Pacific Journal of Mathematics

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Vol. 110, No. 2 October 1984

PACIFIC JOURNAL OF MATHEMATICS
Vol. 110, No. 2, 1984

SOME REMARKS ON MEASURES ON NON-COMPACT SEMI-SIMPLE LIE GROUPS

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This paper answers a question posed by K. R. Parthasarathy: Let X be a symmetric space of non-compact type and G the connected component of the group of isometries of X. Let m be the canonical G-invariant measure on X and E a Borel set in X such that $\bar{E}$ is compact and $0 < m(E) < \infty$. If μ, ν are probability measures on X such that $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then is $\mu = \nu$? We answer the question in the affirmative (Theorem A) and also find that the condition "$\bar{E}$ is compact" is unnecessary. A special case of this problem (under the condition that μ and ν are K-invariant probabilities on X, where K is a maximal compact subgroup of G) was settled by I. K. Rana.

1. It is interesting to consider the corresponding problem on the real line: If E is a Borel subset of $\mathbf{R}$ such that $0 < m(E) < \infty$ (where m is the Lebesgue measure on $\mathbf{R}$) and μ, ν are two probabilities on $\mathbf{R}$ such that $\mu(x + E) = \nu(x + E)$ for all $x \in \mathbf{R}$, then is $\mu = \nu$? The answer to this is 'yes' under some additional conditions on E — for example $\bar{E}$ compact *or* $E \subset \mathbf{R}^+$ *or* E becomes "very thin at ∞". (See [**5**].) However in general the answer does not seem to be known. It is in view of this that Theorem A is interesting because in the case of a symmetric space of non-compact type all we require is $0 < m(E) < \infty$. We should also point out that Theorem A does not hold in the case of symmetric spaces of compact type — see [**1**]. Finally we take up briefly: (a) the question of what happens if the measures are allowed to be infinite and get a strong negative result (Theorem B) — (for more information on this problem see [**1**]); and (b) the corresponding question for the group G itself and again get a negative result (Theorem C).

2. Preliminaries. A symmetric space X of non-compact type is of the form G/K where G is the connected component of the group of isometries of X and K is a maximal compact subgroup of G. Moreover G is semi-simple, non-compact and with finite centre. Thus instead of working with measures on X we work with right K-invariant measures on G and we can therefore state all our results in terms of the group G. We now fix some notation that will be used in the sequel — for any unexplained concepts see [**2**] or [**3**]. Throughout this paper G is an arbitrary connected, non-compact, semi-simple Lie group with finite centre and K a fixed maximal compact subgroup of G. Let m be a fixed Haar measure on G. $L^1(G)$ will denote the set of complex valued functions on G integrable with respect to m. A function f on G is said to be right K-invariant (respectively left K-invariant) iff $f(xk) = f(x)$ (respectively $f(kx) = f(x)$), $x \in G$, $k \in K$. Let $L^1(G/K) = \{f \in L^1(G);\ f \text{ right } K\text{-invariant}\}$ and $L^1(K\backslash G/K) = \{f \in L^1(G);\ f \text{ both left and right } K\text{-invariant}\}$. For a set $E \subset G$, let 1_E denote its indicator function. A set $E \subset G$ is said to be right K-invariant (resp. left K-invariant) iff 1_E is right K-invariant (resp. left K-invariant). A measure μ on G is said to be right K-invariant iff $\mu(Ek) = \mu(E)$ for all Borel sets $E \subset G$ and all $k \in K$. If $f \in L^1(G)$ define $f^K \in L^1(G/K)$ by $f^K(x) = \int_K f(xk)\, dk$ where dk is the normalized Haar measure on the compact group K. If f is a function on G, let $\tilde{f}$ be defined by $\tilde{f}(x) = f(x^{-1})$. (Note that if f is right K-invariant $\tilde{f}$ will be left K-invariant and vice-versa.) If $f_1, f_2 \in L^1(G)$ define $f_1 * f_2 \in L^1(G)$ by

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1}) f_2(y)\, dm(y).$$

It is easy to see that $(f_1 * f_2)^K = f_1 * f_2^K$.

Let $G = KAN$ be a fixed Iwasawa decomposition of G (see [3]) and let a be the Lie algebra of A, a^* the dual of a and a_c^* the complexification of a^*. For each $\lambda \in a^*$ let π_λ be the irreducible unitary representation of G on H_λ where $\{(\pi_\lambda, H_\lambda)\}_{\lambda \in a^*}$ is the class-1 principal series representation of G (see [2], p. 59). Then each H_λ contains a vector v_λ, $\|v_\lambda\| = 1$ and $\pi_\lambda(k)v_\lambda = v_\lambda$ for all $k \in K$ and, moreover, v_λ is unique up to a scalar multiple of modulus one. If (π, H) is a unitary representation of G, then π "lifts" to a representation of $L^1(G)$ and we also denote this by π. (Thus $\pi(f) = \int_G f(x)\pi(x)\, dm(x)$, where the integral on the right has to be suitably interpreted.) For each $\lambda \in a_c^*$, let φ_λ be the elementary spherical function corresponding to λ (see [2] or [3]) and if $f \in L^1(K\backslash G/K)$ define its spherical Fourier transform $\hat{f}$ on a^* by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1})\, dm(x).$$

We now make three basic observations which will be needed in the next section.

