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THE JORDAN DECOMPOSITION AND HALF-NORMS

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DEREK W. ROBINSON AND SADAYUKI YAMAMURO

Let $\mathfrak{B}$ be a Banach space, with norm $\|\cdot\|$, ordered by a positive cone $\mathfrak{B}_+$ and order the dual $\mathfrak{B}^*$ by the dual cone $\mathfrak{B}^*_+$. We prove that, if $\mathfrak{B}$ is orthogonally generated, each $f \in \mathfrak{B}^*$ has an orthogonal, and norm-unique, Jordan decomposition $f = f_+ - f_-$ with $f_\pm \in \mathfrak{B}^*$,

$$\|f\| = \|f_+\| + \|f_-\|,$$

if, and only if, the norm on $\mathfrak{B}$ has the order theoretic property

$$\|a\| = \inf\{\lambda \geq 0;\ -\lambda u \leq a \leq \lambda v \text{ for some } u, v \in \mathfrak{B}_1\},$$

when $\mathfrak{B}_1$ is the unit ball of $\mathfrak{B}$. Various characterizations of the canonical half-norm associated with $\mathfrak{B}_+$ are also given.

0. Introduction. Let $\mathfrak{B}$ be a Banach space with a positive cone $\mathfrak{B}_+$ i.e., a norm-closed proper convex cone, and introduce the dual cone $\mathfrak{B}^*_+$, in the dual $\mathfrak{B}^*$ of $\mathfrak{B}$, by

$$\mathfrak{B}^*_+ = \{f \in \mathfrak{B}^*; f(a) \geq 0, a \in \mathfrak{B}_+\}.$$

It follows that $\mathfrak{B}^*_+$ is a norm-closed convex cone and if $\mathfrak{B}_+$ is weakly generating in the sense that $\mathfrak{B} = \overline{\mathfrak{B}_+ - \mathfrak{B}_+}$, where the bar denotes the closure, then $\mathfrak{B}^*_+$ is proper. We shall call $\mathfrak{B}_+$ *orthogonally generating* if every $a \in \mathfrak{B}$ admits a decomposition $a = a_1 - a_2$ with $a_i \in \mathfrak{B}_+$ $(i = 1, 2)$ and

$$\|a_1 + a_2\| = \|a_1 - a_2\|.$$

Clearly, every Banach lattice and the hermitian part of a C^*-algebra have orthogonally generating positive cones with $a_1 = a_+$ and $a_2 = a_-$ where $a_\pm$ denote the usual positive and negative components of a.

In general, the cones $\mathfrak{B}_+$ and $\mathfrak{B}^*_+$ define order relations on $\mathfrak{B}$ and $\mathfrak{B}^*$ respectively. If $a, b \in \mathfrak{B}$, one sets $a \geq b$ whenever $a - b \in \mathfrak{B}_+$. Similarly, if $f, g \in \mathfrak{B}^*$, one sets $f \geq g$ whenever $f - g \in \mathfrak{B}^*_+$.

The main purpose of this note is to determine conditions under which a general $f \in \mathfrak{B}^*$ has an orthogonal norm-unique Jordan decomposition, i.e., a decomposition of the form $f = f_+ - f_-$ with $f_\pm \in \mathfrak{B}^*_+$ such that

(1) (Jordan decomposition) $\|f\| = \|f_+\| + \|f_-\|$;

(2) (Orthogonality) $\|f_+ + f_-\| = \|f_+ - f_-\|$;

(3) (Norm-uniqueness) If $f = g_1 - g_2$ is another decomposition with the property (1), then

$$\|f_+\| = \|g_1\| \quad \text{and} \quad \|f_-\| = \|g_2\|.$$

Our principal result is the following:

THEOREM 1. *If $\mathfrak{B}_+$ is orthogonally generating, the following conditions are equivalent*:

1. *For every $a \in \mathfrak{B}$*

$$\|a\| = \inf\{\lambda \geq 0; -\lambda u \leq a \leq \lambda v, u, v \in \mathfrak{B}_1\},$$

where $\mathfrak{B}_1$ denotes the unit ball of $\mathfrak{B}$.

