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# ENDOSCOPIC GROUPS AND BASE CHANGE C/R

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**We consider a real reductive group $G$ with complex points $G(\mathbf{C})$, Galois automorphism $\sigma$, and real points $G(\mathbf{R}) = \{g \in G(\mathbf{C}): \sigma(g) = g\}$. In general, an irreducible admissible representation $\Pi$ of $G(\mathbf{C})$ equivalent to its Galois conjugate $\Pi \circ \sigma$ need not be a lift from $G(\mathbf{R})$, even if $G$ is quasi-split over R. Following the results of $L$-indistinguishability we might expect this phenomenon to be related to the fact that $\sigma$-twisted conjugacy on $G(\mathbf{C})$ need not be "stable", and therefore attempt to match the various "unstable" combinations of $\sigma$-twisted orbital integrals on $G(\mathbf{C})$ with stable orbital integrals on certain groups $H(\mathbf{R})$. The principle of functoriality in the $L$-group would then suggest, with reservations in the nontempered case, a relation between the $\sigma$-twisted characters of representations of $G(\mathbf{C})$ fixed up to equivalence by $\sigma$ and the "dual lifts" to $G(\mathbf{C})$ of stable characters on the groups $H(\mathbf{R})$.**

**In this paper we define the relevant groups $H$... they turn out to be the endoscopic groups from $L$-indistinguishability... and prove a matching theorem for orbital integrals. As a preliminary to the proposed dual liftings of characters we also study the "factoring" of Galois-invariant Langlands parameters for $G(\mathbf{C})$.**

**1. Introduction.** We begin with two simple examples. Let $G(\mathbf{C}) = \mathbf{C}^x$ and $\sigma(z) = \bar{z}^{-1}$, $z \in \mathbf{C}^x$, so that $G(\mathbf{R}) = \{g \in G(\mathbf{C}): \sigma(g) = g\}$ is the unit circle in $\mathbf{C}^x$. A quasicharacter on $\mathbf{C}^x$ fixed by $\sigma$, i.e., trivial on the positive reals, need not be of the form $z \to \chi(z\sigma(z)) = \chi(z/\bar{z})$, with $\chi$ a character on the unit circle. At the same time $z \in \mathbf{C}^x$ is stably $\sigma$-conjugate to $-z$, but not $\sigma$-conjugate (see **[Sh6]** for definitions). Let $f \in C_c^\infty(\mathbf{C}^x)$ and write $f(r, \theta)$ for $f(re^{i\theta})$. Set $H_1 = H_2 = G$, so that $H_1(\mathbf{R}) = S^1$. Let

$$f_1(e^{i\theta}) = \frac{1}{2}\int_0^\infty (f(r, \theta/2) + f(r, \theta/2 + \pi))\, dr/r$$

and

$$f_2(e^{i\theta}) = \frac{1}{2}e^{i\theta/2}\int_0^\infty (f(r, \theta/2) - f(r, \theta/2 + \pi))\, dr/r$$

for $-\pi < \theta < \pi$. Then both $f_1$ and $f_2$ extend smoothly to $S^1$. If $\chi$ is a character on $S^1$ then $f \to \int_{-\pi}^{\pi} \chi(e^{i\theta})f_1(e^{i\theta})\, d\theta$ is a distribution on $\mathbf{C}^x$ representing the usual lift of $\chi$ to $G(\mathbf{C})$, i.e., representing the quasicharacter $z \to \chi(z\sigma(z))$. On the other hand, $f \to \int_{-\pi}^{\pi} \chi(e^{i\theta})f_2(e^{i\theta})\, d\theta$ lifts $\chi$ to the quasicharacter $z = re^{i\theta} \to \chi(z\sigma(z))e^{i\theta}$. We have therefore recovered the remaining Galois-invariant quasicharacters on $\mathbf{C}^x$.

For a general group, however, there are difficulties more akin to those for $L$-indistinguishability. Consider $G = \mathrm{SL}_2$. Let

$$H(\mathbf{R}) = \left\{ r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \right\}.$$

Note that if $\theta \not\equiv 0 \pmod{\pi}$ then $r(\theta)$ and $r(\theta + \pi)$ are stably $\sigma$-conjugate in $G(\mathbf{C})$ but not $\sigma$-conjugate (see [**Sh6**, Lemma 2.5.2]). For $f \in C_c^\infty(\mathrm{SL}_2(\mathbf{C}))$, define

$$f_H(r(\theta)) = e^{i\theta/2}(e^{i\theta} - e^{-i\theta})\big(\Phi_f^\sigma(\theta/2) + \Phi_f^\sigma(\theta/2 + \pi)\big),$$

for $-\pi < \theta < \pi$, where

$$\Phi_f^\sigma(\theta) = \int_{G(\mathbf{C})/H(\mathbf{R})} f(\sigma(g) r(\theta) g^{-1}) \frac{dg}{d\theta},$$

$dg$ denoting a Haar measure on $G(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})$. It can be shown that $f_H$ extends to a $C^\infty$ function on $H(\mathbf{R})$. Then $f \to \int_{H(\mathbf{R})} \chi f_H$ is a distribution on $\mathrm{SL}_2(\mathbf{C})$ (see [**Sh6**, §5.4] for an explicit formula). L. Clozel has shown that this distribution is, up to a constant, the twisted character of a Galois-fixed equivalence class of representations of $\mathrm{SL}_2(\mathbf{C})$. It is easily verified that all such classes of (irreducible, admissible) representations of $\mathrm{SL}_2(\mathbf{C})$ which are not lifts from $\mathrm{SL}_2(\mathbf{R})$ are lifts in this way.

Returning to the general problem, we find it convenient to consider $G(\mathbf{C})$ as the group of real points on a group $\tilde{G}$, and $\sigma$ as the restriction to $\tilde{G}(\mathbf{R})$ of an algebraic automorphism $\alpha$ of $\tilde{G}$ (cf. §2). Also, since $(\tilde{G}, \alpha)$ is our starting point, rather than $G$ itself, we may as well assume that $G$ is quasi-split over $\mathbf{R}$.

In this paper we will be concerned with the matchings for $\alpha$-twisted orbital integrals on $\tilde{G}(\mathbf{R})$; this includes the problem of determining what it is they should match. Theorem 7.1 is our main result, and §§2 to 6 are preparation for it. Also, as both a check on our definitions and a preliminary to the proposed dual liftings, we will consider the question of "factoring" Galois-invariant Langlands parameters for $G(\mathbf{C})$ or, equivalently [**L1**] $\alpha$-invariant parameters for $\tilde{G}(\mathbf{R})$. Theorem 8.1 is the main result.

In [**Sh6**] we started a study of the matching problem for $\alpha$-twisted orbital integrals. We found that, despite various "technical" difficulties, the jump formulas for twisted orbital integrals on $\tilde{G}(\mathbf{R})$ are closed related to those for ordinary orbital integrals on $G(\mathbf{R})$. Making convenient technical assumptions, we then put together a matching theorem involving the endoscopic groups from $L$-indistinguishability. In this paper we start afresh, making none of the technical assumptions of [**Sh6**]. We first define the notion of *endoscopic group for* $(\tilde{G}, \alpha)$. This turns out to be the same as the notion of endoscopic group in $L$-indistinguishability **[L3]**, **[Sh4]**. However, there is new information in the data for an endoscopic group $H$ for $(\tilde{G}, \alpha)$ and it is this information which allows us to formulate a matching theorem without the assumption (4.3.2) of **[Sh6]**. Moreover in relating the embeddings ${}^LH \hookrightarrow {}^L\tilde{G}$ relevant to our present problem to the embeddings ${}^LH \hookrightarrow {}^LG$ from $L$-indistinguishability we find a remarkable quasicharacter on $\tilde{H}(\mathbf{R}) \simeq H(\mathbf{C})$ which allows us to dispense with the "cross-section for the norm" in **[Sh6]** (cf. Lemma 6.4).

As always, the twisted orbital integrals must be normalized. The normalization factors will be written in a form suitable for global applications **[L3]** and, more specifically, in a form to reflect the connection with $L$-indistinguishability for real groups. The proof of Theorem 7.1 itself relies heavily on the proof of the matching theorem for $L$-indistinguishability (see **[Sh5]** for an outline of the latter proof).

We will follow the notation of **[Sh1]**–**[Sh7]** as closely as possible, especially with respect to $L$-group data. However, we now write $G(\mathbf{C})$ and $G(\mathbf{R})$ in place of $\mathbf{G}$ and $G$. The definitions in this paper may be presented in greater generality (cf. **[Sh7]**); in the general case there is no such intimate tie with $L$-indistinguishability.

**2. The groups $G$, $\tilde{G}$ and the automorphism $\alpha$.** Let $G$ be a connected reductive linear algebraic group defined over $\mathbf{R}$. Assume that $G$ is quasi-split over $\mathbf{R}$. In fixing the usual $L$-group data, we take $G$ itself for $G^*$, a quasi-split inner form of $G$, and the identity map for $\psi$, an inner twist from $G$ to $G^*$. Then $B^*$ will be a Borel subgroup over $\mathbf{R}$ in $G$, and $T^*$ a maximal torus over $\mathbf{R}$ in $B^*$. We form the dual $({}^LG^0, {}^LB^0, {}^LT^0, \{X_r\})$ with $r \in \Sigma({}^LB^0, {}^LT^0)$, the set of simple roots of ${}^LT^0$ in ${}^LB^0$. In fact it will be convenient to have fixed a root vector $X_r$, for any root $r$ of ${}^LT^0$ in ${}^LG^0$. We therefore fix a Chevalley basis and take for $\{X_r, r \in \Sigma({}^LB^0, {}^LT^0)\}$ the vectors so provided. Then ${}^LG = {}^LG^0 \rtimes W$, with $\sigma_G$ denoting the action of $1 \times \sigma \in W$ on ${}^LG^0$. See **[Sh 3, 4, or 5]** for further explanation of the notation.

