SOME REMARKS ON MEASURES ON NONCOMPACT SEMISIMPLE LIE GROUPS

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This paper answers a question posed by K. R. Parthasarathy: Let $X$ be a symmetric space of non-compact type and $G$ the connected component of the group of isometries of $X$. Let $m$ be the canonical $G$-invariant measure on $X$ and $E$ a Borel set in $X$ such that $E$ is compact and $0 < m(E) < \infty$. If $\mu$, $\nu$ are probability measures on $X$ such that $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then is $\mu = \nu$? We answer the question in the affirmative (Theorem A) and also find that the condition “$E$ is compact” is unnecessary. A special case of this problem (under the condition that $\mu$ and $\nu$ are $K$-invariant probabilities on $X$, where $K$ is a maximal compact subgroup of $G$) was settled by I. K. Rana.

1. It is interesting to consider the corresponding problem on the real line: If $E$ is a Borel subset of $\mathbb{R}$ such that $0 < m(E) < \infty$ (where $m$ is the Lebesgue measure on $\mathbb{R}$) and $\mu$, $\nu$ are two probabilities on $\mathbb{R}$ such that $\mu(x + E) = \nu(x + E)$ for all $x \in \mathbb{R}$, then is $\mu = \nu$? The answer to this is ‘yes’ under some additional conditions on $E$ — for example $E$ compact or $E \subset \mathbb{R}^+$ or $E$ becomes “very thin at $\infty$”. (See [5].) However in general the answer does not seem to be known. It is in view of this that Theorem A is interesting because in the case of a symmetric space of non-compact type all we require is $0 < m(E) < \infty$. We should also point out that Theorem A does not hold in the case of symmetric spaces of compact type — see [1]. Finally we take up briefly: (a) the question of what happens if the measures are allowed to be infinite and get a strong negative result (Theorem B) — (for more information on this problem see [1]); and (b) the corresponding question for the group $G$ itself and again get a negative result (Theorem C).

2. Preliminaries. A symmetric space $X$ of non-compact type is of the form $G/K$ where $G$ is the connected component of the group of isometries of $X$ and $K$ is a maximal compact subgroup of $G$. Moreover $G$ is semi-simple, non-compact and with finite centre. Thus instead of working with measures on $X$ we work with right $K$-invariant measures on $G$ and we can therefore state all our results in terms of the group $G$. We now fix some notation that will be used in the sequel — for any unexplained concepts see [2] or [3]. Throughout this paper $G$ is an arbitrary
connected, non-compact, semi-simple Lie group with finite centre and $K$ a fixed maximal compact subgroup of $G$. Let $m$ be a fixed Haar measure on $G$. $L^1(G)$ will denote the set of complex valued functions on $G$ integrable with respect to $m$. A function $f$ on $G$ is said to be right $K$-invariant (respectively left $K$-invariant) iff $f(kx) = f(x)$ (respectively $f(kx) = f(x)$), $x \in G$, $k \in K$. Let $L^1(G/K) = \{ f \in L^1(G); f$ right $K$-invariant$\}$ and $L^1(K \setminus G/K) = \{ f \in L^1(G); f$ both left and right $K$-invariant$\}$. For a set $E \subset G$, let $1_E$ denote its indicator function. A set $E \subset G$ is said to be right $K$-invariant (resp. left $K$-invariant) iff $1_E$ is right $K$-invariant (resp. left $K$-invariant). A measure $\mu$ on $G$ is said to be right $K$-invariant iff $\mu(Ek) = \mu(E)$ for all Borel sets $E \subset G$ and all $k \in K$. If $f \in L^1(G)$ define $f^K \in L^1(G/K)$ by $f^K(x) = \int_k f(kx) \, dk$ where $dk$ is the normalized Haar measure on the compact group $K$. If $f$ is a function on $G$, let $\tilde{f}$ be defined by $\tilde{f}(x) = f(x^{-1})$. (Note that if $f$ is right $K$-invariant $\tilde{f}$ will be left $K$-invariant and vice-versa.) If $f_1, f_2 \in L^1(G)$ define $f_1 \ast f_2 \in L^1(G)$ by

$$( f_1 \ast f_2)(x) = \int_G f_1(xy^{-1})f_2(y) \, dm(y).$$

It is easy to see that $(f_1 \ast f_2)^K = f_1 \ast f_2^K$.