2. *Every $f \in \mathfrak{B}^*$ has an orthogonal norm-unique Jordan decomposition.*

3. *If $a = a_1 - a_2$ is an orthogonal decomposition of $a \in \mathfrak{B}$, then*

$$\|a\| = \|a_1\| \vee \|a_2\| = N(a) \vee N(-a)$$

where N is the canonical half-norm associated with $\mathfrak{B}_+$.

Before giving the definition of half-norms, we note that condition 1 is easily verified if $\mathfrak{B}$ is the hermitian part of a C^*-algebra. First set

$$\|a\|_1 = \inf\{\lambda \geq 0; -\lambda u \leq a \leq \lambda v, u, v \in \mathfrak{B}_1\}$$

and note that $\|a\|_1 \leq \|a\|$. Next adjoin an identity element $\mathbf{1}$ if necessary, and remark that in principle this reduces $\|\cdot\|_1$. But, if $-\lambda u \leq a \leq \lambda v$ with $u, v \in \mathfrak{B}_1$, then $(\mathbf{1} - u) \leq (\mathbf{1} + a/\lambda) \leq \mathbf{1} + v$ and $0 \leq \mathbf{1} + a/\lambda \leq 2\mathbf{1}$. Therefore, $\|a\| \leq \lambda$ and $\|a\| = \|a\|_1$.

The situation is quite different for order complete Banach lattices. Theorem 1 then implies (see [7], Example 1.5) that $\mathfrak{B}^*$ has such a Jordan decomposition if, and only if, $\mathfrak{B}$ is an AM-space.

The proof of Theorem 1 is based upon the notion of a *half-norm*, i.e., a function N over $\mathfrak{B}$ with the properties:

(N1) $0 \leq N(a) \leq k\|a\|$ for some $k > 0$,

(N2) $N(a_1 + a_2) \leq N(a_1) + N(a_2)$,

(N3) $N(\lambda a) = \lambda N(a)$ for all $\lambda > 0$,

(N4) $N(a) \vee N(-a) = 0$ if, and only if, $a = 0$.

The existence of a half-norm over $\mathfrak{B}$ is equivalent to the existence of a positive cone $\mathfrak{B}_+$ in $\mathfrak{B}$. In fact, if N is a half-norm on $\mathfrak{B}$, then

$$\mathfrak{B}_+ = \{a \in \mathfrak{B}; N(-a) = 0\}$$

is a positive cone. Conversely, if $\mathfrak{B}_+$ is a positive cone in $\mathfrak{B}$, then

$$N(a) = \inf\{\|a + b\|; b \in \mathfrak{B}_+\}$$

defines a half-norm over $\mathfrak{B}$. Following Arendt, Chernoff and Kato [2] we call this latter half-norm the *canonical half-norm* associated with $\mathfrak{B}_+$. Note that it automatically satisfies

$$0 \le N(a) \le \|a\|.$$

Half-norms are particularly useful for studying positive semigroups [2], [3], [9]. We derive various properties of half-norms in §2, after discussing the Jordan decomposition property in §1.

1. The Jordan decomposition. Throughout this section, let $\mathfrak{B}$ be a Banach space ordered by a positive cone $\mathfrak{B}_+$ and let N be a half-norm associated with $\mathfrak{B}_+$, i.e., N is such that

$$\mathfrak{B}_+ = \{a; N(-a) = 0\}.$$

LEMMA 2. *Let f be a linear functional on $\mathfrak{B}$. If there exists a constant $\alpha > 0$ such that*

$$f(a) \le \alpha N(a) \quad \textit{for all } a \in \mathfrak{B}$$

then f is positive and continuous. Conversely, if N is the canonical half-norm associated with $\mathfrak{B}_+$ and $f \in \mathfrak{B}^*$ is positive, then

$$f(a) \le \|f\| N(a) \quad \textit{for all } a \in \mathfrak{B}.$$

Proof. If $f(a) \le \alpha N(a)$ for all $a \in \mathfrak{B}$, then $-f(a) \le \alpha N(-a)$, and f is obviously positive. But by condition (N1),

$$|f(a)| \le \alpha N(a) \vee N(-a) \le \alpha k \|a\|$$

i.e., f is continuous. Conversely, if $f \in \mathfrak{B}^*$ is positive and N is canonical, we choose $b_n \in \mathfrak{B}_+$ such that $\|a + b_n\| < N(a) + 1/n$. Then,

$$f(a) \le f(a + b_n) \le \|f\| \, \|a + b_n\| \le \|f\| \left(N(a) + \frac{1}{n} \right).$$

