $a_n = c_n - d_n$. Then, $a_n \leq c_n$ and, hence, $N(a - c_n) \rightarrow 0$. This shows $a \in \mathcal{C}^N$. Therefore, $\mathcal{C}^N = L(\mathcal{C})^N$. To prove

the equality $\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$, let $a \in \mathcal{C}^N$ and choose $c_n \in \mathcal{C}$ such that $N(a - c_n) \rightarrow 0$. By the definition of the canonical half-norm, there exist $b_n \in \mathfrak{B}_+$ such that $\|a - c_n + b_n\| \rightarrow 0$. Therefore, $a \in \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$. Conversely, if $a \in \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}$, then $a \in \mathfrak{B}_+$ and there exist $c_n \in \mathcal{C}$ and $b_n \in \mathfrak{B}$ such that $\|a - c_n + b_n\| \rightarrow 0$. Then, $N(a - c_n) \rightarrow 0$ and, hence, $a \in \mathcal{C}^N$.

Finally, to prove (4), let \mathcal{J} be an order ideal and $a \leq x \leq b$ for $a, b \in \mathcal{J}^N$. Then, for $b_n \in \mathcal{J}$ such that $N(b - b_n) \rightarrow 0$, we have $0 \leq N(x - b_n) \leq N(b - b_n) \rightarrow 0$. Hence $x \in \mathcal{J}^N$. The remaining equality can be proved in the same manner as in (3).

When \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ , we have

$$\mathcal{C} = \mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+),$$

by Lemma 1. Then

$$\overline{\mathcal{C}} = \overline{\mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)}$$

If $\overline{\mathcal{C}}$ is hereditary, again by Lemma 1,

$$\overline{\mathcal{C}} = \mathfrak{B}_+ \cap (\overline{\mathcal{C}} - \mathfrak{B}_+),$$

whereas by Lemma 3,

$$\mathcal{C}^N = \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)},$$

and, in general, we only have

$$\overline{\mathfrak{B}_+ \cap (\mathcal{C} - \mathfrak{B}_+)} \subset \mathfrak{B}_+ \cap (\overline{\mathcal{C}} - \mathfrak{B}_+) \subset \mathfrak{B}_+ \cap \overline{(\mathcal{C} - \mathfrak{B}_+)}.$$

Similar relations hold for order ideals. To obtain relations $\overline{\mathcal{C}} = \mathcal{C}^N$ and $\overline{\mathcal{J}} \cap \mathfrak{B}_+ = \mathcal{J}^N$, we introduce new types of hereditary cones and order ideals.

A subcone \mathcal{C} of \mathfrak{B}_+ is said to be *topologically hereditary* if the following condition is satisfied: if $a_n \leq c_n, c_n \in \mathcal{C}, a \in \mathfrak{B}_+$ and $\|a - a_n\| \rightarrow 0$, then $a \in \overline{\mathcal{C}}$. An order ideal \mathcal{J} is called a *topological order ideal* if the following condition is satisfied: if $x_n \leq a_n, a_n \in \mathcal{J}, a \in \mathfrak{B}_+$ and $\|a - x_n\| \rightarrow 0$, then $a \in \overline{\mathcal{J}}$. An example of a topologically hereditary cone is \mathcal{C}^N for a hereditary cone \mathcal{C} . It will be shown in Lemma 10, that archimedean order ideals are topological order ideals.

LEMMA 4. (1) *Let \mathcal{C} be an hereditary subcone of \mathfrak{B}_+ . Then \mathcal{C} is topologically hereditary if, and only if, $\overline{\mathcal{C}} = \mathcal{C}^N$.*

(2) *Let \mathcal{J} be an order ideal. Then \mathcal{J} is a topological order ideal if, and only if, $\overline{\mathcal{J}} \cap \mathfrak{B}_+ = \mathcal{J}^N$.*

Proof. To prove (1), let \mathcal{C} be an hereditary subcone of \mathfrak{B}_+ . If \mathcal{C} is topologically hereditary and $a \in \mathcal{C}^N$, by Lemma 3, there exist $c_n \in \mathcal{C}$ and $b_n \in \mathfrak{B}_+$ such that $\|a - c_n + b_n\| \rightarrow 0$. Then, for $a_n = c_n - b_n$, we have $a_n \leq c_n$ and $\|a - a_n\| \rightarrow 0$. Hence, $a \in \overline{\mathcal{C}}$. Therefore, $\mathcal{C}^N \subset \overline{\mathcal{C}}$. But $\overline{\mathcal{C}} \subset \mathcal{C}^N$ follows from (1) and (2) of Lemma 3. Conversely, assume $\overline{\mathcal{C}} = \mathcal{C}^N$, $a_n \leq c_n$, $c_n \in \mathcal{C}$, $a \in \mathfrak{B}_+$ and $\|a - a_n\| \rightarrow 0$. Then, since

$$0 \leq N(a - c_n) \leq N(a - a_n) \leq \|a - a_n\| \rightarrow 0,$$

we have $a \in \mathcal{C}^N = \overline{\mathcal{C}}$.