Let $\tilde{G}$ be the group obtained from $G$ by restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$. We realize $\tilde{G}$ as $G \times G$ with Galois automorphism $\sigma_{\tilde{G}}$: $(x, y) \to (\sigma_G(y), \sigma_G(x))$. Then $\tilde{B}^* = B^* \times B^*$ will be the distinguished Borel subgroup defined over $\mathbf{R}$ and $\tilde{T}^* = T^* \times T^*$. We realize the $L$-group ${}^L\tilde{G}$ of $\tilde{G}$ as follows. Set ${}^L\tilde{G}^0 = {}^LG^0 \times {}^LG^0$, ${}^L\tilde{B}^0 = {}^LB^0 \times {}^LB^0$, ${}^L\tilde{T}^0 = {}^LT^0 \times {}^LT^0$, $X_{(r,r')} = (X_r, X_{r'})$ for all roots $r$, $r'$ of ${}^LT^0$ in ${}^LG^0$, and define $\sigma_{\tilde{G}}$: ${}^L\tilde{G}^0 \to {}^L\tilde{G}^0$ by $\sigma_{\tilde{G}}(g, h) = (\sigma_G(h), \sigma_G(g))$, $g, h \in {}^LG^0$. Then ${}^L\tilde{G} = {}^L\tilde{G}^0 \rtimes W$, with $\mathbf{C}^\times \times 1$ acting trivially and $1 \times \sigma$ by $\sigma_{\tilde{G}}$.

Let $\alpha\colon \tilde{G} \to \tilde{G}$ be the automorphism $(x, y) \to (y, x)$. We take the standard dual automorphism (cf. [**Sh7**]) of $\alpha$, and denote it by $\alpha$ also. Thus:

$$\alpha((g, h) \times w) = (h, g) \times w, \qquad g, h \in {}^LG^0, w \in W.$$

**3. Endoscopic groups for** $(\tilde{G}, \alpha)$. The following is a special case of the definitions in [**Sh7**]. Let $s \in {}^L\tilde{G}^0$. Then we set $N(s) = s\alpha(s)$, $\mathrm{Cent}(N(s), {}^L\tilde{G}^0) = \{g \in {}^L\tilde{G}^0\colon g^{-1}N(s)g = N(s)\}$ and $\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0) = \{g \in {}^L\tilde{G}^0\colon g^{-1}s\alpha(g) = s\}$. Call $s$ *$\alpha$-semisimple* if $\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0)$ is reductive. In §4 we will observe that $s$ is $\alpha$-semisimple if and only if $N(s)$ is semisimple (cf. Lemma 4.2). Let $\tilde{Z}^W$ be the group of $W$-invariants in the center of ${}^L\tilde{G}^0$. Thus $\tilde{Z}^W = {}^L\tilde{G}^0 \cap \mathrm{Center}({}^L\tilde{G}) = \{(g, \sigma_G(g)) \times 1 \times 1\colon g \in \mathrm{Center}({}^LG^0)\}$. Also

$$\mathrm{Cent}_\alpha(sz, {}^L\tilde{G}^0) = \mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0), \qquad s \in {}^L\tilde{G}^0, z \in \tilde{Z}^W.$$

*We will now use $s$ to denote a coset of $\tilde{Z}^W$ in ${}^L\tilde{G}^0$ and* $\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0)$ *to denote* $\mathrm{Cent}_\alpha(a, {}^L\tilde{G}^0)$ *for $a$ in the coset $s$.* Following [**Sh7**], we consider tuples

$$\left(s, {}^LH_s^0, {}^LB_s^0, {}^LT_s^0, \{Y\}, \rho_s\right)$$

where

(i) $s \in {}^L\tilde{G}^0$ is a coset of $\tilde{Z}^W$ consisting of $\alpha$-semisimple elements,

(ii) ${}^LH_s^0 = (\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0))^0$,

(iii) ${}^LB_s^0$ is a Borel subgroup of ${}^LH_s^0$,

(iv) ${}^LT_s^0 \subset {}^LB_s^0$ is a maximal torus in ${}^LH_s^0$,

(v) $\{Y\}$ is a set of root vectors for the simple roots of ${}^LT_s^0$ in ${}^LB_s^0$,

(vi) $\rho_s\colon W \to \mathrm{Aut}({}^LH_s^0, {}^LB_s^0, {}^LT_s^0, \{Y\})$ is a homomorphism which factors through $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ and is "*realized in* $\mathrm{Cent}_\alpha(s, {}^L\tilde{G})$", i.e. $\rho_s(w) = \mathrm{ad}\, n(w)|_{{}^LH_s^0}$, $w \in W$, for some $n(w) \in {}^L\tilde{G}^0 \times w$ such that $n(w)^{-1}a\alpha(n(w)) = a$ for each $a$ in the coset $s$.

Let ${}^LH_s = {}^LH_s^0 \rtimes W$, the action of $W$ on ${}^LH_s^0$ being that defined by $\rho_s$. Often we will write $\sigma_s$ for the automorphism $\rho_s(1 \times \sigma)$, and abbreviate $(s, {}^LH_s^0, \ldots, \rho_s)$ by $(s, {}^LH_s)$.

Two tuples

$$\left(s, {}^LH_s^0, {}^LB_s^0, {}^LT_s^0, \{Y\}, \rho_s\right) \quad \text{and} \quad \left(s', {}^LH_{s'}^0, {}^LB_{s'}^0, {}^LT_{s'}^0, \{Y'\}, \rho_{s'}\right)$$

are *equivalent* if there exists $g \in {}^L\tilde{G}^0$ such that ${}^LH_{s'}^0 = g^{-1}\,{}^LH_s^0 g$, ${}^LB_{s'}^0 = g^{-1}\,{}^LB_s^0 g$, ${}^LT_{s'}^0 = g^{-1}\,{}^LT_s^0 g$, $\{Y'\} = \{\mathrm{Ad}\, g^{-1}(Y)\}$ and if $n(w) \in \mathrm{Cent}_\alpha(s, {}^L\tilde{G})$ realizes $\rho_s(w)$ then $g^{-1}n(w)g$ lies in $\mathrm{Cent}_\alpha(s', {}^L\tilde{G})$ and realizes $\rho_{s'}(w)$, $w \in W$. The set of all equivalence classes will be denoted

$\mathfrak{S}(\tilde{G}, \alpha)$. Using the results of the next section and Lemma 2.3.3 of **[Sh4]**, we may show that $\mathfrak{S}(\tilde{G}, \alpha)$ is a finite set. Since this fact will not be needed we omit the proof.

Finally, we call a quasi-split group $H$ over **R** an *endoscopic group for* $(\tilde{G}, \alpha)$ if some ${}^L H_s$ as above is an $L$-group for $H$.

**4. The relation between endoscopic groups for $(\tilde{G}, \alpha)$ and endoscopic groups for $G$.** By the endoscopic groups for $G$ we mean the groups "$H$" of **[Sh4]**, i.e. essentially the groups of **[L1]**. The set $\mathfrak{S}(G)$, or $\mathfrak{S}(G, 1)$ in the more general notation of **[Sh7]**, and the tuples used in its definition will be taken from **[Sh4]** (...there is a small difference in the definitions of **[L3]**).

We embed ${}^L G$ "diagonally" in ${}^L\tilde{G}$, i.e. by the map $g \times w \to (g, g) \times w$, $g \in {}^L G^0$, $w \in W$, and will frequently *identify* ${}^L G$ *with its image in* ${}^L\tilde{G}$. As in **[Sh4]**, $Z^W$ will denote the set of $W$-invariants in the center of ${}^L G^0$.

By an $\alpha$-conjugacy class in ${}^L\tilde{G}^0$, we will mean a set $\{g^{-1} a\alpha(g);\ g \in {}^L\tilde{G}^0\}$, where $a \in {}^L\tilde{G}^0$.

LEMMA 4.1.

(i) *Each $\alpha$-conjugacy class in ${}^L\tilde{G}^0$ contains an element of the form* $(x, 1)$, $x \in {}^L G^0$.

(ii) *For* $x \in {}^L G^0$, $\mathrm{Cent}_\alpha((x, 1), {}^L\tilde{G}^0) = \mathrm{Cent}(x, {}^L G^0)$.

Here, of course, $\mathrm{Cent}(x, {}^L G^0)$ has been identified with its image in ${}^L\tilde{G}$ under the diagonal map.

*Proof.* Let $a = (g_1, g_2) \in {}^L\tilde{G}^0$, $g = (1, g_2)$. Then $g^{-1} a\alpha(g) = (1, g_2^{-1})(g_1, g_2)(g_2, 1) = (g_1 g_2, 1)$, so that (i) is proved. (ii) is also a simple calculation.

LEMMA 4.2. *$a \in {}^L\tilde{G}^0$ is $\alpha$-semisimple if and only if $N(a) = a\alpha(a)$ is semisimple.*

*Proof.* Let $a \in {}^L\tilde{G}^0$. Choose $g \in {}^L\tilde{G}^0$ such that $g^{-1} a\alpha(g) = (x, 1)$, for suitable $x \in {}^L G^0$. Then

$$\mathrm{Cent}_\alpha(a, {}^L\tilde{G}^0) = g\,\mathrm{Cent}_\alpha((x, 1), {}^L\tilde{G}^0) g^{-1} = g\,\mathrm{Cent}(x, {}^L G^0) g^{-1}.$$

On the other hand, $N(a) = g(x, x)g^{-1}$, so that

$$\mathrm{Cent}(N(a), {}^L\tilde{G}^0) = g\big(\mathrm{Cent}(x, {}^L G^0) \times \mathrm{Cent}(x, {}^L G^0)\big) g^{-1}.$$

The lemma then follows from standard facts.

LEMMA 4.3. *Let s be a coset of* $\tilde{Z}^W$ *in* ${}^L\tilde{G}^0$ *consisting of* $\alpha$*-semisimple elements. Then there exists* $g \in {}^L\tilde{G}^0$ *such that* $s' = g^{-1}s\alpha(g)$ *has the property that* $\{a\alpha(a): a \in s'\}$ *is contained in* ${}^LG^0$. *Then* $\{a\alpha(a): a \in s'\}$ *is contained in a unique coset of* $Z^W$ *in* ${}^LG^0$. *This coset, to be denoted* $N(s')$, *consists of semisimple elements.*

*Proof*. Let $a \in s$. Choose $g \in {}^L\tilde{G}^0$ such that $g^{-1}a\alpha(g) = (x, 1)$, where $x \in {}^LG^0$ is semisimple. Let $s' = (x, 1)\tilde{Z}^W$. Then if $b \in s'$, $b\alpha(b) = (x, x)(z\sigma_G(z), z\sigma_G(z))$, for some $z \in \mathrm{Cent}({}^LG^0)$. Thus, with our identifications, $b\alpha(b) \in xZ^W$, a coset of $Z^W$ in ${}^LG^0$ consisting of semisimple elements. The rest is clear.