Hence, $f(a) \le \|f\| N(a)$.

We denote by $\mathfrak{B}_N^*$ the set of all $f \in \mathfrak{B}^*$ such that $f(a) \le N(a)$ for all $a \in \mathfrak{B}$. The importance of this set is due to the following N-extension theorem

LEMMA 3. *For every $a \in \mathfrak{B}$, there exists $f \in \mathfrak{B}_N^*$ such that $f(a) = N(a)$.*

Proof. We may assume that $N(a) \ne 0$. Let $\mathfrak{M}$ be the linear space spanned by a and define a linear functional g on $\mathfrak{M}$ by

$$g(\xi a) = \xi N(a) \quad \text{for all } \xi \in \mathbf{R}.$$

Then, it is easy to see that

$$g(b) \leq N(b) \quad \text{for all } b \in \mathfrak{M}.$$

It now follows from the subadditivity and homogeneity of N that g has a linear extension f to $\mathfrak{B}$ satisfying the properties of the lemma (see, for example, [4], pages 65–66).

We remark that this lemma implies

$$N(a) = \sup\{f(a); f \in \mathfrak{B}_N^*\}.$$

Next, we define the conjugate N^* of N by

$$N^*(f) = \sup\{f(a);\, a \in \mathfrak{B}_+ ,\, \|a\| \leq 1\}$$

for every $f \in \mathfrak{B}^*$. Then, N^* has the following properties;

(N1)* $0 \leq N^*(f) \leq \|f\|$

(N2)* $N^*(f+g) \leq N^*(f) + N^*(g)$,

(N3)* $N^*(\lambda f) = \lambda N^*(f)$ for $\lambda \geq 0$.

In order that N^* is a half-norm on $\mathfrak{B}^*$ it must also satisfy the condition

(N4)* $N^*(f) \vee N^*(-f) = 0$ if, and only if, $f = 0$.

For this we need an assumption on the positive cone $\mathfrak{B}_+$. The positive cone $\mathfrak{B}_+$ is said to be *generating* if every $a \in \mathfrak{B}$ has a decomposition $a = a_1 - a_2$ with $a_i \in \mathfrak{B}_+$ $(i = 1, 2)$. Ando [1], has proved that when $\mathfrak{B}_+$ is generating there exists a constant $\rho > 0$ such that each $a \in \mathfrak{B}$ has a decomposition $a = a_1 - a_2$ with $a_i \in \mathfrak{B}_+$ $(i = 1, 2)$, and

$$\|a_1\| \vee \|a_2\| \leq \rho\|a\|.$$

When this is the case, we shall say that $\mathfrak{B}_+$ is *ρ-generating*.

LEMMA 4. *When $\mathfrak{B}_+$ is generating, N^* is a half-norm on $\mathfrak{B}^*$ and, for $f \in \mathfrak{B}^*$, f is positive if, and only if, $N^*(-f) = 0$.*

If $\mathfrak{B}_+$ is ρ-generating then

$$\|f\| \leq \rho(N^*(f) + N^*(-f)).$$

Proof. If $N^*(f) + N^*(-f) = 0$, we have $f(a) = 0$ when $a \geq 0$ or $a \leq 0$. Since $\mathfrak{B}_+$ is generating, this implies $f = 0$ and, hence, N^* is a half-norm on $\mathfrak{B}^*$. It is obvious that $N^*(-f) = 0$ if f is positive. The converse follows from

$$-f(a) \leq N^*(-f)\|a\| \quad \text{for } a \geq 0.$$

Now, to prove the last statement, assume that $\alpha > N^*(f) + N^*(-f)$ and choose α_i $(i = 1, 2)$ such that $\alpha = \alpha_1 + \alpha_2$, $N^*(f) < \alpha_1$ and $N^*(-f) < \alpha_2$.

