

# Pacific Journal of Mathematics

**DIOPHANTINE DETERMINATIONS OF  $3^{(p-1)/8}$  AND  $5^{(p-1)/4}$**

RICHARD HOWARD HUDSON

# DIOPHANTINE DETERMINATIONS OF $3^{(p-1)/8}$ AND $5^{(p-1)/4}$

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Let  $p$  be a prime  $= 24f + 1$ . The author and Kenneth S. Williams derived a criteria for 3 to be an eighth power (mod  $p$ ) in terms of the parameters in the Diophantine systems  $a^2 + b^2$  and  $x^2 + 3y^2$ . A new proof of this result is given which uses Jacobi sums. This proof is simpler in that it does not require summing 36 cyclotomic numbers; moreover, it leads simultaneously to new necessary and sufficient criteria for  $3^{(p-1)/8}$  to be congruent to  $b/a$  (mod  $p$ ),  $a \equiv 1$  (mod 4),  $b > 0$ . Using this result, criteria for  $3^{(p-1)/8} \equiv 1, b/a, -1, \text{ or } -b/a$  (mod  $p$ ) are given in terms of the parameters in other well-known quadratic partitions of  $p$  or of  $4p$ .

Let  $p$  be a prime  $= 20f + 1$ ,  $16p = x^2 + 50u^2 + 50v^2 + 125w^2$ ,  $xw = v^2 - 4uv - u^2$ . It is shown that  $5^{(p-1)/4} \equiv 1$  (mod  $p$ ) if and only if  $16 \mid w$  or  $uv \equiv 2$  (mod 4). This result is of interest in relation to criteria given by Emma Lehmer for 2 to be a fifth power (mod  $p$ ) and for  $p$  to be a hyperartiad.

**1. Introduction and preliminaries.** For a prime  $p = 24f + 1$  we have the following quadratic partitions of  $p$  or of  $4p$ :

- (1)  $p = a^2 + b^2, a \equiv 1$  (mod 4),      (2)  $p = c^2 + 2d^2, c \equiv 1$  (mod 4),  
 (3)  $p = x^2 + 3y^2, x \equiv 1$  (mod 3),      (4)  $p = u^2 + 6v^2, u \equiv 1$  (mod 4),  
 (5)  $4p = A^2 + 27B^2, A \equiv 1$  (mod 3).

Using the law of octic reciprocity given by A. E. Western [8], the value of  $3^{(p-1)/8}$  has been given in terms of the Diophantine systems (1) and (2); specifically, we have

(a)

$$3^{(p-1)/8} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} a \equiv c \pmod{3} & \text{if } p \equiv 1 \pmod{48}, \\ a \equiv -c \pmod{3} & \text{if } p \equiv 25 \pmod{48}, \end{cases}$$

(b)

$$3^{(p-1)/8} \equiv b/a \pmod{p} \Leftrightarrow \begin{cases} b \equiv c \pmod{3} & \text{if } p \equiv 1 \pmod{48}, \\ b \equiv -c \pmod{3} & \text{if } p \equiv 25 \pmod{48}. \end{cases}$$

Throughout we fix  $b$  to be positive in case (b), as in [1, p. 3.7], by fixing a primitive root  $g(p)$  such that  $g^{6f} \equiv b/a$  (mod  $p$ ) for  $b > 0$ .

Using cyclotomic numbers of order 12 [9] and an index formula due to Muskat [7], Hudson and Williams [5] gave necessary and sufficient criteria for 3 to be an eighth power modulo  $p$  (case (a) above) in terms of the parameters in systems (1) and (3).

In this note we show that the Davenport-Hasse relation in a form given by Yamamoto [11] and certain relations between Jacobi sums of order 24 lead simultaneously to the result of Hudson and Williams [4] (and more neatly as the proof does not necessitate summing 36 cyclotomic numbers) and to a new criteria in case (b) (3 is not a fourth power (mod  $p$ )); see Theorem 1. Using this theorem, we obtain in this paper similar criteria in terms of parameters in (4) and (5), see Theorems 2 and 3. Finally, in (3.3) and Theorem 4, we delineate criteria for 5 to be a quartic residue (mod  $p = 20f + 1$ ) in terms of the parameters in (3.1) in relation to Lehmer's [6] criteria for 2 to be a quintic residue.