Property (2) can be proved in the same manner, using (4) in Lemma 3.

There are two cases where some points of \mathcal{C}^N automatically belong to $\overline{\mathcal{C}}$.

PROPOSITION 5. *Let \mathfrak{M} be an hereditary subcone of \mathfrak{B}_+ , or an order ideal in \mathfrak{B} . If \mathfrak{B}_+ contains an interior point u , $u \in \mathfrak{M}^N$ implies $u \in \overline{\mathfrak{M}}$.*

Proof. If $u \in \mathfrak{M}^N$, there exist $a_n \in \mathfrak{M}$ and $x_n \in \mathfrak{B}$ such that $x_n \leq a_n$ and $\|u - x_n\| \rightarrow 0$. Since u is an interior point of \mathfrak{B}_+ , we may suppose $x_n \in \mathfrak{B}_+$. Then, by assumption, $x_n \in \mathfrak{M}$ and, hence, $u \in \overline{\mathfrak{M}}$.

PROPOSITION 6. *Let \mathfrak{B} be a normed lattice. Then, every hereditary subcone of \mathfrak{B}_+ is topologically hereditary, and every positively generated order ideal is a topological order ideal.*

Proof. Let \mathfrak{M} be an hereditary subcone, or a positively generated order ideal, and suppose $x_n \leq a_n$, $a_n \in \mathfrak{M}$, $a \in \mathfrak{B}_+$, and $\|a - x_n\| \rightarrow 0$. Then, $0 \leq x_n^+ \leq a_n^+$ and $\|a - x_n^+\| \rightarrow 0$, where x_n^+ and a_n^+ denote the positive parts of x_n and a_n . In both cases, $a_n^+ \in \mathfrak{M}$ and, hence $x_n^+ \in \mathfrak{M}$. Then we have $a \in \overline{\mathfrak{M}}$.

REMARK. When \mathcal{J} is a positively generated order ideal in a vector lattice \mathfrak{B} , every $a \in \mathcal{J}$ has a decomposition $a = a_1 - a_2$ with $a_i \in \mathcal{J} \cap \mathfrak{B}_+$. Then $a \leq a^+ \leq a_1$, which implies $a^+ \in \mathcal{J}$. This fact was used in the above proof and it also means that \mathcal{J} is a sublattice.

3. Positive bipolars. Let \mathfrak{B} be an ordered normed space with positive cone \mathfrak{B}_+ and let N be the canonical half-norm associated with \mathfrak{B}_+ . Let \mathfrak{B}^* be the dual of \mathfrak{B} and denote by $f(x)$ the value of $f \in \mathfrak{B}^*$ at $x \in \mathfrak{B}$. Then $f \in \mathfrak{B}^*$ is defined to be positive if $f(x) \geq 0$ for all $x \in \mathfrak{B}_+$. The dual cone \mathfrak{B}_+^* consists of all positive elements of \mathfrak{B}^* .

We recall a result, proved in [7], Lemma 2, which states that $f \in \mathfrak{B}^*$ is positive if, and only if,

$$f(x) \leq \|f\|N(x) \quad \text{for all } x \in \mathfrak{B}.$$

We start with a proposition which illustrates the relevance of the N -closure.

PROPOSITION 7. *Let \mathfrak{N} be a linear subspace of \mathfrak{B} and let $a \in \mathfrak{B}_+$. Then, there exists $f \in \mathfrak{B}_+^*$ such that*

$$f(a) > 0 \quad \text{and} \quad f(x) = 0 \quad \text{for all } x \in \mathfrak{N}$$

if, and only if, $a \notin \mathfrak{N}^N$.