Observation 1. If $f \in L^1(G/K)$ and $\pi_\lambda(f) = 0$ for almost all $\lambda \in a^*$ (with respect to Lebesgue measure on a^*), then $f = 0$ a.e. with respect to the Haar measure on G.

(This follows from the Plancherel theorem for K-invariant functions and the fact that the Plancherel measure on a^* is absolutely continuous with respect to Lebesgue measure on a^* — see [2].)

Observation 2. Let $f \in L^1(G/K)$. Let v_λ and H_λ be as before. Then $\pi_\lambda(f) = 0$ iff $\pi_\lambda(f)v_\lambda = 0$.

(This follows from the fact that if f is right K-invariant and if $v \in H_\lambda$ transforms according to a non-trivial irreducible representation of K, then $\pi_\lambda(f)v = 0$. Thus "all the information about $\pi_\lambda(f)$ is contained in $\pi_\lambda(f).v_\lambda$". See [2].)

Observation 3. If $f \in L^1(K\backslash G/K)$, then $\pi_\lambda(f)v_\lambda = \hat{f}(\lambda)v_\lambda$. Moreover if $0 \neq f$, $\hat{f}$ is nonzero a.e. on a^* with respect to Lebesgue measure on a^*.

(For the first part see the discussion on pp. 69–70 of [2]. The second part follows from the fact that $\hat{f}$ extends to a holomorphic function in a certain "tube" in a_c^* containing a^* — see [2].)

3. The main results. We are now in a position to prove the assertion made in the introduction.

THEOREM A. *Let E be a right K-invariant Borel set in G such that $0 < m(E) < \infty$. If μ is a complex (finite) right K-invariant measure on G such that $\mu(g \cdot E) = 0$ for all $g \in G$, then $\mu \equiv 0$.*

(This theorem can be interpreted as follows: Let X be the symmetric space G/K and let G act (as isometries) on X in the usual manner. If μ, ν are probabilities on G/K, E a Borel set in X of finite G-invariant measure and $\mu(g \cdot E) = \nu(g \cdot E)$ for all $g \in G$, then $\mu = \nu$.)

Proof. It is enough to prove the theorem for $\mu = f \in L^1(G/K)$. (Then an easy approximate identity argument can be used to deduce the theorem for a general right K-invariant complex measure μ.) We have to prove that if $\int_{g \cdot E} f(x)\, dm(x) = 0$ for all $g \in G$, then $f = 0$ a.e. (m). The above condition implies $f * \tilde{1}_E \equiv 0$. Now $(f * \tilde{1}_E)^K = f * \tilde{1}_E^K$ and hence $f * \tilde{1}_E^K = 0$. Since 1_E is right K-invariant, observe that $\tilde{1}_E$ is left K-invariant and hence $\tilde{1}_E^K$ is K-bi-invariant. To prove the theorem it is enough to show (by Observation 1 in §2) that $\pi_\lambda(f) = 0$ for almost all $\lambda \in a^*$. Let v_λ and H_λ be as in §2. So by Observation 2, it is enough to show $\pi_\lambda(f)v_\lambda = 0$ a.e. (λ). Since $f * \tilde{1}_E^K \equiv 0$ we have $\pi_\lambda(f * \tilde{1}_E^K)v_\lambda = 0$ for all λ, i.e. $\pi_\lambda(f)\pi_\lambda(\tilde{1}_E^K)v_\lambda = 0$ for all λ. Thus using the K-bi-invariance of $\tilde{1}_E^K$ and using Observation 3 we have $(\tilde{1}_E^K)\hat{}(\lambda)\pi_\lambda(f)v_\lambda = 0$ for all λ. But by the second part of Observation 3, $(\tilde{1}_E^K)\hat{}(\lambda) \neq 0$ a.e. (λ) and hence we have $\pi_\lambda(f)v_\lambda = 0$ a.e. (λ) and the proof of the theorem is complete.

However the situation changes drastically if we do not assume f to be integrable in the above theorem — (of course in this case we have to restrict ourselves to sets E with $\bar{E}$ compact). In fact we have the following negative result.