As preliminaries, we require an easy modification of Wilson's theorem giving for a prime  $p = mnf + 1$ ,

$$(1) \quad mnf!nf! \equiv (-1)^{mf-1} \equiv (-1)^{n-1} \pmod{p}.$$

Next, see, e.g. [5], for  $1 \leq s < r \leq 23$ ,  $p = 24f + 1$ , we have

$$(1.2) \quad \binom{rf}{sf} \equiv (-1)^{sf} \binom{(24-r+s)f}{sf} \pmod{p}.$$

Finally, for a prime  $p = mnf + 1$  we have from the Davenport-Hasse relation in the form given by Yamamoto [11, p. 488] that

$$(1.3) \quad (n^{(p-1)/m})^t \equiv \frac{ntf! \prod_{j=1}^{n-1} (mjf)!}{\prod_{j=0}^{n-1} (mj+t)f!} \pmod{p}.$$

Our notation for Jacobi sums is as follows. Let  $\chi_{24}$  be a character (mod  $p$ ) of order 24, let  $\phi_{24} = e^{2\pi i/24}$ , and let  $g$  be a primitive root of  $p$  with  $g^f \equiv \phi_{24} \pmod{\Omega}$  where  $\Omega$  is a prime ideal divisor of  $p$  in  $Q(\phi_{24})$ . For  $x \not\equiv 0 \pmod{p}$ , let  $\text{ind}_g(x)$  be the unique integer  $b$  such that  $x \equiv g^b \pmod{p}$ ,  $0 \leq b \leq p-2$ . Then the Jacobi sum  $J_{24}(r, s)$  of order 24 is defined by

$$J_{24}(r, s) = \sum_{x=0}^{p-1} \chi_{24}^r(x) \chi_{24}^s(1-x) = \sum_{x=2}^{p-1} \phi_{24}^{r \text{ind}_g(x) + s \text{ind}_g(1-x)}.$$

**2. Diophantine determinations of  $3^{(p-1)/8}$ .**

**THEOREM 1.** *Let  $p = 24f + 1 = a^2 + b^2 = x^2 + 3y^2$ ,  $a \equiv 1 \pmod{4}$ ,  $b > 0$ . Then we have*

$$(a) \quad 3^{(p-1)/8} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} a \equiv 1 \pmod{3} & \text{and } y \equiv 0 \pmod{8}, \\ a \equiv 2 \pmod{3} & \text{and } y \equiv 4 \pmod{8}, \end{cases}$$

and

$$(b) \quad 3^{(p-1)/8} \equiv b/a \pmod{p} \Leftrightarrow \begin{cases} b \equiv 1 \pmod{3} & \text{and } y \equiv 0 \pmod{8} \\ b \equiv 2 \pmod{3} & \text{and } y \equiv 4 \pmod{8}. \end{cases}$$

*Proof.* From [5, Th. 15.1] we have

$$(2.1) \quad \begin{pmatrix} 8f \\ 2f \end{pmatrix} \equiv \begin{cases} +1 \text{ or } -1 \pmod{p} & \text{according as } a \equiv 1 \text{ or } 2 \pmod{3}, \\ b/a \text{ or } -b/a \pmod{p} & \text{according as } b \equiv 1 \text{ or } 2 \pmod{3}. \end{cases}$$

Moreover, it follows from Gauss [3] that

$$(2.2) \quad \begin{pmatrix} 12f \\ 6f \end{pmatrix} \equiv 2a \pmod{p} \quad \text{for } a \equiv 1 \pmod{4}.$$

Using (1.1), (1.2), and (1.3) we have

$$2^{(p-1)/4} = (2^{(p-1)/12})^3 \equiv \frac{6f!12f!}{3f!15f!} \equiv \frac{\begin{pmatrix} 18f \\ 3f \end{pmatrix}}{\begin{pmatrix} 18f \\ 6f \end{pmatrix}} \equiv \frac{(-1)^f \begin{pmatrix} 9f \\ 3f \end{pmatrix}}{\begin{pmatrix} 12f \\ 6f \end{pmatrix}} \pmod{p},$$

$$3^{(p-1)/8} \equiv \frac{3f!8f!16f!}{f!9f!17f!} \equiv (-1)^f \frac{3f!7f!}{f!9f!} \equiv (-1)^f \frac{\begin{pmatrix} 7f \\ f \end{pmatrix}}{\begin{pmatrix} 9f \\ 3f \end{pmatrix}},$$

from which it follows that

$$(2.3) \quad \frac{\begin{pmatrix} 7f \\ f \end{pmatrix}}{\begin{pmatrix} 12f \\ 6f \end{pmatrix}} \equiv (-1)^{b/4} 3^{(p-1)/8} \pmod{p}.$$

From Berndt [1, pp. 3.17, 3.25, 3.23] we have

$$(2.4) \quad (-1)^{b/4+y/4} = (-1)^{v/2} \quad \text{and} \quad J_{24}(1, 7) = (-1)^{v/2} J_{24}(1, 1).$$

