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**SOME REMARKS ON MEASURES ON NONCOMPACT
SEMISIMPLE LIE GROUPS**

ALLADI SITARAM

SOME REMARKS ON MEASURES ON NON-COMPACT SEMI-SIMPLE LIE GROUPS

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This paper answers a question posed by K. R. Parthasarathy: Let X be a symmetric space of non-compact type and G the connected component of the group of isometries of X . Let m be the canonical G -invariant measure on X and E a Borel set in X such that \bar{E} is compact and $0 < m(E) < \infty$. If μ, ν are probability measures on X such that $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then is $\mu = \nu$? We answer the question in the affirmative (Theorem A) and also find that the condition “ \bar{E} is compact” is unnecessary. A special case of this problem (under the condition that μ and ν are K -invariant probabilities on X , where K is a maximal compact subgroup of G) was settled by I. K. Rana.

1. It is interesting to consider the corresponding problem on the real line: If E is a Borel subset of \mathbf{R} such that $0 < m(E) < \infty$ (where m is the Lebesgue measure on \mathbf{R}) and μ, ν are two probabilities on \mathbf{R} such that $\mu(x + E) = \nu(x + E)$ for all $x \in \mathbf{R}$, then is $\mu = \nu$? The answer to this is ‘yes’ under some additional conditions on E — for example \bar{E} compact or $E \subset \mathbf{R}^+$ or E becomes “very thin at ∞ ”. (See [5].) However in general the answer does not seem to be known. It is in view of this that Theorem A is interesting because in the case of a symmetric space of non-compact type all we require is $0 < m(E) < \infty$. We should also point out that Theorem A does not hold in the case of symmetric spaces of compact type — see [1]. Finally we take up briefly: (a) the question of what happens if the measures are allowed to be infinite and get a strong negative result (Theorem B) — (for more information on this problem see [1]); and (b) the corresponding question for the group G itself and again get a negative result (Theorem C).

2. Preliminaries. A symmetric space X of non-compact type is of the form G/K where G is the connected component of the group of isometries of X and K is a maximal compact subgroup of G . Moreover G is semi-simple, non-compact and with finite centre. Thus instead of working with measures on X we work with right K -invariant measures on G and we can therefore state all our results in terms of the group G . We now fix some notation that will be used in the sequel — for any unexplained concepts see [2] or [3]. Throughout this paper G is an arbitrary

connected, non-compact, semi-simple Lie group with finite centre and K a fixed maximal compact subgroup of G . Let m be a fixed Haar measure on G . $L^1(G)$ will denote the set of complex valued functions on G integrable with respect to m . A function f on G is said to be right K -invariant (respectively left K -invariant) iff $f(xk) = f(x)$ (respectively $f(kx) = f(x)$), $x \in G$, $k \in K$. Let $L^1(G/K) = \{f \in L^1(G); f \text{ right } K\text{-invariant}\}$ and $L^1(K \backslash G/K) = \{f \in L^1(G); f \text{ both left and right } K\text{-invariant}\}$. For a set $E \subset G$, let 1_E denote its indicator function. A set $E \subset G$ is said to be right K -invariant (resp. left K -invariant) iff 1_E is right K -invariant (resp. left K -invariant). A measure μ on G is said to be right K -invariant iff $\mu(Ek) = \mu(E)$ for all Borel sets $E \subset G$ and all $k \in K$. If $f \in L^1(G)$ define $f^K \in L^1(G/K)$ by $f^K(x) = \int_K f(xk) dk$ where dk is the normalized Haar measure on the compact group K . If f is a function on G , let \tilde{f} be defined by $\tilde{f}(x) = f(x^{-1})$. (Note that if f is right K -invariant \tilde{f} will be left K -invariant and vice-versa.) If $f_1, f_2 \in L^1(G)$ define $f_1 * f_2 \in L^1(G)$ by

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1})f_2(y) dm(y).$$

It is easy to see that $(f_1 * f_2)^K = f_1 * f_2^K$.