Let $G = KAN$ be a fixed Iwasawa decomposition of $G$ (see [3]) and let $a$ be the Lie algebra of $A$, $a^*$ the dual of $a$ and $a^*_c$ the complexification of $a^*$. For each $\lambda \in a^*$ let $\pi_{\lambda}$ be the irreducible unitary representation of $G$ on $H_{\lambda}$ where $\{(\pi_{\lambda}, H_{\lambda})\}_{\lambda \in a^*}$ is the class-1 principal series representation of $G$ (see [2], p. 59). Then each $H_{\lambda}$ contains a vector $v_{\lambda}$, $\|v_{\lambda}\| = 1$ and $\pi_{\lambda}(k)v_{\lambda} = v_{\lambda}$ for all $k \in K$ and, moreover, $v_{\lambda}$ is unique up to a scalar multiple of modulus one. If $(\pi, H)$ is a unitary representation of $G$, then $\pi$ "lifts" to a representation of $L^1(G)$ and we also denote this by $\pi$. (Thus $\pi(f) = \int_G f(x)\pi(x) \, dm(x)$, where the integral on the right has to be suitably interpreted.) For each $\lambda \in a^*_c$, let $\varphi_{\lambda}$ be the elementary spherical function corresponding to $\lambda$ (see [2] or [3]) and if $f \in L^1(K \setminus G/K)$ define its spherical Fourier transform on $a^*$ by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_{\lambda}(x^{-1}) \, dm(x).$$

We now make three basic observations which will be needed in the next section.

**Observation 1.** If $f \in L^1(G/K)$ and $\pi_{\lambda}(f) = 0$ for almost all $\lambda \in a^*$ (with respect to Lebesgue measure on $a^*$), then $f = 0$ a.e. with respect to the Haar measure on $G$. 
Remark 2. Let $f \in L^1(G/K)$. Let $\nu_\lambda$ and $H_\lambda$ be as before. Then $\pi_\lambda(f) = 0$ iff $\pi_\lambda(f)\nu_\lambda = 0$.

(This follows from the fact that if $f$ is right $K$-invariant and if $\nu \in H_\lambda$ transforms according to a non-trivial irreducible representation of $K$, then $\pi_\lambda(f)\nu = 0$. Thus “all the information about $\pi_\lambda(f)$ is contained in $\pi_\lambda(f)\nu_\lambda$”. See [2].)

Observation 3. If $f \in L^1(K \backslash G/K)$, then $\pi_\lambda(f)\nu_\lambda = \hat{f}(\lambda)\nu_\lambda$. Moreover if $0 \neq f, \hat{f}$ is nonzero a.e. on $a^*$ with respect to Lebesgue measure on $a^*$.

(For the first part see the discussion on pp. 69–70 of [2]. The second part follows from the fact that $\hat{f}$ extends to a holomorphic function in a certain “tube” in $a^*_c$ containing $a^*$ — see [2].)

3. The main results. We are now in a position to prove the assertion made in the introduction.

Theorem A. Let $E$ be a right $K$-invariant Borel set in $G$ such that $0 < m(E) < \infty$. If $\mu$ is a complex (finite) right $K$-invariant measure on $G$ such that $\mu(g \cdot E) = 0$ for all $g \in G$, then $\mu \equiv 0$.

(This theorem can be interpreted as follows: Let $X$ be the symmetric space $G/K$ and let $G$ act (as isometries) on $X$ in the usual manner. If $\mu, \nu$ are probabilities on $G/K$, $E$ a Borel set in $X$ of finite $G$-invariant measure and $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then $\mu = \nu$.)

Proof. It is enough to prove the theorem for $\mu = f \in L^1(G/K)$. (Then an easy approximate identity argument can be used to deduce the theorem for a general right $K$-invariant complex measure $\mu$.). We have to prove that if $\int_{g \cdot E} f(x) \, dm(x) = 0$ for all $g \in G$, then $f = 0$ a.e. $(m)$. The above condition implies $f \ast \tilde{1}_E \equiv 0$. Now $(f \ast \tilde{1}_E)^K = f \ast \tilde{1}_E^K$ and hence $f \ast \tilde{1}_E^K = 0$. Since $1_E$ is right $K$-invariant, observe that $\tilde{1}_E$ is left $K$-invariant and hence $\tilde{1}_E^K$ is $K$-bi-invariant. To prove the theorem it is enough to show (by Observation 1 in §2) that $\pi_\lambda(f) = 0$ for almost all $\lambda \in a^*$. Let $\nu_\lambda$ and $H_\lambda$ be as in §2. So by Observation 2, it is enough to show $\pi_\lambda(f)\nu_\lambda = 0$ a.e. $(\lambda)$. Since $f \ast \tilde{1}_E^K \equiv 0$ we have $\pi_\lambda(f \ast \tilde{1}_E^K)\nu_\lambda = 0$ for all $\lambda$, i.e. $\pi_\lambda(f)\pi_\lambda(\tilde{1}_E^K)\nu_\lambda = 0$ for all $\lambda$. Thus using the $K$-bi-invariance of $\tilde{1}_E^K$ and
using Observation 3 we have \((\tilde{1}_E)^*(\lambda)\pi_\lambda(f)v_\lambda = 0\) for all \(\lambda\). But by the second part of Observation 3, \((\tilde{1}_E)^*(\lambda) \neq 0\) a.e. \((\lambda)\) and hence we have \(\pi_\lambda(f)v_\lambda = 0\) a.e. \((\lambda)\) and the proof of the theorem is complete.