Fixing a primitive root  $g(p)$  so that  $g^{6f} \equiv b/a \pmod{p}$ ,  $b > 0$ , it follows from [10, Lemma 6] that

$$\binom{8f}{f} \equiv (-1)^{v/2} \binom{2f}{f} \equiv (-1)^{b/4+y/4} \binom{2f}{f} \pmod{p}.$$

But clearly  $\binom{8f}{f} \binom{7f}{f} = \binom{2f}{f} \binom{8f}{2f}$  so that, using (2.3),

$$3^{(p-1)/8} \equiv \frac{(-1)^{b/2+y/4} \binom{8f}{2f}}{\binom{12f}{6f}} \equiv \begin{cases} (-1)^{y/4} \pmod{p} \Leftrightarrow a \equiv 1 \pmod{3}, \\ (-1)^{y/4} b/a \pmod{p} \Leftrightarrow b \equiv 1 \pmod{3}. \end{cases}$$

This completes the proof of Theorem 1.

**THEOREM 2.** *Let  $p = 24f + 1 = a^2 + b^2 = u^2 + 6v^2$ ,  $a \equiv 1 \pmod{4}$ ,  $b > 0$ . Then we have*

$$(1) \quad 3^{(p-1)/8} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} b \equiv 2v \pmod{8} & \text{and } a \equiv 1 \pmod{3}, \\ b \equiv -2v \pmod{8} & \text{and } a \equiv 2 \pmod{3}, \end{cases}$$

and

$$(2) \quad 3^{(p-1)/8} \equiv b/a \pmod{p} \Leftrightarrow \begin{cases} b \equiv 2v \pmod{8} & \text{and } b \equiv 1 \pmod{3}, \\ b \equiv -2v \pmod{8} & \text{and } b \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Theorem 2 is an immediate consequence of Theorem 1 and the left-hand-side of (2.4).

**THEOREM 3.** *Let  $p = 24f + 1 = a^2 + b^2$ ,  $a \equiv 1 \pmod{4}$  and  $b > 0$ ,  $4p = A^2 + 27B^2$  with  $A \equiv 1 \pmod{2}$ . Then we have*

$$(a) \quad 3^{(p-1)/8} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} B \equiv \pm 3 \pmod{8} & \text{and } a \equiv (-1)^f \pmod{3}, \\ B \equiv \pm 1 \pmod{8} & \text{and } a \equiv (-1)^{f+1} \pmod{3}, \end{cases}$$

(b)

$$3^{(p-1)/8} \equiv b/a \pmod{p} \Leftrightarrow \begin{cases} B \equiv \pm 3 \pmod{8} & \text{and } b \equiv (-1)^f \pmod{3}, \\ B \equiv \pm 1 \pmod{8} & \text{and } b \equiv (-1)^{f+1} \pmod{3}. \end{cases}$$

*Proof.* Note that  $(-1)^f = +1 \Leftrightarrow x \equiv 1 \pmod{8}$  (and  $(-1)^f = -1 \Leftrightarrow x \equiv 5 \pmod{8}$ ) as

$$(2.5) \quad x \equiv 1 \pmod{8} \Leftrightarrow x^2 + 3y^2 \equiv 1 \pmod{16}.$$

It is easily seen that

$$(2.6) \quad \begin{aligned} B &= \pm \frac{1}{3}(x - y) & \text{if } y \equiv 1 \pmod{3}, \\ B &= \pm \frac{1}{3}(x + y) & \text{if } y \equiv 2 \pmod{3}. \end{aligned}$$

Theorem 3 now follows from (2.5), (2.6), and Theorem 1.

**THEOREM 4.** *Let  $p = 24f + 1 = a^2 + b^2$ ,  $a \equiv 1 \pmod{4}$  and  $b > 0$ ,  $4p = A^2 + 27B^2$  with  $A \equiv 0 \pmod{2}$ . Then we have*

$$(a) \quad 3^{(p-1)/8} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} B \equiv 0 \pmod{16} & \text{and } a \equiv 1 \pmod{3}, \\ B \equiv 8 \pmod{16} & \text{and } a \equiv 2 \pmod{3}, \end{cases}$$

$$(b) \quad 3^{(p-1)/8} \equiv b/a \pmod{p} \Leftrightarrow \begin{cases} B \equiv 0 \pmod{16} & \text{and } b \equiv 1 \pmod{3}, \\ B \equiv 8 \pmod{16} & \text{and } b \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* As  $B = \pm 2y$  if  $A \equiv 0 \pmod{2}$ , we have  $B \equiv 0 \pmod{16}$  if  $y \equiv 0 \pmod{8}$  and  $B \equiv 8 \pmod{16}$  if  $y \equiv 4 \pmod{8}$  so that Theorem 4 follows from Theorem 1.