Proof. If such an $f \in \mathfrak{B}_+^*$ exists, then, for any $b \in \mathfrak{N}$,

$$0 < f(a) = f(a - b) \leq \|f\|N(a - b),$$

which implies $a \notin \mathfrak{N}^N$. Conversely, assume $a \notin \mathfrak{N}^N$ and set

$$\alpha = \inf_{b \in \mathfrak{N}} N(a - b) > 0.$$

Let \mathfrak{N}_a be the linear subspace spanned by a and \mathfrak{N} , and define a linear functional g on \mathfrak{N}_a by

$$g(\xi a + b) = \alpha \xi$$

for all $\xi \in \mathbf{R}$ and $b \in \mathfrak{N}$. Then one has

$$g(c) \leq N(c) \quad \text{for all } c \in \mathfrak{N}_a,$$

because, if $c = \xi a + b$ with $\xi \leq 0$,

$$g(c) = \alpha \xi \leq 0 \leq N(\xi a + b),$$

and, if $c = \xi a + b$ with $\xi > 0$,

$$g(c) = \alpha \xi \leq \xi N(a + b/\xi) = N(\xi a + b).$$

Hence g can be extended to an $f \in \mathfrak{B}^*$ such that

$$f(c) \leq N(c) \quad \text{for all } c \in \mathfrak{B}.$$

Therefore, f is positive and, by construction, $f(a) > 0$ and $f(c) = 0$ for all $c \in \mathfrak{N}$.

An immediate consequence is a characterization of the positive bipolars $\mathcal{C}^{\perp\perp}$ of an hereditary subcone \mathcal{C} , and $\mathcal{F}^{\perp\perp}$ of an order ideal \mathcal{F} . The

definitions of these polars are as follows:

$$\begin{aligned}\mathcal{C}^\perp &= \{f \in \mathfrak{B}_+^* ; f(c) = 0 \text{ for every } c \in \mathcal{C}\}, \\ \mathcal{C}^{\perp\perp} &= \{a \in \mathfrak{B}_+ ; f(a) = 0 \text{ for every } f \in \mathcal{C}^\perp\}, \\ \mathcal{F}^\perp &= \{f \in \mathfrak{B}_+^* ; f(x) = 0 \text{ for every } x \in \mathcal{F}\}, \\ \mathcal{F}^{\perp\perp} &= \{a \in \mathfrak{B} ; f(a) = 0 \text{ for every } f \in \mathcal{F}^\perp\}.\end{aligned}$$

THEOREM 8. (1) *If \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ , then $\mathcal{C}^{\perp\perp} = \mathcal{C}^N$.*
 (2) *If \mathcal{F} is an order ideal, $\mathcal{F}^{\perp\perp} \cap \mathfrak{B}_+ = \mathcal{F}^N$.*

Proof. When \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ , $\mathcal{C}^{\perp\perp} \subset \mathcal{C}^N$ follows from Proposition 7. Conversely, if $a \in \mathcal{C}^N$ and $f \in \mathcal{C}^\perp$, there exist $c_n \in \mathcal{C}$ such that $N(a - c_n) \rightarrow 0$ and

$$0 \leq f(a) = f(a - c_n) \leq \|f\|N(a - c_n) \rightarrow 0.$$

Therefore, $\mathcal{C}^N \subset \mathcal{C}^{\perp\perp}$. The second part can be proved in the same manner.

Now we can characterize the subcones \mathcal{C} which satisfy the equality $\mathcal{C} = \mathcal{C}^{\perp\perp}$.

COROLLARY 9. *Let \mathcal{C} be a subcone of \mathcal{C}_+ . Then $\mathcal{C} = \mathcal{C}^{\perp\perp}$ if, and only if, \mathcal{C} is closed and topologically hereditary.*

Proof. By Lemma 4, if \mathcal{C} is closed and topologically hereditary, we have $\mathcal{C} = \mathcal{C}^N$. Conversely, if $\mathcal{C} = \mathcal{C}^{\perp\perp}$, then \mathcal{C} is hereditary and $\mathcal{C} = \mathcal{C}^N$. Hence, by Lemma 4, $\mathcal{C} = \bar{\mathcal{C}}$ and \mathcal{C} is topologically hereditary.

In [2] it was proved that if \mathfrak{B} is the hermitian part of a C^* -algebra, \mathfrak{B}_+ the positive elements of the algebra, and \mathcal{C} an hereditary subcone of \mathfrak{B}_+ , then one has the bipolar property $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$. A similar property was established for separable, countably order-complete, Banach lattices. But this latter result can be improved by the foregoing.

COROLLARY 10. *Let \mathfrak{B} be a normed lattice and \mathcal{C} an hereditary subcone of \mathfrak{B}_+ . Then $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ and hence $\bar{\mathcal{C}}$ is hereditary.*

Proof. It follows from Proposition 6 that \mathcal{C} is topologically hereditary and hence $\bar{\mathcal{C}} = \mathcal{C}^N$ by Lemma 4. But $\mathcal{C}^N = \mathcal{C}^{\perp\perp}$ by Theorem 8 and hence $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$.