LEMMA 4.4. *Each element of* $\mathfrak{S}(\tilde{G}, \alpha)$ *has a representative* $(s, {}^LH_s)$ *such that* $(N(s), {}^LH_s)$ *is a representative for an element of* $\mathfrak{S}(G)$ *i.e. such that* $\{a\alpha(a): a \in s\}$ *is contained in* ${}^LG^0$ (*...so that* $N(s)$ *is defined*), ${}^LH_s^0$ *coincides with* $(\mathrm{Cent}(N(s), {}^LG^0))^0$, *and* $\rho_s$ *is "realized in* $\mathrm{Cent}(N(s), {}^LG)$."

*Proof*. We may take $s = (x, 1)\tilde{Z}^W$, some $x \in {}^LT^0$. Then $N(s) = xZ^W$ and $\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0) = \mathrm{Cent}(N(s), {}^LG^0)$. We may also assume that ${}^LT_s^0 = {}^LT^0$, ${}^LB_s^0 = {}^LB^0 \cap {}^LH_s^0$ (...${}^LT^0$ and ${}^LB^0$ being identified with their images in ${}^L\tilde{G}^0$) and that $\{Y\} = \{X_r: r \in \Sigma({}^LB^0 \cap {}^LH_s^0, {}^LT^0)\}$. Then $\rho_s$ is a homomorphism of $W$ into $\mathrm{Aut}({}^LH_s^0, {}^LB^0 \cap {}^LH_s^0, {}^LT^0, \{Y\})$. Suppose that $\rho_s(w) = \mathrm{ad}\, n(w)|_{{}^LH_s^0}$, where $n(w) \in {}^L\tilde{G}^0 \times w$ satisfies $n(w)^{-1}(x, 1)\alpha(n(w)) = (x, 1)$ (cf. (vi) in §3). Then $n(w)^{-1}(x, x)n(w) = (x, x)$. Also, if $n(w) = (n_1(w), n_2(w)) \times w$ then calculation shows that for $w \in \mathbf{C}^x \times 1$ we have $n_1(w) = n_2(w)$ lies in the center of ${}^LH_s^0$ and for $w = 1 \times \sigma$ we have $n_1(w) = xn_2(w)$. Thus for all $w \in W$, $\rho_s(w) = \mathrm{ad}\, m(w)|_{{}^LH_s^0}$ where $m(w) = (n_1(w), n_1(w)) \times w \in {}^LG$. Also, $m(w)$ centralizes $(x, x)$. Thus $\rho_s$ is "realized in $\mathrm{Cent}(N(s), {}^LG)$" and the lemma is proved.

LEMMA 4.5. *The correspondence in Lemma* 4.4 *induces a map*

$$\mathfrak{N}: \mathfrak{S}(\tilde{G}, \alpha) \to \mathfrak{S}(G).$$

*Proof*. We have to show that if $(s, {}^LH_s)$ and $(s', {}^LH_{s'})$ are as in Lemma 4.4, representing the same element of $\mathfrak{S}(\tilde{G}, \alpha)$, then the 5-tuples defining ${}^LH_s$ and ${}^LH_{s'}$ are conjugate under ${}^LG^0$. They are conjugate under ${}^L\tilde{G}^0$, by definition. It is easily checked that this conjugation may be replaced by one from ${}^LG^0$.

The map $\mathfrak{N}$ need not be injective, as the example that $G$ is a compact torus shows. However $\mathfrak{N}$ does have finite fibers (which implies that $\mathfrak{S}(\tilde{G}, \alpha)$ is finite, as asserted in the last section). Reversing the construction in the proof of Lemma 4.4 shows that $\mathfrak{N}$ is surjective.

**5. Allowed embeddings of $^LH_s$ in $^L\tilde{G}$.** Fix an element of $\mathfrak{S}(\tilde{G}, \alpha)$, with representative $(s, {}^LH_s)$ chosen as in the proof of Lemma 4.4. In particular, $s = (x, 1)\tilde{Z}^W$, $x \in {}^LT^0$, and ${}^LH_s^0 = (\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0))^0 = (\mathrm{Cent}(N(s), {}^LG^0))^0$. We may further assume that ${}^LH_s^0$ is in standard position (cf. [**Sh3**, §2.2, Ex. 4.3.1]).

Suppose that $\xi$: ${}^LH_s \hookrightarrow {}^LG$ is an admissible embedding, as in $L$-indistinguishability [**L1**], [**Sh3**]. Here we regard ${}^LH_s^0$ as a subgroup of ${}^LG^0$ yet to be embedded diagonally in ${}^L\tilde{G}^0$, and assume that $\xi|_{{}^LH_s^0}$ is the inclusion map. The "diagonal" embedding of ${}^LG$ in ${}^L\tilde{G}$ then yields an embedding of ${}^LH_s$ in ${}^L\tilde{G}$, again denoted $\xi$. Explicitly, $\xi$ is of the form:

$$\xi(h \times 1 \times 1) = (h, h) \times 1 \times 1, \qquad h \in {}^LH_s^0,$$
$$\xi(1 \times z \times 1) = (\xi_0(z), \xi_0(z)) \times z \times 1, \qquad z \in \mathbf{C}^x,$$

where $\xi_0$: $\mathbf{C}^x \to \mathrm{Cent}({}^LH_s^0)$ is a homomorphism satisfying $\xi_0(\bar{z}) = \sigma_s(\xi_0(z))$, $z \in \mathbf{C}^x$, and

$$\xi(1 \times 1 \times \sigma) = (n_0, n_0) \times 1 \times \sigma,$$

where $n_0 \in {}^LG^0$ normalizes ${}^LT^0$, $n_0\sigma_G(n_0) = \xi_0(-1)$ and $n_0 \times 1 \times \sigma \in {}^LG$ acts on ${}^LH_s^0$ as $\sigma_s = \rho_s(1 \times \sigma)$. It follows immediately that $\xi({}^LH_s) \subset \mathrm{Cent}(N(s), {}^LG)$. However, our present problem dictates (cf. §8) that we consider embeddings for which the image of ${}^LH_s$ is contained in $\mathrm{Cent}_\alpha(s, {}^L\tilde{G})$. That this is a quite different condition is indicated even by the example that $G$ is a compact torus.

DEFINITION 5.1. Let $(s, {}^LH_s)$ be a representative for an element of $\mathfrak{S}(\tilde{G}, \alpha)$. Then $\tilde{\xi}$: ${}^LH_s \hookrightarrow {}^L\tilde{G}$ is an allowed embedding if:

(i) $\tilde{\xi}$ is an admissible homomorphism, i.e. $\tilde{\xi}$ is a homomorphism such that $\tilde{\xi}({}^LH_s^0 \times w) \subset {}^L\tilde{G}^0 \times w$, $w \in W$,

(ii) on ${}^LH_s^0$, $\tilde{\xi}$ is the inclusion mapping, and

(iii) $\tilde{\xi}({}^LH_s) \subset \mathrm{Cent}_\alpha(s, {}^L\tilde{G})$.

We return to our choice $s = (x, 1)\tilde{Z}^W$, etc. Once again it is more convenient to regard ${}^LH_s^0$ as a subgroup of ${}^LG^0$ yet to be embedded diagonally in ${}^L\tilde{G}^0$. Then an allowed embedding $\tilde{\xi}\colon {}^LH_s \hookrightarrow {}^L\tilde{G}$ is of the form:

$$\tilde{\xi}(h \times 1 \times 1) = (h, h) \times 1 \times 1, \qquad h \in {}^LH_s^0,$$
$$\tilde{\xi}(1 \times z \times 1) = (\tilde{\xi}_0(z), \tilde{\xi}_0(z)) \times z \times 1, \qquad z \in \mathbf{C}^\times,$$

where $\tilde{\xi}_0$ satisfies the same conditions as $\xi_0$ earlier, and

$$\tilde{\xi}(1 \times 1 \times \sigma) = (xm_0, m_0) \times 1 \times \sigma$$

where $m_0 \in {}^LG^0$ normalizes ${}^LT^0$, $xm_0\sigma_G(m_0) = \tilde{\xi}_0(-1)$, and $m_0 \times 1 \times \sigma \in {}^LG$ acts on ${}^LH_s^0$ as $\sigma_s$ (... then also $xm_0 \times 1 \times \sigma$ acts on ${}^LH_s^0$ as $\sigma_s$, as we have already used in the proof of Lemma 4.2).

Let ${}^L\tilde{H}_s^0 = {}^LH_s^0 \times {}^LH_s^0$. We of course regard ${}^L\tilde{H}_s^0$ as a subgroup of ${}^L\tilde{G}^0$. Define an action of $W$ on ${}^L\tilde{H}_s^0$ by requiring $\mathbf{C}^\times \times 1$ to act trivially and $1 \times \sigma$ to act by the automorphism $(h_1, h_2) \to (\sigma_s(h_2), \sigma_s(h_1))$. If ${}^LH_s$ is the $L$-group of $H$ then ${}^L\tilde{H}_s$ is the $L$-group of $\tilde{H} = \operatorname{Res}_{\mathbf{R}}^{\mathbf{C}} H$.

LEMMA 5.2. *Let $\tilde{\xi}$ be an allowed embedding of ${}^LH_s$ in ${}^L\tilde{G}$ and $\xi$ be an admissible embedding of ${}^LH_s$ in ${}^LG \subset {}^L\tilde{G}$. Then*

$$\tilde{\xi}(h \times w) = a(w)\xi(h \times w), \qquad h \in {}^LH_s^0, w \in W,$$

*where $a(w)$ is a 1-cocycle of $W$ in* $\operatorname{Cent}({}^L\tilde{H}_s^0)$.

*Proof*. This follows easily from our explicit description of $\tilde{\xi}$ and $\xi$. The details are omitted.