Let $G = KAN$ be a fixed Iwasawa decomposition of G (see [3]) and let \mathfrak{a} be the Lie algebra of A , \mathfrak{a}^* the dual of \mathfrak{a} and \mathfrak{a}_c^* the complexification of \mathfrak{a}^* . For each $\lambda \in \mathfrak{a}^*$ let π_λ be the irreducible unitary representation of G on H_λ where $\{(\pi_\lambda, H_\lambda)\}_{\lambda \in \mathfrak{a}^*}$ is the class-1 principal series representation of G (see [2], p. 59). Then each H_λ contains a vector v_λ , $\|v_\lambda\| = 1$ and $\pi_\lambda(k)v_\lambda = v_\lambda$ for all $k \in K$ and, moreover, v_λ is unique up to a scalar multiple of modulus one. If (π, H) is a unitary representation of G , then π "lifts" to a representation of $L^1(G)$ and we also denote this by π . (Thus $\pi(f) = \int_G f(x)\pi(x) dm(x)$, where the integral on the right has to be suitably interpreted.) For each $\lambda \in \mathfrak{a}_c^*$, let φ_λ be the elementary spherical function corresponding to λ (see [2] or [3]) and if $f \in L^1(K \backslash G/K)$ define its spherical Fourier transform \hat{f} on \mathfrak{a}^* by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1}) dm(x).$$

We now make three basic observations which will be needed in the next section.

Observation 1. If $f \in L^1(G/K)$ and $\pi_\lambda(f) = 0$ for almost all $\lambda \in \mathfrak{a}^*$ (with respect to Lebesgue measure on \mathfrak{a}^*), then $f = 0$ a.e. with respect to the Haar measure on G .

(This follows from the Plancherel theorem for K -invariant functions and the fact that the Plancherel measure on a^* is absolutely continuous with respect to Lebesgue measure on a^* — see [2].)

Observation 2. Let $f \in L^1(G/K)$. Let v_λ and H_λ be as before. Then $\pi_\lambda(f) = 0$ iff $\pi_\lambda(f)v_\lambda = 0$.

(This follows from the fact that if f is right K -invariant and if $v \in H_\lambda$ transforms according to a non-trivial irreducible representation of K , then $\pi_\lambda(f)v = 0$. Thus “all the information about $\pi_\lambda(f)$ is contained in $\pi_\lambda(f)v_\lambda$ ”. See [2].)

Observation 3. If $f \in L^1(K \backslash G/K)$, then $\pi_\lambda(f)v_\lambda = \hat{f}(\lambda)v_\lambda$. Moreover if $0 \neq f, \hat{f}$ is nonzero a.e. on a^* with respect to Lebesgue measure on a^* .

(For the first part see the discussion on pp. 69–70 of [2]. The second part follows from the fact that \hat{f} extends to a holomorphic function in a certain “tube” in a_c^* containing a^* — see [2].)

3. The main results. We are now in a position to prove the assertion made in the introduction.

THEOREM A. *Let E be a right K -invariant Borel set in G such that $0 < m(E) < \infty$. If μ is a complex (finite) right K -invariant measure on G such that $\mu(g \cdot E) = 0$ for all $g \in G$, then $\mu \equiv 0$.*

(This theorem can be interpreted as follows: Let X be the symmetric space G/K and let G act (as isometries) on X in the usual manner. If μ, ν are probabilities on G/K , E a Borel set in X of finite G -invariant measure and $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then $\mu = \nu$.)

Proof. It is enough to prove the theorem for $\mu = f \in L^1(G/K)$. (Then an easy approximate identity argument can be used to deduce the theorem for a general right K -invariant complex measure μ .) We have to prove that if $\int_{g \cdot E} f(x) dm(x) = 0$ for all $g \in G$, then $f = 0$ a.e. (m). The above condition implies $f * \tilde{1}_E \equiv 0$. Now $(f * \tilde{1}_E)^K = f * \tilde{1}_E^K$ and hence $f * \tilde{1}_E^K = 0$. Since 1_E is right K -invariant, observe that $\tilde{1}_E$ is left K -invariant and hence $\tilde{1}_E^K$ is K -bi-invariant. To prove the theorem it is enough to show (by Observation 1 in §2) that $\pi_\lambda(f) = 0$ for almost all $\lambda \in a^*$. Let v_λ and H_λ be as in §2. So by Observation 2, it is enough to show $\pi_\lambda(f)v_\lambda = 0$ a.e. (λ). Since $f * \tilde{1}_E^K \equiv 0$ we have $\pi_\lambda(f * \tilde{1}_E^K)v_\lambda = 0$ for all λ , i.e. $\pi_\lambda(f)\pi_\lambda(\tilde{1}_E^K)v_\lambda = 0$ for all λ . Thus using the K -bi-invariance of $\tilde{1}_E^K$ and

using Observation 3 we have $(\tilde{1}_E^K)^\wedge(\lambda)\pi_\lambda(f)v_\lambda = 0$ for all λ . But by the second part of Observation 3, $(\tilde{1}_E^K)^\wedge(\lambda) \neq 0$ a.e. (λ) and hence we have $\pi_\lambda(f)v_\lambda = 0$ a.e. (λ) and the proof of the theorem is complete.