Of course in the \mathcal{C}^* -algebra case this argument could be reversed. If \mathcal{C} is an hereditary subcone of \mathfrak{B}_+ then $\bar{\mathcal{C}} = \mathcal{C}^{\perp\perp}$ by [2] and hence $\bar{\mathcal{C}} = \mathcal{C}^N$. Thus each hereditary subcone is topologically hereditary.

The case of order ideals is discussed in the next section.

4. Archimedean order ideals. As an application of the above results, we derive the positive bipolar property for archimedean order ideals (see [8]).

According to Kadison [5], an archimedean ordered vector space is an ordered vector space \mathfrak{B} with an order unit u whose order is archimedean, i.e., if $x \leq \lambda u$ for all $\lambda > 0$, then $x \leq 0$.

When \mathfrak{B} is an archimedean ordered vector space, its positive cone \mathfrak{B}_+ is a proper cone. We set

$$N(x) = \inf\{\lambda > 0; x \leq \lambda u\}$$

and

$$\|x\| = N(x) \vee N(-x).$$

Under the assumption that the order is archimedean, this defines a norm and \mathfrak{B}_+ is closed with respect to the corresponding norm topology. Furthermore, N is the canonical half-norm of \mathfrak{B}_+ . (For more details about the canonical half-norms defined by order units, see [6].) Therefore, an archimedean ordered vector space in the sense of Kadison is an ordered normed space equipped with the canonical half-norm defined by an order unit.

An *archimedean order ideal* is a closed, positively generated, order ideal \mathfrak{J} such that the quotient $\mathfrak{B}/\mathfrak{J}$ is archimedean with respect to the order defined by the cone $\theta(\mathfrak{B}_+)$, where $\theta: \mathfrak{B} \mapsto \mathfrak{B}/\mathfrak{J}$ is the canonical map. When \mathcal{A} is a C^* -algebra, \mathfrak{J} is an archimedean ideal if, and only if, \mathfrak{J} is the hermitian part of a two sided ideal of \mathcal{A} .

When \mathcal{A} is an archimedean order ideal in \mathfrak{B} , since $\mathfrak{B}/\mathfrak{J}$ is archimedean, the positive cone $\theta(\mathfrak{B}_+)$ is closed with respect to the quotient norm:

$$\|\theta(x)\| = \inf\{\|y\|; y \in \theta(x)\}.$$

Evidently, $\theta: \mathfrak{B} \mapsto \mathfrak{B}/\mathfrak{J}$ is continuous with respect to this norm topology.

LEMMA 11. *Archimedean order ideals are topological order ideals.*

Proof. Let \mathfrak{J} be an archimedean order ideal and suppose $x_n \leq a_n$, $a_n \in \mathfrak{J}$, $a \in \mathfrak{B}_+$ and $\|a - x_n\| \rightarrow 0$. Then, $\theta(x_n) \leq \theta(a_n) = 0$ and $\|\theta(a) - \theta(x_n)\| \rightarrow 0$. Hence, $\theta(a) \leq 0$. On the other hand, $\theta(a) \geq 0$ because $a \geq 0$. Therefore, $\theta(a) = 0$, i.e., $a \in \mathfrak{J}$.

We are now ready to reproduce a result in [8], Theorem 2.4 from the previous results.

PROPOSITION 12. For an archimedean order ideal \mathcal{F} , $\mathcal{F} = \mathcal{F}^{\perp\perp}$.

Proof. By Lemma 11, \mathcal{F} is a closed topological order ideal. Hence, by Lemma 4, $\mathcal{F} \cap \mathbb{B}_+ = \mathcal{F}^N$. Therefore, by Theorem 8, $\mathcal{F}_+ = \mathcal{F} \cap \mathbb{B}_+ = \mathcal{F}^{\perp\perp} \cap \mathbb{B}_+$. Let $a \in \mathcal{F}^{\perp\perp}$. Then, by a version of a result of Effros [3], Theorem 2.4, and Størmer [8], Lemma 2.3, there exists $b \in \mathbb{B}_+$ such that $b \geq a$ and $b \in \mathcal{F}^{\perp\perp}$. Since \mathcal{F} is positively generated, $\mathcal{F} = \mathcal{F}_+ - \mathcal{F}_+$ and $\mathcal{F}^\perp = \mathcal{F}_+^\perp$. Now, since $a = b - (b - a)$ and $b, b - a \in \mathcal{F}^{\perp\perp} \cap \mathbb{B}_+ = \mathcal{F}^N = \mathcal{F} \cap \mathbb{B}_+ = \mathcal{F}_+$ by Lemma 4 and Theorem 8, we conclude that $a \in \mathcal{F}$.

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