Suppose that $\tilde{\xi}$, $\tilde{\xi}'$ are both allowed embeddings of ${}^LH_s$ in ${}^L\tilde{G}$. Then $\tilde{\xi}'(w) = b(w)\tilde{\xi}(w)$, $w \in W$, where $w \to b(w)$ is a 1-cocycle of $W$ in the center of ${}^LH_s^0$ embedded diagonally in ${}^L\tilde{G}^0$. We conclude then that the image of ${}^LH_s$ under an allowed embedding is independent of the choice of embedding; we write thus simply "Image ${}^LH_s$." Suppose next that $(s, {}^LH_s)$ and $(s', {}^LH_{s'})$ are equivalent in the sense of §3. Fix $g \in {}^L\tilde{G}^0$ as in the definition. Suppose that $\tilde{\xi}$ is an allowed embedding of ${}^LH_s$ in ${}^L\tilde{G}$. Then ad $g$ and $\tilde{\xi}$ determine an allowed embedding of ${}^LH_{s'}$ in ${}^L\tilde{G}$. We conclude then that there is an allowed embedding of ${}^LH_s$ in ${}^L\tilde{G}$ if and only if there is such an embedding of ${}^LH_{s'}$. Moreover, when embeddings exist we have $g^{-1}(\text{Image}\,{}^LH_s)g = \text{Image}\,{}^LH_{s'}$.

We defer a study of the existence of allowed embeddings. Recall, however, that if the center of ${}^LG^0$ is connected then ${}^LH_s$ embeds admissibly in ${}^LG$ **[L1]**. The proof of this result can be used to show also that there is an allowed embedding of ${}^LH_s$ in ${}^L\tilde{G}$.

**6. Ingredients for the matching theorem.** Fix an element of $\mathfrak{S}(\tilde{G}, \alpha)$ with representative $(s, {}^LH_s)$ satisfying $s = (x, 1)\tilde{Z}^W$, etc., as in the last section. We assume that $\tilde{\xi}\colon {}^LH_s \hookrightarrow {}^L\tilde{G}$ is an allowed embedding. The main purpose of this section is to attach to $\tilde{\xi}$ normalizing factors to appear in the matching theorem of the next section. We will assume also that there is an admissible embedding of ${}^LH_s$ in ${}^LG$, say $\xi$. The choice of $\xi$ will not affect the normalization factors (cf. Lemma 6.2), but we write individual terms in the factors in a way that involves $\xi$, in order to make clear the relation with the factors from $L$-indistinguishability.

Let $H$ be an endoscopic group for $(\tilde{G}, \alpha)$ with $L$-group ${}^LH_s$. We fix a Borel subgroup $B_H$ over $\mathbf{R}$ containing the maximal torus $T_H$ over $\mathbf{R}$, and assume that $X^*(T_H) = X_*({}^LT^0) = X^*(T^*)$ and that $\Sigma(B_H, T_H)$ is the dual of $\Sigma({}^LB_s^0, {}^LT^0)$. The group $\tilde{H} = \mathrm{Res}_{\mathbf{R}}^{\mathbf{C}} H$ will also play a role. We set $\tilde{B}_H = B_H \times B_H$ and $\tilde{T}_H = T_H \times T_H$; ${}^L\tilde{H}_s$, which appeared in the last section, is an $L$-group for $\tilde{H}$.

Since $H$ is also an endoscopic group for $G$ we may invoke many of the definitions from $L$-indistinguishability (cf. **[L1]**, **[Sh4]**). Let $T$ be a maximal torus over $\mathbf{R}$ in $G$. A pseudodiagonalization (p.d.) $\eta$ of $T$ is a map from $T$ to $T^*$ of the form $T \overset{\mathrm{ad}\,x}{\to} T_0 \overset{\mathrm{ad}\,m}{\to} T^*$, where $x \in \mathfrak{A}(T)$ **[L1]**, $T_0 = xTx^{-1}$ is standard (i.e. the maximal $R$-split torus in $T_0$ lies in $T^*$) and $m$ belongs to the Levi group attached to $T_0$. Then $\sigma_{(T,\eta)}$ denotes the transfer, by $\eta$, of the Galois action on $T$ to $T^*$, and to $X^*(T^*) = X_*({}^LT^0)$, $X_*(T^*) = X^*({}^LT^0)$ and ${}^LT^0 = X_*({}^LT^0) \otimes \mathbf{C}^\times$.

The set $\mathfrak{T}_H(G) = \{(T, \eta)\colon \sigma_{(T,\eta)} \in \Omega({}^LH_s^0, {}^LT^0)\sigma_s\}$, where $\Omega({}^LH_s^0, {}^LT^0)$ denotes the Weyl group of $({}^LH_s^0, {}^LT^0)$, is the starting point for the definitions of **[Sh4**, §2.4]. We will use it again. First, because $G$ is quasi-split over $\mathbf{R}$, for each maximal torus $T'$ over $\mathbf{R}$ in $H$ there exists $h \in H(\mathbf{C})$ and $(T, \eta) \in \mathfrak{T}_H(G)$ such that $hT'h^{-1} = T_H$ and

$$X^*(T') \overset{\mathrm{ad}\,h}{\to} X^*(T_H) = X^*(T^*) \overset{\eta^{-1}}{\to} X^*(T)$$

lifts to an isomorphism $i(h, \eta)\colon T' \to T$ over $\mathbf{R}$. We say that $\gamma' \in H(\mathbf{R})$ *originates from* $\gamma \in G(\mathbf{R})$ *via* $(T, \eta)$ if $\gamma'$ is the preimage of $\gamma$ under some such map $i(h, \eta)$.

Recall that $s = (x, 1)\tilde{Z}^W$. Any element of this coset is of the form $a = (xz, \sigma_G(z))$, where $z$ is in the center of ${}^LG^0$. But $a\alpha(a) = (xz\sigma_G(z), xz\sigma_G(z))$, an element of ${}^LT^0 = \mathrm{Hom}(X^*({}^LT^0), \mathbf{C}^\times)$. Also $\sigma_s(x) = x$. Thus $\{a\alpha(a)\colon a \in s\}$ defines a family of quasicharacters on $X^*({}^LT^0)$, each invariant under $\sigma_{(T,\eta)}$, for any $(T, \eta) \in \mathfrak{T}_H(G)$. Fix $(T, \eta) \in \mathfrak{T}_H(G)$. Then, on transfer to $T$ via $\eta$, we get a family of quasicharacters on $X_*(T)$, each invariant under $\sigma_T$. On $X_*(T_{\mathrm{sc}})$, the span of the coroots of $T$ in $G$, these quasicharacters all coincide and so we have defined a single quasicharacter of the type used in $L$-indistinguishability (cf. **[L1]**, also **[Sh4**, §2.4]). Moreover on $\{\lambda^\vee \in X_*(T)\colon \sigma_T\lambda^\vee = -\lambda^\vee\}$, the quasicharacters coincide again. We therefore obtain a single character on

$$\left\{\lambda^\vee \in X_*(T) \colon \sigma_T\lambda^\vee = -\lambda^\vee\right\} / \left\{\mu^\vee - \sigma_T\mu^\vee : \mu^\vee \in X_*(T)\right\}$$

and thence by Tate-Nakayama duality, a character on $H^1(T) = H^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), T(\mathbf{C}))$. Unless otherwise indicated, $\kappa$ *will denote both the quasicharacter on* $X_*(T_{\mathrm{sc}})$ *and the character on* $H^1(T)$ *attached to s and the pair* $(T, \eta) \in \mathfrak{T}_H(G)$.

With $G$ embedded diagonally in $\tilde{G}$, we have $\tilde{T} = \mathrm{Res}_{\mathbf{R}}^{\mathbf{C}} T$ naturally embedded in $\tilde{G}$ as $\mathrm{Cent}(T, \tilde{G}) = T \times T$, for any maximal torus $T$ over $\mathbf{R}$ in $G$. The norm from $\tilde{T}$ to $T$ is obtained from the map $\tilde{T}(\mathbf{R}) \to T(\mathbf{R})$ defined by $\delta = (t, \sigma_G(t)) \to \delta\alpha(\delta) = (t\sigma_G(t), t\sigma_G(t))$. As in **[Sh6]** we regard the norm from $\tilde{G}$ to $G$ (...or from $\tilde{T}$ to $T$) as an (injective) map from the set of stable regular $\alpha$-semisimple twisted conjugacy classes in $\tilde{G}(\mathbf{R})$ (...or in $\tilde{T}(\mathbf{R})$) to the set of stable regular semisimple conjugacy classes in $G(\mathbf{R})$ (...or to $T(\mathbf{R})$). by Lemma 2.4.3(ii) of **[Sh6]** this norm from $\tilde{G}$ to $G$ can be recovered from the norms from $\tilde{T}$ to $T$, as $T$ ranges over the maximal tori over $\mathbf{R}$ in $G$.

Note that if $\eta\colon T \to T^*$ is a p.d., then so is $\eta \times \eta\colon \tilde{T} \to \tilde{T}^*$. Thus we can use $\eta$ to transfer data from $\tilde{T}$ to $\tilde{T}^*$ or from $\tilde{T}^*$ to $\tilde{T}$.

We come then to the normalizing factors. The admissible embedding $\xi\colon {}^LH_s \hookrightarrow {}^LG$ has been fixed, and ${}^LH_s$ chosen to satisfy the conditions of **[Sh3, Sh4]**. We may therefore write $\xi = \xi(\mu^*, \lambda^*)$, for suitable $\mu^*, \lambda^* \in X_*({}^LT^0) \otimes \mathbf{C}$, and define the attached correction (quasi) characters $\Lambda_{(T,\eta)}$ on $T(\mathbf{R})$, for $(T, \eta) \in \mathfrak{T}_H(G)$. Although the notation does not reflect it, $\Lambda_{(T,\eta)}$ depends on the choice of $\xi$.