Then, for every $a \in \mathfrak{B}$ and its decomposition $a = a_1 - a_2$ with $a_i \in \mathfrak{B}_+$ $(i = 1, 2)$,

$$f(a) = f(a_1) - f(a_2) < \alpha_1 \|a_1\| + \alpha_2 \|a_2\|$$

and

$$-f(a) = f(a_2) - f(a_1) < \alpha_1 \|a_2\| + \alpha_2 \|a_1\|.$$

Therefore,

$$|f(a)| < \alpha_1 \rho \|a\| + \alpha_2 \rho \|a\| < \alpha \rho \|a\|,$$

and, hence, $\| f \| \leq \alpha\rho$.

We remark that, if every element of $\mathfrak{B}$ admits a Jordan decomposition one has $\| f \| = N^*(f) + N^*(-f)$.

Now, we start the proof of Theorem 1. We begin with a result of Grosberg and Krein [5].

LEMMA 5. (*Grosberg-Krein*). *If N is the canonical half-norm associated with $\mathfrak{B}_+$ the following two conditions are equivalent*;

(1) $\| a \| = N(a) \vee N(-a)$ *for all* $a \in \mathfrak{B}$.

(2) *Every element of* $\mathfrak{B}^*$ *admits a Jordan decomposition.*

Proof. Assume that the condition 1 holds and set $P = \{ f \in \mathfrak{B}^*; \| f \| \leq 1, f \geq 0 \}$. Then, since $\mathfrak{B}_N^* \subset P$, we can conclude from Lemma 3 that the polar P^0 of P coincides with the closed unit ball $\mathfrak{B}_1$ of $\mathfrak{B}$. Hence, the closed unit ball $\mathfrak{B}_1^*$ of $\mathfrak{B}^*$ coincides with the bipolar P^{00}. Therefore Grothendieck's argument [6] leads us to condition 2. Conversely, if condition 2 holds and $N(a) \vee N(-a) < 1$, we choose an arbitrary $f \in \mathfrak{B}^*$ such that $\| f \| = 1$. Then, for $f_\pm \geq 0$ such that $f = f_+ - f_-$ and $\| f \| = \| f_+ \| + \| f_- \|$, we have

$$|f_+(a)| \leq \|f_+\| \quad \text{and} \quad |f_-(a)| \leq \|f_-\|$$

In fact, since N is canonical, we can find $b, c \in \mathfrak{B}_+$ such that

$$\|a + b\| < 1 \quad \text{and} \quad \|-a + c\| < 1.$$

Then,

$$f_+(a) \leq f_+(a + b) \leq \|f_+\| \quad \text{and} \quad -f_+(a) \leq f_+(-a + c) \leq \|f_+\|.$$

Therefore, $|f_+(a)| \leq \| f_+ \|$. Similarly, $|f_-(a)| \leq \| f_- \|$. Then, $|f(a)| \leq |f_+(a)| + |f_-(a)| \leq \| f \| = 1$ and, hence, $\| a \| \leq 1$.

It was proved in [7], Lemma 1.1, that the canonical half-norm N associated with $\mathfrak{B}_+$ satisfies

$$N(a) = \inf\{\lambda \geq 0; a \leq \lambda u, u \in \mathfrak{B}_1\}.$$

Therefore, condition 1 in Theorem 1 is another expression of $\|a\| = N(a) \vee N(-a)$. Hence, by the Grosberg-Krein theorem, Lemma 5, every element of $\mathfrak{B}^*$ admits a Jordan decomposition. We are going to show that this decomposition is orthogonal and norm-unique. Note that, if $\mathfrak{B}_+$ is orthogonally generating, it is 1-generating.In fact, if $a = a_1 - a_2$ is an orthogonal decomposition, then, since $\|a\| = \|a_1 + a_2\|$,

$$\begin{aligned} 2\|a_1\| &= \|(a_1 + a_2) + (a_1 - a_2)\| \\ &\leq \|a_1 + a_2\| + \|a_1 - a_2\| = 2\|a\|, \end{aligned}$$

which implies $\|a_1\| \leq \|a\|$. Similarly, $\|a_2\| \leq \|a\|$. Therefore, the following two lemmas, together with Lemma 5, prove that condition 1 implies condition 2 in Theorem 1.