THEOREM B. *Let E be a K-bi-invariant Borel set in G with $\bar{E}$ compact and $m(E) > 0$. Then there exists an elementary spherical function φ such that $\int_{g\cdot E}\varphi(x)\,dm(x) = 0$ for all $g \in G$.*

Proof. It is well known that if $h \in L^1(K\backslash G/K)$ and if h is of *compact support* then $\hat{h}$ extends to an entire function on a_c^* (where we identify a_c^* with $\mathbf{C}^n$, $n = \operatorname{rank}(G/K)$). Further $\hat{h}$ satisfies an estimate of the following type:

$$|\hat{h}(z)| \leq Ae^{B\|z\|}, \qquad z \in \mathbf{C}^n \;(= a_c^*)$$

i.e., $\hat{h}$ is an entire function of exponential type. Also, since $h \in L^1(K\backslash G/K)$, $\hat{h}$ restricted to a^* vanishes at ∞ on a^*. Using the Hadamard factorization theorem one can easily show that such a function must necessarily have a zero, i.e., $\exists\lambda_0 \in a_c^*$ such that $\hat{h}(\lambda_0) = 0$. If we apply this discussion to $\tilde{1}_E$, we have $(\tilde{1}_E)\hat{}(\lambda_0) = 0$. (Note that we have assumed E is K-bi-invariant and $\bar{E}$ is compact.) Thus:

$$(*) \qquad \int_G \tilde{1}_E(g)\varphi_{\lambda_0}(g^{-1})\,dm(g) = \int_G 1_E(g)\varphi_{\lambda_0}(g)\,dm(g) = 0.$$

Now

$$(\varphi_{\lambda_0} * \tilde{1}_E)(x) = \int_G \varphi_{\lambda_0}(xy)\tilde{1}_E(y^{-1})\,dm(y)$$

$$= \int_G \varphi_{\lambda_0}(xy)1_E(y)\,dm(y).$$

Making use of the left K-invariance of E and the fact

$$\int_K \varphi_{\lambda_0}(xky)\,dk = \varphi_{\lambda_0}(x)\varphi_{\lambda_0}(y)$$

we get $\forall x \in G$,

$$\left(\varphi_{\lambda_0} * \tilde{1}_E\right)(x) = \varphi_{\lambda_0}(x) \int \varphi_{\lambda_0}(y) 1_E(y)\, dm(y) = 0$$

by (*). Thus the theorem is proved since we can take $\varphi = \varphi_{\lambda_0}$.

(Again Theorem B can be interpreted as follows: Let E be a K-invariant set in G/K such that E has positive G-invariant measure and $\bar{E}$ is compact. Then there exist distinct positive infinite measures μ, ν on G/K such that $\mu(gE) = \nu(gE)$ for all $g \in G$. The "Euclidean" version of this theorem (i.e. $G =$ the set of rigid motions and $X = \mathbf{R}^n$) was proved by Brown-Schreiber-Taylor — see reference [**4**] in [**1**].

The problem considered in Theorem B is a special case of what is known as the Pompeiu problem. For more information on this problem we refer the reader to [**1**].)

A meaningful question to ask at the group level is: Let G be a semi-simple, connected, non-compact Lie group (without compact factors). If E is a Borel set in G with $0 < m(E) < \infty$, $f \in L^1(G)$ and $\int_{g \cdot E} f(x)\, dm(x) = \int_{E \cdot g} f(x)\, dm(x) = 0$ for all $g \in G$, then is $f = 0$ a.e.? The answer to this turns out to be negative as the following theorem shows:

THEOREM C. *Let G be the group* SL(2, **R**) *and E a K-bi-invariant Borel set in G with $0 < m(E) < \infty$. Then there exists a non-trivial $f \in L^1(G)$ such that*

$$\int_{g \cdot E} f(x)\, dm(x) = \int_{E \cdot g} f(x)\, dm(x) = 0 \quad \textit{for all } g \in G.$$

Proof. Let $0 \neq f$ be the matrix element of an *integrable* discrete series representation π of G. (It is known that such a π exists.) Then $f \in L^1(G) \cap L^2(G)$ and it is also known that such an f is orthogonal to $L^2(G/K)$ and $L^2(K \backslash G)$. Using this and the K-bi-invariance of E it easily follows that $\int_{g \cdot E} f(x)\, dm(x) = \int_{E \cdot g} f(x)\, dm(x) = 0$.

We would like to end this article with the following question: What can you say about the above problem if G does *not* have discrete series representations (for example if G is a complex group)?.

Acknowledgements. The author would like to thank Professor K. R. Parthasarathy for getting him intersted in the problems considered in this paper. The author also thanks Dr. B. Bagchi and Dr. S. C. Bagchi for useful conversations.

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Received May 5, 1982 and in revised form December 21, 1982.

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The Pacific Journal of Mathematics ISSN 0030-8730 is published monthly by the Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: Send address changes to Pacific Journal of Mathematics, P. O. Box 969, Carmel Valley, CA 93924.

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Pacific Journal of Mathematics

Vol. 110, No. 2 October, 1984