Since $\tilde{\xi}\colon {}^LH_s \hookrightarrow {}^L\tilde{G}$ has also been fixed, we have the 1-cocycle $a(w)$ of $W$ in $\mathrm{Center}({}^L\tilde{H}_s^0)$ from Lemma 5.2. A procedure in **[L2]** attaches to $a(w)$ a quasicharacter on $\tilde{H}(\mathbf{R})$. This quasicharacter determines a pair $(\tilde{\mu}_0, \tilde{\lambda}_0)$ of elements from $X^*(\tilde{T}_H) \otimes \mathbf{C} = X_*({}^L\tilde{T}^0) \otimes \mathbf{C}$. We may also recover $(\tilde{\mu}_0, \tilde{\lambda}_0)$ directly from the 1-cocycle $a(w)$. Thus define $\tilde{\mu}_0, \tilde{\lambda}_0$ by

$$\lambda^\vee(a(z \times 1)) = z^{\langle \tilde{\mu}_0, \lambda^\vee \rangle} \bar{z}^{\langle \sigma_s\tilde{\mu}_0, \lambda^\vee \rangle}, \qquad z \in \mathbf{C}^x,$$

$$\lambda^\vee(a(1 \times \sigma)) = e^{2\pi i \langle \tilde{\lambda}_0, \lambda^\vee \rangle}$$

for $\lambda^\vee \in X^*({}^L\tilde{T}^0)$. Then $\tilde{\mu}_0$ is uniquely determined and $\tilde{\lambda}_0$ is uniquely determined modulo

$$X_*({}^L\tilde{T}^0) + \left\{\tilde{\lambda} - \tilde{\sigma}_s\tilde{\lambda} \colon \tilde{\lambda} \in X_*({}^L\tilde{T}^0) \otimes \mathbf{C}\right\}.$$

Also

$$\tilde{\mu}_0 - \tilde{\sigma}_s\tilde{\mu}_0 \in X_*({}^L\tilde{T}^0), \qquad 1/2(\tilde{\mu}_0 - \tilde{\sigma}_s\tilde{\mu}_0) \equiv \tilde{\lambda}_0 + \tilde{\sigma}_s\tilde{\lambda}_0 \mod X_*({}^L\tilde{T}^0),$$

and

$$\langle \tilde{\mu}_0, \lambda^\vee \rangle = 0, \qquad \langle \tilde{\lambda}_0, \lambda^\vee \rangle \in \mathbf{Z}$$

whenever $\lambda^\vee$ lies in the span of the roots of ${}^L\tilde{T}^0$ in ${}^L\tilde{H}_s^0$ (cf. §9.1 of [**Sh3**]). Here we have used $\tilde{\sigma}_s$ to denote the action of $1 \times 1 \times \sigma \in {}^L\tilde{H}_s$.

Let $(T, \eta) \in \mathfrak{T}_H(G)$. Then on transferring $(\tilde{\mu}_0, \tilde{\lambda}_0)$ to $\tilde{T}$ using $\eta$ we obtain the data also denoted $(\tilde{\mu}_0, \tilde{\lambda}_0)$ for a quasicharacter on $\tilde{T}(\mathbf{R})$ (cf. [**Sh3**, §4.1]). This quasicharacter will be denoted $a_{(T,\eta)}$.

LEMMA 6.1.

$$a_{(T,\eta)} \textit{ is } \alpha\textit{-invariant}.$$

*Proof*. We describe $a_{(T,\eta)}$ explicitly. Let $\delta = (t, \sigma_T(t)) \in \tilde{T}(\mathbf{R})$. Write $t$ as $\exp X$, $X \in \mathrm{Lie}(T(\mathbf{C})) = X_*(T) \otimes \mathbf{C}$. Then $\sigma_T(t) = \exp \sigma_T(\bar{X})$, where if $X = \sum_{i=1}^n \lambda_i^\vee \otimes z_i$ then $\sigma_T(\bar{X}) = \sum_{i=1}^n \sigma_T(\lambda_i^\vee) \otimes \bar{z}_i$. Because $a(\mathbf{C}^\times \times 1)$ lies in the diagonal subgroup of Center$({}^L\tilde{H}_s^0)$, as is evident from the form of the embeddings $\xi$ and $\tilde{\xi}$ (cf. last section), we must have $\tilde{\mu}_0$ lying in the diagonal subspace of $X_*({}^L\tilde{T}^0) \otimes \mathbf{C} = (X_*({}^LT^0) \otimes \mathbf{C}) \times (X_*({}^LT^0) \otimes \mathbf{C})$. Thus we write $\tilde{\mu}_0$ as $(\mu_0, \mu_0)$, $\mu_0 \in X_*({}^LT^0) \otimes \mathbf{C}$. As usual, we transfer $\mu_0$ to $X^*(T) \otimes \mathbf{C}$ via $\eta$ without change in notation. Then

$$a_{(T,\eta)}(\delta) = e^{\mu_0(X + \sigma_T(\bar{X}))}.$$

Since $\alpha(\delta) = (\exp \sigma_T(\bar{X}),\ \exp X)$ it is now clear that $a_{(T,\eta)}(\alpha(\delta)) = a_{(T,\eta)}(\delta)$, and the lemma is proved.

Note that $a_{(T,\eta)}$ is uniquely determined by the class of $a(w)$ in $H^1(W, \mathrm{Center}({}^L\tilde{H}_s^0))$, but is affected by a change in $\xi$ or $\tilde{\xi}$. The dependence on $\tilde{\xi}$ of our normalization factors is to be expected; the dependence on $\xi$ is not.

LEMMA 6.2. *Fix* $(T, \eta) \in \mathfrak{T}_H(G)$ *and* $\delta \in \tilde{T}(\mathbf{R})$. *Then* $\alpha_{(T,\eta)}(\delta)\Lambda_{(T,\eta)}(\delta\alpha(\delta))$ *depends on* $\tilde{\xi}$ *alone.*

*Proof*. The embedding $\xi$ may be replaced only by $h \times w \to a_0(w)\xi(h \times w)$, where $a_0(w)$ is a 1-cocycle of $W$ in the center of ${}^LH_s^0$ embedded diagonally in the center of ${}^L\tilde{H}_s^0$. Then $a(w)$ is replaced by $a_0(w)^{-1}a(w)$. The cocycle $a_0(w)$ defines first a quasicharacter $\chi$ on $H(\mathbf{R})$ and second a quasicharacter $\tilde{\chi}$ on $\tilde{H}(\mathbf{R})$. As before, we use $\eta$ to transfer data and define quasicharacters $\chi_{(T,\eta)}$ on $T(\mathbf{R})$ and $\tilde{\chi}_{(T,\eta)}$ on $\tilde{T}(\mathbf{R})$. Since $\Lambda_{(T,\eta)}$ is replaced by $\chi_{(T,\eta)}\Lambda_{(T,\eta)}$ and $a_{(T,\eta)}$ by $\tilde{\chi}_{(T,\eta)}^{-1}a_{(T,\eta)}$, we have only to show that $\tilde{\chi}_{(T,\eta)}(\delta) = \chi_{(T,\eta)}(\delta\alpha(\delta))$. Define parameters $\mu_1, \lambda_1 \in X_*({}^LT^0) \otimes \mathbf{C}$ for $\chi$ as usual; use the same symbols for their transfer to $X^*(T) \otimes \mathbf{C}$

via $\eta$. For $\tilde{\chi}$ we can use parameters $\tilde{\mu}_1 = (\mu_1, \mu_1)$, $\tilde{\lambda}_1 = (\lambda_1, \lambda_1)$ in $X_*({}^L\tilde{T}^0) \otimes \mathbf{C}$ (...or $X^*(\tilde{T}) \otimes \mathbf{C}$, after transfer). Since $\tilde{\chi}$ is clearly $\alpha$-invariant (see the last proof), we may take $\delta = (\exp X, \exp X)$, $X \in \mathrm{Lie}(T(\mathbf{R}))$. Then $\tilde{\chi}(\delta) = e^{\langle 2\mu_1, X\rangle}$ and $\chi(\delta\alpha(\delta)) = \chi(\delta^2) = e^{\langle \mu_1, 2X\rangle}$, so that the lemma is proved.

The next lemma is simple but very useful (cf. proof of Lemma 6.4). Each element of $H^1(T)$ can be represented by a cocycle $\sigma \to \exp i\pi\lambda^\vee$, where $\lambda^\vee \in X_*(T)$ and $\sigma_T\lambda^\vee = -\lambda^\vee$. We will use $\exp i\pi\lambda^\vee$ to denote this cocycle *and* its class in $H^1(T)$; of course, $\exp i\pi\lambda^\vee$ also denotes an element of $T(\mathbf{R}) \subset \tilde{T}(\mathbf{R})$. Recall that to $(T, \eta) \in \mathfrak{T}_H(G)$ and our fundamental datum $s = (x, 1)\tilde{Z}^W$ we have attached a character $\kappa$ on $H^1(T)$.

LEMMA 6.3.

$$a_{(T,\eta)}(\exp i\pi\lambda^\vee) = \kappa(\exp i\pi\lambda^\vee)$$

*for all* $\lambda^\vee \in X_*(T)$ *such that* $\sigma_T\lambda^\vee = -\lambda^\vee$.

Note that the left side alone appears to depend on the choice of $\xi$ and $\tilde{\xi}$. However a quasicharacter $\tilde{\chi}$ as in the last proof annihilates $\exp i\pi\lambda^\vee$, if $\lambda^\vee \in X_*(T)$ and $\sigma_T\lambda^\vee = -\lambda^\vee$. Indeed we then have $i\pi\lambda^\vee \in \mathrm{Lie}(T(\mathbf{R}))$, so that $\tilde{\chi}(\exp i\pi\lambda^\vee) = e^{2\pi i\langle \mu_1, \lambda^\vee\rangle} = 1$, since $\frac{1}{2}(\mu_1 - \sigma_T\mu_1) \equiv (\lambda_1 + \sigma_T\lambda_1)$ mod $X^*(T)$ implies that $\langle \frac{1}{2}(\mu_1 - \sigma_T\mu_1), \lambda^\vee\rangle = \langle \mu_1, \lambda^\vee\rangle$ lies in $\mathbf{Z}$. It then follows that neither side of the formula depends on $\xi$ or $\tilde{\xi}$.

*Proof of Lemma* 6.3. First we evaluate the right side. The cocycle $\sigma \to \exp i\pi\lambda^\vee$ corresponds under the Tate-Nakayama isomorphism to the coset of $\lambda^\vee$ in

$$H^{-1}(X_*(T)) = \{\mu^\vee \in X_*(T) : \sigma_T\mu^\vee = -\mu^\vee\} / \{\nu^\vee - \sigma_T\nu^\vee : \nu^\vee \in X_*(T)\}.$$

Thus $\kappa(\exp i\pi\lambda^\vee) = \lambda^\vee(x)$, where $s = (x, 1)\tilde{Z}^W$ was used to define $\kappa$. Note that we have transferred $\lambda^\vee$ to ${}^LT^0$ via $\eta$.