However the situation changes drastically if we do not assume \(f\) to be integrable in the above theorem — (of course in this case we have to restrict ourselves to sets \(E\) with \(\overline{E}\) compact). In fact we have the following negative result.

**THEOREM B.** Let \(E\) be a \(K\)-bi-invariant Borel set in \(G\) with \(\overline{E}\) compact and \(m(E) > 0\). Then there exists an elementary spherical function \(\varphi\) such that \(\int_{g \cdot E} \varphi(x) dm(x) = 0\) for all \(g \in G\).

**Proof.** It is well known that if \(h \in L^1(K \backslash G/K)\) and if \(h\) is of compact support then \(\hat{h}\) extends to an entire function on \(a^*_c\) (where we identify \(a^*_c\) with \(\mathbb{C}^n, n = \text{rank}(G/K)\)). Further \(\hat{h}\) satisfies an estimate of the following type:

\[ |\hat{h}(z)| \leq Ae^{B||z||}, \quad z \in \mathbb{C}^n (= a^*_c) \]

i.e., \(\hat{h}\) is an entire function of exponential type. Also, since \(h \in L^1(K \backslash G/K), \) \(\hat{h}\) restricted to \(a^*_c\) vanishes at \(\infty\) on \(a^*_c\). Using the Hadamard factorization theorem one can easily show that such a function must necessarily have a zero, i.e., \(\exists \lambda_0 \in a^*_c\) such that \(\hat{h}(\lambda_0) = 0\). If we apply this discussion to \(\tilde{1}_E\), we have \((\tilde{1}_E)^*(\lambda_0) = 0\). (Note that we have assumed \(E\) is \(K\)-bi-invariant and \(\overline{E}\) is compact.) Thus:

\[ (*) \quad \int_G \tilde{1}_E(g)\varphi_{\lambda_0}(g^{-1}) dm(g) = \int_G 1_E(g)\varphi_{\lambda_0}(g) dm(g) = 0. \]

Now

\[ (\varphi_{\lambda_0} * \tilde{1}_E)(x) = \int_G \varphi_{\lambda_0}(xy)\tilde{1}_E(y^{-1}) dm(y) \]

\[ = \int_G \varphi_{\lambda_0}(xy)1_E(y) dm(y). \]

Making use of the left \(K\)-invariance of \(E\) and the fact

\[ \int_K \varphi_{\lambda_0}(xky) dk = \varphi_{\lambda_0}(x)\varphi_{\lambda_0}(y) \]
we get $\forall x \in G,$

$$(\varphi_{\lambda_0} \ast \tilde{1}_E)(x) = \varphi_{\lambda_0}(x) \int \varphi_{\lambda_0}(y) 1_E(y) \, dm(y) = 0$$

by $(\ast)$. Thus the theorem is proved since we can take $\varphi = \varphi_{\lambda_0}$.

(Again Theorem B can be interpreted as follows: Let $E$ be a $K$-invariant set in $G/K$ such that $E$ has positive $G$-invariant measure and $\tilde{E}$ is compact. Then there exist distinct positive infinite measures $\mu, \nu$ on $G/K$ such that $\mu(gE) = \nu(gE)$ for all $g \in G$. The "Euclidean" version of this theorem (i.e. $G = $ the set of rigid motions and $X = \mathbb{R}^n$) was proved by Brown-Schreiber-Taylor — see reference [4] in [1].

The problem considered in Theorem B is a special case of what is known as the Pompeiu problem. For more information on this problem we refer the reader to [1].)

A meaningful question to ask at the group level is: Let $G$ be a semi-simple, connected, non-compact Lie group (without compact factors). If $E$ is a Borel set in $G$ with $0 < m(E) < \infty$, $f \in L^1(G)$ and $\int_{g \cdot E} f(x) \, dm(x) = \int_{E \cdot g} f(x) \, dm(x) = 0$ for all $g \in G$, then is $f = 0$ a.e.? The answer to this turns out to be negative as the following theorem shows:

**Theorem C.** Let $G$ be the group $\text{SL}(2, \mathbb{R})$ and $E$ a $K$-bi-invariant Borel set in $G$ with $0 < m(E) < \infty$. Then there exists a non-trivial $f \in L^1(G)$ such that

$$\int_{g \cdot E} f(x) \, dm(x) = \int_{E \cdot g} f(x) \, dm(x) = 0 \quad \text{for all } g \in G.$$

**Proof.** Let $0 \neq f$ be the matrix element of an integrable discrete series representation $\pi$ of $G$. (It is known that such a $\pi$ exists.) Then $f \in L^1(G) \cap L^2(G)$ and it is also known that such an $f$ is orthogonal to $L^2(G/K)$ and $L^2(K \setminus G)$. Using this and the $K$-bi-invariance of $E$ it easily follows that $\int_{g \cdot E} f(x) \, dm(x) = \int_{E \cdot g} f(x) \, dm(x) = 0$.

We would like to end this article with the following question: What can you say about the above problem if $G$ does not have discrete series representations (for example if $G$ is a complex group)?

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