LEMMA 6. *Assume that $\mathfrak{B}_+$ is 1-generating and $f = f_1 - f_2$ is a Jordan decomposition of $f \in \mathfrak{B}^*$. Then $\|f_1\| = N^*(f)$ and $\|f_2\| = N^*(-f)$.*

Proof. By Lemma 4, we have

$$\|f\| \leq N^*(f) + N^*(-f).$$

On the other hand, we have $N^*(f) \leq \|f_1\|$, because

$$f(a) = f_1(a) - f_2(a) \leq f_1(a) \leq \|f_1\|$$

if $a \geq 0$ and $\|a\| \leq 1$. Similarly, $N^*(-f) \leq \|f_2\|$. Then, since $\|f\| = \|f_1\| + \|f_2\|$, we must have $N^*(f) = \|f_1\|$ and $N^*(-f) = \|f_2\|$.

LEMMA 7. *Assume that $\mathfrak{B}_+$ is orthogonally generating and $f = f_1 - f_2$ is a Jordan decomposition of $f \in \mathfrak{B}^*$. Then f_1 and f_2 are orthogonal, i.e.,*

$$\|f_1 + f_2\| = \|f_1 - f_2\|$$

Proof. Let $a = a_1 - a_2$ be an orthogonal decomposition of $a \in \mathfrak{B}$. Then

$$\pm f(a) \leq (f_1 + f_2)(a_1 + a_2).$$

Hence,

$$|f(a)| \leq \|f_1 + f_2\| \; \|a_1 + a_2\| \leq \|f_1 + f_2\| \; \|a\|$$

which implies $\|f\| \leq \|f_1 + f_2\|$. On the other hand, if follows from the definition of Jordan decomposition that $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\| = \|f\|$. Therefore f_1 and f_2 are orthogonal.

Next, we prove that condition 2 implies condition 3 in Theorem 1. First, we note that we have

$$\|a\| = N(a) \vee N(-a)$$

by Lemma 5. Now, let $a = a_1 - a_2$ be an orthogonal decomposition. Then, as we have shown above, we have $\|a_i\| \leq \|a\|$ $(i = 1, 2)$. On the other hand, since $a \leq a_1$, we have $N(a) \leq N(a_1) \leq \|a_1\|$ and, similarly, $N(-a) \leq \|a_2\|$. Hence,

$$\|a\| \geq \|a_1\| \vee \|a_2\| \geq N(a) \vee N(-a) = \|a\|.$$

That condition 3 implies condition 1 in Theorem 1 is trivial.

2. Half-norms. Let $\mathfrak{B}$ be an ordered Banach space with a positive cone $\mathfrak{B}_+$. The equality

$$N(a) = \inf\{\|a + b\|;\, b \in \mathfrak{B}_+\} = \inf\{\lambda \geq 0;\, a \leq \lambda u,\, u \in \mathfrak{B}_1\}$$

referred to in §1, gives an order theoretic characterization of the canonical half-norm N associated with $\mathfrak{B}_+$.

The next theorem gives a criterion for another order theoretic characterization of N.