For the left side, we write $a(z \times 1) = (a_0(z), a_0(z))$, $z \in \mathbf{C}^x$, and $a(1 \times \sigma) = (xb_0, b_0)$, where $a_0(z)$, $b_0$ lie in the center of ${}^LH_s^0$. Since $i\pi\lambda^\vee \in \mathrm{Lie}(T(\mathbf{R}))$, we have $a_{(T,\eta)}(\exp i\pi\lambda^\vee) = e^{2\pi i\langle \mu_0, \lambda^\vee\rangle} = \lambda^\vee(a_0(-1))$, where again we have transferred $\lambda^\vee$ to ${}^LT^0$ without change in notation (cf. proof of Lemma 6.1). On the other hand, $a(1 \times \sigma)\tilde{\sigma}_s(a(1 \times \sigma)) = a(-1)$ implies that $a_0(-1) = xb_0\sigma_s(b_0) = xb_0\sigma_{(T,\eta)}(b_0)$. Since $\sigma_{(T,\eta)}\lambda^\vee = -\lambda^\vee$, we have that $\lambda^\vee(a_0(-1)) = \lambda^\vee(x)$, and the lemma is proved.

We continue with $(T, \eta) \in \mathfrak{T}_H(G)$ and associated character $\kappa$ on $H^1(T)$. Fix a set $\{u = \exp i\pi\lambda^\vee : \lambda^\vee \in X_*(T), \sigma_T\lambda^\vee = -\lambda^\vee\}$ such that the cocycles $\sigma \to \exp i\pi\lambda^\vee$ form a complete set of (noncohomologous) representatives for the elements of $H^1(T)$.

For $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$, and Haar measures $dt$ on $T(\mathbf{R})$, $d\tilde{g}$ on $\tilde{G}(\mathbf{R})$ form (cf. **[Sh6]**):

$$\Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}) = \sum_u \kappa(u) \int_{\tilde{G}(\mathbf{R})/T(\mathbf{R})} f(\alpha(\tilde{g}) u \delta \tilde{g}^{-1}) \frac{d\tilde{g}}{dt},$$

for $\delta \in \tilde{T}(\mathbf{R})$ such that $\delta\alpha(\delta)$ is regular. Note that for all $\delta \in \tilde{T}(\mathbf{R})$, $\delta\alpha(\delta)$ lies in $T(\mathbf{R})^0$, the identity component of $T(\mathbf{R})$.

LEMMA 6.4.

$$\gamma \to a_{(T,\eta)}(\delta) \Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}),$$

*if* $\delta\alpha(\delta) = \gamma$, $\gamma \in T(\mathbf{R})^0_{\mathrm{reg}} = T(\mathbf{R})^0 \cap G_{\mathrm{reg}}$, *is a well-defined function on* $T(\mathbf{R})^0_{\mathrm{reg}}$.

*Proof*. By Lemma 6.3,

$$a_{(T,\eta)}(\delta) \Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}) = \sum_u a_{(T,\eta)}(u\delta) \int_{\tilde{G}(\mathbf{R})/T(\mathbf{R})} f(\alpha(\tilde{g}) u \delta \tilde{g}^{-1}) \frac{d\tilde{g}}{dt}$$

which we will write as $\Phi(\delta)$. If $\delta\alpha(\delta) = \delta'\alpha(\delta')$ then $\delta' = v\delta$, where $v\alpha(v) = 1$, $v \in \tilde{T}(\mathbf{R})$. Then it is easily seen that $v = t^{-1}\alpha(t)u$ for some $t \in \tilde{T}(\mathbf{R})$ and $u$ as in the summation. Since $a_{(T,\eta)}$ is $\alpha$-invariant we then have $\Phi(\delta') = \Phi(v\delta) = \Phi(u\delta)$ which clearly coincides with $\Phi(\delta)$. Thus the lemma is proved.

Finally, suppose that $(T, \eta) \in \mathfrak{T}_H(G)$ and that $i(h, \eta)$: $T' \to T$ is defined over $\mathbf{R}$. Then the Haar measure $dt$ on $T(\mathbf{R})$ is transported via $i(h, \eta)$ to a Haar measure $dt'$ on $T'(\mathbf{R})$; $dt'$ is independent of the choice of $h$. Also, we say that $\gamma' \in T'(\mathbf{R})_{\mathrm{reg}}$ *is not a norm* if it is not in the image of the norm map from $\tilde{T}' = \mathrm{Res}_{\mathbf{R}}^{\mathbf{C}} T'$ to $T'$, i.e. $\gamma'$ does not lie in the identity component of $T'(\mathbf{R})$. Then if $\gamma'$ originates from $\gamma \in T(\mathbf{R})_{\mathrm{reg}}$ via $(T, \eta)$, $\gamma$ is not in the image of the norm from $\tilde{T}$ to $T$ (and conversely...).

We have not assumed that $\xi$ or $\tilde{\xi}$ is of "unitary type" **[Sh3]**. It is easily checked that there is a quasicharacter $\chi$ on $H(\mathbf{R})$ such that $|\chi(\gamma')\Lambda_{(T,\eta)}(\gamma) a_{(T,\eta)}(\delta)| = 1$ if $\gamma'$ originates from $\gamma = \delta\alpha(\delta)$ via $(T, \eta)$. We then define $\mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ to be the set of functions $f$ on $H(\mathbf{R})$ such that $f\chi$ belongs to $\mathcal{C}(H(\mathbf{R}))$, the Schwartz space of $H(\mathbf{R})$. As the notation indicates, this space does not depend on the choice of $\chi$. For $f \in \mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ the stable orbital integrals $\Phi_f^{(T',1)}(\gamma', dt', dh)$, $\gamma' \in T'(\mathbf{R}) \cap H_{\text{reg}}$ (cf. [**Sh4**] etc.) are well-defined.

It remains now to recall the factor $\Delta_{(T,\eta)}$ from $L$-indistinguishability. Thus

$$\Delta_{(T,\eta)} = (-1)^{q(G,H)} \varepsilon(T,\eta) \Lambda_{(T,\eta)}{}' \Delta_{(T,\eta)},$$

where $q(G, H)$ is an integer, $(-1)^{q(G,H)}$ being inserted only for convenience, $\varepsilon(T, \eta) = \pm 1$ is defined implicitly, $\Lambda_{(T,\eta)}$ is as earlier in this section and $'\Delta_{(T,\eta)}$ is a discriminant function (see [**Sh4**, §3] for further details).

## 7. The matching theorem.

THEOREM 7.1. *Let $H$ be an endoscopic group for $(\tilde{G}, \alpha)$, with $L$-group ${}^LH_s$ chosen as earlier. Suppose that $\tilde{\xi}$: ${}^LH_s \hookrightarrow {}^L\tilde{G}$ is an allowed embedding and that $\xi$: ${}^LH_s \hookrightarrow {}^LG$ is admissible (for L-indistinguishability). Then for each $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$ there exists $f_H \in \mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ such that*:

$$\Phi_{f_H}^{(T',1)}(\gamma', dt', dh) = \begin{cases} \Delta_{(T,\eta)}(\gamma) a_{(T,\eta)}(\delta) \Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}), \\ \text{if } \gamma' \text{ originates from } \quad \gamma = \delta\alpha(\delta) \\ \text{via } (T,\eta) \in \mathcal{T}_H(G), \\ 0 \quad \text{if } \gamma \text{ is not a norm}. \end{cases}$$

Here it is assumed that $\gamma'$ originates from *regular* elements in $G(\mathbf{R})$. Then $\gamma'$ is regular in $H(\mathbf{R})$ [**Sh2**]; $T'$ is the maximal torus containing $\gamma'$. Recall that $\Delta_{(T,\eta)}$ depends on $\xi$ alone, that $a_{(T,\eta)}$ depends on both $\xi$ and $\tilde{\xi}$, and that $\Delta_{(T,\eta)}(\gamma) a_{(T,\eta)}(\delta)$ depends on $\tilde{\xi}$ alone... as long as $(T, \eta)$ and $\delta$ are fixed.

REMARK. We have used $C_c^\infty(\tilde{G}(\mathbf{R}))$ instead of the more natural $\mathcal{C}(\tilde{G}(\mathbf{R}))$ since the necessary analysis of "twisted $F_f$" (cf. [**Sh6**]), for $f$ a Schwartz function, has not been carried out. Work of L. Clozel now in progress should settle this matter and allow us to replace $C_c^\infty(\tilde{G}(\mathbf{R}))$ by $\mathcal{C}(\tilde{G}(\mathbf{R}))$.

*Proof of the theorem*. Let $\gamma' \in H(\mathbf{R})$. Suppose that $\gamma'$ originates from $\gamma \in G_{\text{reg}}$ via $(T, \eta)$ and from $\bar{\gamma}$ via $(\bar{T}, \bar{\eta})$. Choose $\delta$ so that $\delta\alpha(\delta) = \gamma$. Write $\bar{\gamma}$ as $y\gamma y^{-1}$ and $\bar{\eta}$ as $\omega_H \circ \eta \circ \text{ad}\, y^{-1}$, where $\omega_H \in \Omega(H, T_H) \subset \Omega(G, T^*)$ and $y \in \mathfrak{A}(T)$ (cf. [**Sh4**, §3]). Then for $\bar{\delta}$ such that $\bar{\delta}\alpha(\bar{\delta}) = \bar{\gamma}$ we may take $y\delta y^{-1}$, where $y \in G$ has been identified with its image in $\tilde{G}$ under the diagonal embedding. With this choice of $\bar{\delta}$ we have $a_{(\bar{T},\bar{\eta})}(\bar{\delta}) = a_{(T,\eta)}(\delta)$. The relation between $\Delta_{(\bar{T},\bar{\eta})}(\bar{\gamma})$ and $\Delta_{(T,\eta)}(\gamma)$ is described in [**Sh4**, §3].

For *fixed* $(T, \eta) \in \mathcal{T}_H(G)$ the function

$$\gamma' \to a_{(T,\eta)}(\delta)\Delta_{(T,\eta)}(\gamma)\Phi_f^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}),$$

if $\gamma'$ originates from $\gamma = \delta\alpha(\delta)$ via $(T, \eta)$, is well-defined and invariant under $\mathfrak{A}(T')$. To prove this we invoke [**Sh4**, Propositions 2.4.5 and 3.1.2] and [**Sh6**, Lemma 4.3.2]. These results show that we have only to check that $a_{(T,\eta)}(\delta^\omega) = a_{(T,\eta)}(\delta)$ for $\omega$ an element of the Weyl group $\Omega(G, T)$ of $(G, T)$ which commutes with the Galois action on $T$ and "comes from $H$" (i.e. $\omega \in \Omega_0(G, T) \cap \Omega^{(\kappa)}(G, T)$ as in [**Sh4**, Proposition 2.4.5]). But this invariance of $a_{(T,\eta)}$ follows easily from the fact that $\langle \tilde{\mu}_0, \lambda^\vee \rangle = 0$ for $\lambda^\vee$ in the span of the roots of ${}^L\tilde{T}^0$ in ${}^L\tilde{H}_s^0$ (see the proof of Lemma 6.1).