However the situation changes drastically if we do not assume f to be integrable in the above theorem — (of course in this case we have to restrict ourselves to sets E with \bar{E} compact). In fact we have the following negative result.

THEOREM B. *Let E be a K -bi-invariant Borel set in G with \bar{E} compact and $m(E) > 0$. Then there exists an elementary spherical function φ such that $\int_{g \cdot E} \varphi(x) dm(x) = 0$ for all $g \in G$.*

Proof. It is well known that if $h \in L^1(K \backslash G/K)$ and if h is of compact support then \hat{h} extends to an entire function on a_c^* (where we identify a_c^* with \mathbb{C}^n , $n = \text{rank}(G/K)$). Further \hat{h} satisfies an estimate of the following type:

$$|\hat{h}(z)| \leq Ae^{B\|z\|}, \quad z \in \mathbb{C}^n (= a_c^*)$$

i.e., \hat{h} is an entire function of exponential type. Also, since $h \in L^1(K \backslash G/K)$, \hat{h} restricted to a_c^* vanishes at ∞ on a_c^* . Using the Hadamard factorization theorem one can easily show that such a function must necessarily have a zero, i.e., $\exists \lambda_0 \in a_c^*$ such that $\hat{h}(\lambda_0) = 0$. If we apply this discussion to $\tilde{1}_E$, we have $(\tilde{1}_E)^\wedge(\lambda_0) = 0$. (Note that we have assumed E is K -bi-invariant and \bar{E} is compact.) Thus:

$$(*) \quad \int_G \tilde{1}_E(g)\varphi_{\lambda_0}(g^{-1}) dm(g) = \int_G 1_E(g)\varphi_{\lambda_0}(g) dm(g) = 0.$$

Now

$$\begin{aligned} (\varphi_{\lambda_0} * \tilde{1}_E)(x) &= \int_G \varphi_{\lambda_0}(xy)\tilde{1}_E(y^{-1}) dm(y) \\ &= \int_G \varphi_{\lambda_0}(xy)1_E(y) dm(y). \end{aligned}$$

Making use of the left K -invariance of E and the fact

$$\int_K \varphi_{\lambda_0}(xky) dk = \varphi_{\lambda_0}(x)\varphi_{\lambda_0}(y)$$

we get $\forall x \in G$,

$$(\varphi_{\lambda_0} * \tilde{1}_E)(x) = \varphi_{\lambda_0}(x) \int \varphi_{\lambda_0}(y) 1_E(y) dm(y) = 0$$

by (*). Thus the theorem is proved since we can take $\varphi = \varphi_{\lambda_0}$.

(Again Theorem B can be interpreted as follows: Let E be a K -invariant set in G/K such that E has positive G -invariant measure and \bar{E} is compact. Then there exist distinct positive infinite measures μ, ν on G/K such that $\mu(gE) = \nu(gE)$ for all $g \in G$. The “Euclidean” version of this theorem (i.e. $G =$ the set of rigid motions and $X = \mathbf{R}^n$) was proved by Brown-Schreiber-Taylor — see reference [4] in [1].

The problem considered in Theorem B is a special case of what is known as the Pompeiu problem. For more information on this problem we refer the reader to [1].)

A meaningful question to ask at the group level is: Let G be a semi-simple, connected, non-compact Lie group (without compact factors). If E is a Borel set in G with $0 < m(E) < \infty$, $f \in L^1(G)$ and $\int_{g \cdot E} f(x) dm(x) = \int_{E \cdot g} f(x) dm(x) = 0$ for all $g \in G$, then is $f = 0$ a.e.? The answer to this turns out to be negative as the following theorem shows:

THEOREM C. *Let G be the group $SL(2, \mathbf{R})$ and E a K -bi-invariant Borel set in G with $0 < m(E) < \infty$. Then there exists a non-trivial $f \in L^1(G)$ such that*

$$\int_{g \cdot E} f(x) dm(x) = \int_{E \cdot g} f(x) dm(x) = 0 \quad \text{for all } g \in G.$$

Proof. Let $0 \neq f$ be the matrix element of an integrable discrete series representation π of G . (It is known that such a π exists.) Then $f \in L^1(G) \cap L^2(G)$ and it is also known that such an f is orthogonal to $L^2(G/K)$ and $L^2(K \backslash G)$. Using this and the K -bi-invariance of E it easily follows that $\int_{g \cdot E} f(x) dm(x) = \int_{E \cdot g} f(x) dm(x) = 0$.

We would like to end this article with the following question: What can you say about the above problem if G does *not* have discrete series representations (for example if G is a complex group)?.

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