THEOREM 8. *The following conditions are equivalent*:
(1) $N(a) = \inf\{\lambda \geq 0;\, a \leq \lambda u,\, u \in \mathfrak{B}_1 \cap \mathfrak{B}_+\}$,
(2) *For each* $\varepsilon > 0$ *and* $a \in \mathfrak{B}$ *there is a decomposition*

$$a = a_+ - a_- \quad \textit{with } a_\pm \in \mathfrak{B} \textit{ and } \|a_+\| \leq (1 + \varepsilon)\|a\|.$$

Proof. Assume that condition 1 holds. If $\varepsilon > 0$ and $a \in \mathfrak{B}$, there is $u \in \mathfrak{B}_1 \cap \mathfrak{B}_+$ such that $a \leq N(a)(1 + \varepsilon)u$. Hence, $a = a_+ - a_-$ with $a_+ = N(a)(1 + \varepsilon)u$ and $a_- = a_+ - a$. But, $\|a_+\| \leq N(a)(1 + \varepsilon) \leq (1 + \varepsilon)\|a\|$. Conversely, assume that condition 2 holds. If $a \leq \lambda u$ with $u \in \mathfrak{B}_+$ and u has a decomposition $u = u_+ - u_-$ with $u_\pm \in \mathfrak{B}_+$ and $\|u_+\| \leq 1 + \varepsilon$, then $a \leq \lambda u_+$ and,

$$N(a) < \inf\{\lambda \geq 0;\, a \leq \lambda u,\, u \in \mathfrak{B}_1 \cap \mathfrak{B}_+\} \leq (1 + \varepsilon)N(a).$$

Since this estimate is valid for all $\varepsilon > 0$, one has the desired identification.

It was proved in [7], Proposition 1.6, that condition 2 with $\varepsilon = 0$ is implied by the following three equivalent conditions:

(i) There is a $u \in \mathfrak{B}_1$ such that

$$\{a: \|u - a\| < 1\} \subset \mathfrak{B}_+ ,$$

(ii) $\mathfrak{B}_1$ has a maximal element u,

(iii) there is a $u \in \mathfrak{B}_1$ such that $N = N_u$, where

$$N_u(a) = \inf\{\lambda \geq 0; a \leq \lambda u\}.$$

COROLLARY 9. *If* $(\mathfrak{B}, \mathfrak{B}_+)$ *is the dual of an ordered Banach space* $\mathfrak{B}_*$ *with positive cone* $\mathfrak{B}_{*+}$ *and if* N *is the canonical half-norm associated with* $\mathfrak{B}_+$, *then the following conditions are equivalent*:

(1) $N(a) = \inf\{\lambda \geq 0; a \leq \lambda u, u \in \mathfrak{B}_1 \cap \mathfrak{B}_+\}$,

(2) *each* $a \in \mathfrak{B}$ *has a decomposition* $a = a_+ - a_-$ *with* $a_\pm \in \mathfrak{B}_+$ *and* $\|a_+\| \leq \|a\|$.

Proof. In view of Theorem 8, we only need to show that condition 1 implies condition 2. Now, if condition 1 holds, it follows from Theorem 9 that for $\varepsilon > 0$ and $a \in \mathfrak{B}$ there is a $u_\varepsilon \in \mathfrak{B}_1 \cap \mathfrak{B}_+$ such that $a \leq N(a)(1 + \varepsilon)u_\varepsilon$. But $\mathfrak{B}_1 \cap \mathfrak{B}_+$ is weak* compact and hence u_ε has a weak* limit point u. Therefore

$$N(a)u(\omega) = \lim N(a)(1 + \varepsilon)u_\varepsilon(\omega) \geq a(\omega)$$

for all $\omega \in \mathfrak{B}_{*+}$ and $a \leq N(a)u$ by the definition of a dual cone. Now, $a = a_+ - a_-$ with $a_+ = N(a)u \in \mathfrak{B}_+$, $a_- = a_+ - a \in \mathfrak{B}_+$, and $\|a_+\| \leq N(a) \leq \|a\|$.

If $\mathfrak{B}$ is either a Banach lattice or the hermitian part of a C^*-algebra, then each $a \in \mathfrak{B}$ has a canonical decomposition $a = a_+ - a_-$ into positive and negative components $a_\pm \in \mathfrak{B}_+$ [8], [4]. In both cases, however $\|a_\pm\| \leq \|a\|$ and hence the canonical half-norm has the order theoretic characterization given by condition 1 of Theorem 8.

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