Suppose now that we fix a "framework of Cartan subgroups [**Sh3**], [**Sh4**, §3.2]. Thus we have specified certain pairs $(T_n, \eta_n) \in \mathcal{T}_H(G)$ and embeddings $i_n = i(h_n, \eta_n)$: $T_n' \to T_n$ over $\mathbf{R}$; the set $\{T_n'(\mathbf{R})\}$ provides a complete family of representatives, without redundancy, for the conjugacy classes of Cartan subgroups of $H(\mathbf{R})$. Given $\gamma' \in T_n'(\mathbf{R})$, set $\gamma = i_n(\gamma')$, and choose any $\delta$ such that $\delta\alpha(\delta) = \gamma$. Call $\gamma'$ $G$-regular if $\gamma$ is regular. Then for each $n$ we may consider the function on the $G$-regular elements of $T_n'(\mathbf{R})$ given by

$$\Phi_n(\gamma', dt', dh) = \begin{cases} \varepsilon_n \hat{\Delta}_{(T_n,\eta_n)}(\gamma) a_{(T_n,\eta_n)}(\delta) \Phi_f^{(T_n,\alpha,\kappa_n)}(\gamma, dt, d\tilde{g}) \\ \text{if } \gamma' \in T_n'(\mathbf{R})^0, \\ 0 \quad \text{if } \gamma' \notin T_n'(\mathbf{R})^0, \end{cases}$$

where $\varepsilon_n = \pm 1$ (to be chosen), $\hat{\Delta}_{(T,\eta)} = \varepsilon(T_n, \eta_n)\Delta_{(T_n,\eta_n)}$ (i.e. $\hat{\Delta}_{(T,\eta)}$ is $\Delta_{(T,\eta)}$ with the $\varepsilon(T, \eta)$ removed), and $\kappa_n$ is the "$\kappa$" associated to $(T_n, \eta_n)$. Note that $\{\kappa_n|_{X_*((T_n)_{\mathrm{sc}})}\}$ is exactly the set $\{\kappa_n\}$ from [**Sh2**, §7] and [**Sh3**, §2].

Suppose that we are able to show that there exists $f_H \in \mathcal{C}_{\tilde{\xi}}(H(\mathbf{R}))$ such that

$$(*) \qquad \Phi_{f_H}^{(T_n',1)}(\gamma', dt', dh) = \begin{cases} \Phi_n(\gamma', dt', dh) & \text{if } \gamma' \in T_n'(\mathbf{R})^0, \\ 0 \quad \text{if } \gamma' \notin T_n'(\mathbf{R})^0, \end{cases}$$

for all $G$-regular $\gamma'$ in $T_n'(\mathbf{R})$ and for all $n$ *provided* $\varepsilon_m\varepsilon_n = \varepsilon(m, n)$ whenever $T_m'(\mathbf{R})$ and $T_n'(\mathbf{R})$ are adjacent Cartan subgroups. Here $\varepsilon(m, n)$ is as defined in [**Sh4**, §3.5] (cf. [**Sh2**]). Then we shall take $\varepsilon_n = \varepsilon(T_n, \eta_n)$, so that by the results of $L$-indistinguishability (exp. [**Sh4**, §3.5]) there does exist $f_H$ satisfying $(*)$. It is then routine to verify that $f_H$ satisfies the statement of our theorem (see the first paragraph of this proof; similar arguments for $L$-indistinguishability are given in [**Sh4**, §3]).

Returning to the condition on the existence of $f_H$, we have only to show that our family $\{\Phi_n(\ ,\ ,\ )\}$ behaves like the family $\{\Phi_n\}$ of [**Sh2**, §9] (cf. [**Sh4**, §3.2]). The invariance and growth requirements being satisfied (clearly), only the "jump conditions" remain. Thus we need the jump formulas for the functions $\Psi_{(T,\eta)}$:

$$\gamma \to \begin{cases} a_{(T,\eta)}(\delta)\hat{\Delta}_{(T,\eta)}(\gamma)\Phi^{(T,\alpha,\kappa)}(\delta, dt, d\tilde{g}) & \text{if } \gamma \in T(\mathbf{R})^0_{\text{reg}}, \\ 0 \quad \text{if } \gamma \in T(\mathbf{R})_{\text{reg}} - T(\mathbf{R})^0. \end{cases}$$

These are contained essentially in the analysis of §§4 and 5 of [**Sh6**]. To be more precise, we seek analogues of Lemmas 5.2.2 and 5.2.5 of [**Sh6**], when "$'\Delta_T\Phi^\tau$" is replaced by the function above (with the necessary adjustment in the choice of positive system for the imaginary roots of $T$ used to define the factor $\hat{\Delta}_{(T,\eta)}$). The proof of the analogue of Lemma 5.2.2 is straightforward; because of notational complications we omit further details. Note that the "$\kappa$-signature" [**Sh2**] which appears depends only on $\kappa|_{X_*(T_{sc})}$, i.e. the jump is indeed like that from $L$-indistinguishability. The analogue of Lemma 5.2.5 will be stronger than the original statement, because we no longer need the assumption "$\kappa(\alpha^\vee) = 1$ if (5.2.3) holds." We now have the exact analogue of [**Sh2**, Proposition 9.1] from $L$-indistinguishability. Indeed, let $\gamma_0$ be a semiregular element in $T(\mathbf{R})$ such that $\lambda(\gamma_0) = 1$, where $\lambda$ is an imaginary root such that $\kappa(\lambda^\vee) = -1$. We wish to show that $\Psi_{(T,\eta)}$ is smooth on some neighborhood of $\gamma_0$. We may assume that $\gamma_0 \in T(\mathbf{R})^0$. Fix $\delta_0 \in T(\mathbf{R})^0$ such that $\delta_0^2 = \gamma$. For $\gamma$ close to $\gamma_0$ choose $\delta$ close to $\delta_0$ such that $\delta^2 = \gamma$. It will be sufficient to show that $\delta \to \Psi_{(T,\eta)}(\delta^2)$ is smooth near $\delta_0$. This follows immediately from Lemma 4.3.3 of [**Sh6**]. Note that this type of argument could not be used in the proof of Lemma 5.2.5 of [**Sh6**] because the "cross-section for the norm" was not smooth near $\gamma_0$.

We now complete the proof of Theorem 7.1 by the arguments already indicated.

**8. The dual lifting.** Again we fix an element of $\mathfrak{S}(\tilde{G}, \alpha)$ and choose a convenient representative $(s, {}^LH_s)$ for this element, as in §5. Let $H_s$ be the corresponding endoscopic group. Since $H_s$ is, by definition, quasi-split over $\mathbf{R}$, the set $\Phi(H_s)$ [**L2**] consists of all equivalence classes of admissible homomorphisms $\phi$: $W \to {}^LH_s$. Suppose that $\tilde{\xi}$: ${}^LH_s \hookrightarrow {}^L\tilde{G}$ is an allowed embedding. Then $\tilde{\xi}$ induces a map, also to be denoted $\tilde{\xi}$, from $\Phi(H_s)$ to $\Phi(\tilde{G})$; the image of the class of $\phi$: $W \to {}^LH_s$ is the class of $\tilde{\phi} = \tilde{\xi} \circ \phi$: $W \to {}^L\tilde{G}$. It is easily checked that the image of $\Phi(H_s)$ in $\Phi(\tilde{G})$ is independent of the choice for $\tilde{\xi}$. By the remarks at the end of §5 it is also independent of the choice for $(s, {}^LH_s)$.

On the other hand, the automorphism $\alpha$ of $\tilde{G}$ has a standard dual **[Sh7]**, again denoted $\alpha$:

$$\alpha((g, h) \times w) = (h, g) \times w, \qquad h, g \in {}^L G^0, w \in W.$$

If $\phi$: $W \to {}^L\tilde{G}$ is admissible then so is $\alpha \circ \phi$: $W \to {}^L\tilde{G}$. We write $\{\phi\}$ for the class of $\phi$ and $\{\phi\}^\alpha$ for the class of $\alpha \circ \phi$. Then $\Phi(\tilde{G})^\alpha = \{\{\phi\} \in \Phi(\tilde{G})$: $\{\phi\}^\alpha = \{\phi\}\}$.

For *each* element of $\mathfrak{S}(\tilde{G}, \alpha)$ we fix a representative $(s, {}^L H_s)$ as before, and assume that *each* ${}^L H_s$ has an allowed embedding $\xi$ in ${}^L\tilde{G}$. Also, we will use $\bigcup_{H_s}$ to denote a union over the corresponding endoscopic groups.

THEOREM 8.1.

$$\Phi(\tilde{G})^\alpha = \bigcup_{H_s} \xi(\Phi(H_s)).$$

*Proof.* Let $\phi$: $W \to {}^L H_s$ be admissible. Set $\tilde{\phi} = \xi \circ \phi$. We may assume that $\phi(\mathbf{C}^\times \times 1) \subset {}^L T^0 \times \mathbf{C}^\times \times 1$. Then clearly $\tilde{\phi}$ and $\alpha \circ \tilde{\phi}$ coincide on $\mathbf{C}^\times \times 1$. We write $\phi(1 \times \sigma)$ as $n_H \times 1 \times \sigma \in {}^L H_s$, and $\xi(1 \times 1 \times \sigma)$ as $(xm_0, m_0) \times 1 \times \sigma$ (cf. §5). Then $\tilde{\phi}(1 \times \sigma) = (xn_H m_0, n_H m_0) \times 1 \times \sigma$ and

$$\begin{aligned}(\alpha \circ \tilde{\phi})(1 \times \sigma) &= (n_H m_0, xn_H m_0) \times 1 \times \sigma \\ &= (x^{-1}, x)\tilde{\phi}(1 \times \sigma) = g^{-1}\tilde{\phi}(1 \times \sigma) g,\end{aligned}$$

where $g = (x, 1)$. Then clearly $\alpha \circ \tilde{\phi} = \operatorname{ad} g^{-1} \circ \tilde{\phi}$, and so $\xi(\Phi(H)) \subset \Phi(\tilde{G})^\alpha$.

Suppose now that $\tilde{\phi}$: $W \to {}^L\tilde{G}$ is admissible and that $\{\tilde{\phi}\}^\alpha = \{\tilde{\phi}\}$. Then it is sufficient to show that $\tilde{\phi}$ factors through some ${}^L H_s$ (not necessarily among our fixed representatives) embedded (via an allowed embedding) in ${}^L\tilde{G}$.

Let $S^\alpha_{\tilde{\phi}} = \{a \in {}^L\tilde{G}^0$: $a\tilde{\phi}(w)a^{-1} = (\alpha \circ \tilde{\phi})(w)$, $w \in W\}$. Then $S^\alpha_{\tilde{\phi}}$ is nonempty. If $a_0$ lies in $S^\alpha_{\tilde{\phi}}$ then so does $a_0 z$, for $z \in \tilde{Z}^W$. In fact, then $S^\alpha_{\tilde{\phi}} = a_0 S_{\tilde{\phi}}$, where $S_{\tilde{\phi}}$ is the centralizer of $\tilde{\phi}(W)$ in ${}^L\tilde{G}^0$... recall that the results of **[Sh4]**, with a little extra argument for the case $\tilde{\phi}$ unbounded, show that $S_{\tilde{\phi}} = S^0_{\tilde{\phi}} \tilde{Z}^W$, $S^0_{\tilde{\phi}}$ denoting the identity component in $S_{\tilde{\phi}}$. Choose $s = a_0 \tilde{Z}^W$ contained in $S^\alpha_{\tilde{\phi}}$. Assume that $s$ consists of $\alpha$-semisimple elements (... we will prove below that such an $s$ exists). Then set ${}^L H^0_s = (\operatorname{Cent}_\alpha(s, {}^L\tilde{G}^0))^0$, and select ${}^L B^0_s$, ${}^L T^0_s$ and $\{Y\}$ as in §3. To define a suitable action of $W$ on ${}^L H^0_s$ we have just to give a homomorphism of

Gal($\mathbf{C}/\mathbf{R}$) into Aut$({}^LH_s^0, {}^LB_s^0, {}^LT_s^0, \{Y\})$ such that $\sigma_s$, the image of $\sigma$, is "realized in $\mathrm{Cent}_\alpha(s, {}^L\tilde{G}) = \mathrm{Cent}_\alpha(a_0, {}^L\tilde{G})$". But

$$\tilde{\phi}(1 \times \sigma)^{-1} a_0 \alpha(\tilde{\phi}(1 \times \sigma)) = a_0.$$

Thus $\tilde{\phi}(1 \times \sigma)$ normalizes ${}^LH_s^0$. We may write $\mathrm{ad}\,\tilde{\phi}(1 \times \sigma)|_{{}^LH_s^0}$ as $\omega\sigma_s$, where $\omega$ is an inner automorphism of ${}^LH_s^0$ and $\sigma_s \in \mathrm{Aut}({}^LH_s^0, {}^LB_s^0, {}^LT_s^0, \{Y\})$. Note that $\sigma_s^2 = 1$ and is "realized in $\mathrm{Cent}_\alpha(s, {}^L\tilde{G})$". Using the associated $W$-action we form ${}^LH_s$ and so obtain a representative $(s, {}^LH_s)$ for an element of $\mathfrak{S}(\tilde{G}, \alpha)$. We claim that $\tilde{\phi}$ factors through ${}^LH_s$. Thus, suppose that $\xi$: ${}^LH_s \Rightarrow {}^L\tilde{G}$ is an allowed embedding. Then for each $w \in W$, $\tilde{\phi}(w)$ lies in $\mathrm{Cent}_\alpha(s, {}^L\tilde{G})$ and acts on ${}^LH_s^0 = (\mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0))^0$ as an element $n(w) \times w$ of the image of ${}^LH_s$ in ${}^L\tilde{G}$. By definition, $n(w) \times w \in \mathrm{Cent}_\alpha(s, {}^L\tilde{G})$. Thus $\tilde{\phi}(w) = a(w)(n(w) \times w)$, where $a(w) \in \mathrm{Cent}_\alpha(s, {}^L\tilde{G}^0)$ centralizes ${}^LH_s^0$. But then $a(w)$ lies in the center of ${}^LH_s^0$. Hence $\tilde{\phi}$ factors through ${}^LH_s$.

It remains now to show that $S_{\tilde{\phi}}^\alpha$ contains an $\alpha$-semisimple element. If we replace $\tilde{\phi}$ by $\mathrm{ad}\, g \circ \tilde{\phi}$, $g \in {}^L\tilde{G}^0$, then we must replace $S_{\tilde{\phi}}^\alpha$ by $\alpha(g) S_{\tilde{\phi}}^\alpha g^{-1}$. Therefore we may assume that $S_{\tilde{\phi}}^\alpha$ contains an element $(x^{-1}, 1)$, $x \in {}^LG^0$ (cf. Lemma 4.1). Then we write $\tilde{\phi}(w)$ as $(\phi_1(w), \phi_2(w)) \times w$ and obtain from $(x^{-1}, 1)\tilde{\phi}(w)(x, 1) = \alpha(\tilde{\phi}(w))$, $w \in W$, that $\phi_1(z \times 1) = \phi_2(z \times 1)$, $z \in \mathbf{C}^\times$, and $\phi_1(1 \times \sigma) = x\phi_2(1 \times \sigma)$; also, $x$ lies in the centralizer $S_0$ of the image of the homomorphism $\hat{\phi}_1$: $w \to \phi_1(w) \times w$ of $W$ into ${}^LG$. Write $x = x_u x_s$, where $x_u \in S_0$ is unipotent and $x_s \in S_0$ is semisimple. Then with the same $\phi_1$ and with $x_s$ in place of $x$ we can use the formulas above for $\tilde{\phi}$ to define $\tilde{\phi}_0$: $W \to {}^L\tilde{G}$ such that $S_{\tilde{\phi}_0}^\alpha$ contains $(x_s^{-1}, 1)$. But $\tilde{\phi}_0$ is easily seen to be equivalent to $\tilde{\phi}$ because $x_u^{-1}$, being unipotent and fixed by $\hat{\phi}_1(W)$, can be written as $v(\hat{\phi}_1(1 \times \sigma)v)^{-1}$, $v \in \mathrm{Cent}(\phi_1(\mathbf{C}^\times \times 1), {}^LG^0)$. Since $(x_s^{-1}, 1)$ is $\alpha$-semisimple our proof of Theorem 8.1 is complete.

According to Langlands' functoriality principle this factoring of the $\alpha$-fixed parameters $\{\tilde{\phi}\}$ should be reflected in character theory. Let $\tilde{\phi} \in \Phi(\tilde{G})$ be $\alpha$-fixed (we now drop the $\{\ \}$ from the notation for parameters). Then the $L$-packet $\Pi_{\tilde{\phi}}$ consists of a single infinitesimal equivalence class of irreducible admissible representations fixed by the automorphism $\alpha$: $\tilde{G}(\mathbf{R}) \to \tilde{G}(\mathbf{R})$ (... this is easily checked, see also **[C1]**). Thus the twisted character $\chi_{\tilde{\phi}}^\alpha$ of $\Pi_{\tilde{\phi}}$ is well-defined up to sign (see **[C1]** for a detailed discussion, especially concerning the question of signs). Assume that $\tilde{\phi}$ is bounded, i.e. if $\tilde{\phi}(w) = \tilde{\phi}_0(w) \times w$, $w \in W$, then $\tilde{\phi}_0(W)$ is bounded. Then $\chi_{\tilde{\phi}}^\alpha$ is tempered [**C1**, Theorem 5.12]. On the other hand, suppose that $\tilde{\phi}$ is the lift of $\phi \in \Phi(H)$, in the sense afforded by Theorem

8.1. Then $\phi$ is essentially bounded, so that the $L$-packet $\Pi_\phi$ consists of essentially tempered (equivalence classes of) representations. Thus $\chi_\phi = \Sigma_{\pi \in \Pi_\phi} \chi_\pi$, $\chi_\pi$ denoting the ordinary character of $\pi$, is a stable essentially tempered distribution on $H(\mathbf{R})$ [**Sh1**, Lemma 5.2].

Theorem 7.1 provides a correspondence $(f, f_H)$ between $C_c^\infty(G(\mathbf{R}))$ and $\mathcal{C}_\xi(H(\mathbf{R}))$. As mentioned already, an adequate analysis of the "twisted $F_f$ transform" would provide a correspondence between $\mathcal{C}(\tilde{G}(\mathbf{R}))$ and $\mathcal{C}_\xi(H(\mathbf{R}))$; it would also give a dual lifting of stable tempered distributions on $H(\mathbf{R})$ to twisted-invariant tempered distributions on $\tilde{G}(\mathbf{R})$, with eigendistributions mapping to eigendistributions (see [**Sh4**, §4] for the analogous arguments in the case of $L$-indistinguishability). Nevertheless, with the correspondence of Theorem 7.1 we can define $(\text{Lift } \chi_\phi)(f) = \chi_\phi(f_H)$, $f \in C_c^\infty(\tilde{G}(\mathbf{R}))$. Writing $\chi_\phi(f_H)$ as $\int_{H(\mathbf{R})} f_H(h)\chi_\phi(h)\,dh$, and applying the Weyl Integration Formula, the matching theorem and the twisted analogue of the Weyl Integration Formula, we find that Lift $\chi_\phi$ is a twisted-invariant distribution on $\tilde{G}(\mathbf{R})$ represented by a function explicitly computed in terms of $\chi_\phi$. Moreover, this function transforms under the center of the universal enveloping algebra of $\tilde{G}(\mathbf{C})$ according to the infinitesimal character of $\chi_{\tilde{\phi}}^\alpha$. We may therefore ask if Lift $\chi_\phi$ coincides with $\chi_{\tilde{\phi}}^\alpha$ up to a constant (depending only on $G$ and $H$, once the sign for $\chi_{\tilde{\phi}}^\alpha$ has been suitably fixed). According to [**C1**] with some minor additional arguments, this is true if $H = G$; recall that we are assuming that $\tilde{\phi}$ is bounded, so that $\phi$ is an essentially bounded parameter. Work of L. Clozel now in progress should provide the answer to our question for the case $H \neq G$.

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