**REMARK 1.** Criteria (a) and (b) in Theorems 1 and 4 may be reformulated as

$$3^{(p-1)/8} \equiv (-1)^{y/4+[a/3]} \pmod{p} \quad \text{if } 3 \mid b$$

and

$$3^{(p-1)/8} \equiv (-1)^{y/4+[b/3]+1} b/a \pmod{p} \quad \text{if } 3 \mid a,$$

and for  $A$  even ( $\Leftrightarrow 2^{(p-1)/3} \equiv 1 \pmod{p}$ ) we have

$$3^{(p-1)/8} \equiv (-1)^{B/8+[a/3]} \pmod{p} \quad \text{if } 3 \mid b$$

and

$$3^{(p-1)/8} \equiv (-1)^{B/8+[b/3]+1} b/a \pmod{p} \quad \text{if } 3 \mid a.$$

**REMARK 2.** Putting together the criteria in Theorem 1 and the criteria given at the beginning of this paper we see that the parameters  $c$  and  $y$ ,  $c \equiv 1 \pmod{4}$ , are related for all primes  $p = 24f + 1 = c^2 + 2d^2 = x^2 + 3y^2$  as follows:

$$y \equiv 0 \pmod{8} \Leftrightarrow c \equiv (-1)^f \pmod{3}.$$

**3. Criteria for 5 to be a fourth power (mod  $p$ ).** Let  $p = 20f + 1 = a^2 + b^2 = e^2 + f^2$ ,  $a \equiv e \equiv 1 \pmod{4}$ ;

$$(3.1) \quad 16p = x^2 + 50u^2 + 50v^2 + 125w^2, \quad x \equiv 1 \pmod{5},$$

$$xw = v^2 - 4uv - u^2.$$

Gauss [3] showed that  $5^{(p-1)/4} \equiv 1 \pmod{p} \Leftrightarrow 5 \mid b$ . Recently it has been shown, see [2, p. 382], that  $5^{(p-1)/4} \equiv 1 \pmod{p} \Leftrightarrow 2 \nmid e$ , and that [4]

$$(3.2) \quad 5^{(p-1)/4} \equiv 1 \pmod{p} \Leftrightarrow \begin{cases} x \equiv 4 & \pmod{8}, \\ & \text{or} \\ x \equiv \pm 2w & \pmod{8}. \end{cases}$$

Using results of Emma Lehmer [6] we show that (3.2) can be reformulated as

$$(3.3) \quad 5^{(p-1)/4} \equiv 1 \pmod{p} \Leftrightarrow 16 \mid w \quad \text{or} \quad uv \equiv 2 \pmod{4}.$$

Embodied in (3.3) is considerably more information than in simpler criteria for 5 to be a fourth power (mod  $p$ ) as is seen by the following theorem.

**THEOREM 5.** *Let  $p = 20f + 1$  be a prime satisfying (3.1). Then we have*

(a)

$$5^{(p-1)/4} \equiv 1 \pmod{p} \quad \text{and} \quad 2^{(p-1)/5} \equiv 1 \pmod{p} \Leftrightarrow 16 \mid w,$$

(b)

$$5^{(p-1)/4} \equiv -1 \pmod{p} \quad \text{and} \quad 2^{(p-1)/5} \equiv 1 \pmod{p} \Leftrightarrow 16 \mid x,$$

(c)

$$5^{(p-1)/4} \equiv 1 \pmod{p} \quad \text{and} \quad 2^{(p-1)/5} \not\equiv 1 \pmod{p} \Leftrightarrow uv \equiv 2 \pmod{4},$$

(d)

$$5^{(p-1)/4} \equiv -1 \pmod{p} \quad \text{and} \quad 2^{(p-1)/5} \not\equiv 1 \pmod{p} \Leftrightarrow 4 \mid uv;$$

(e) *in case (c),  $2 \mid v \Leftrightarrow x \equiv 3w \pmod{8}$  and  $2 \mid u \Leftrightarrow x \equiv -3w \pmod{8}$ .*

*Proof.* To prove  $\Rightarrow$  in (a) note that from [6, p. 13] we have  $x = 4a$  and  $w = 4d$ ,  $a \equiv -d \pmod{2}$ , so that  $8 \mid w$  in view of (3.2); moreover,  $u \equiv v \equiv 0 \pmod{4}$ , so that if  $w \not\equiv 0 \pmod{16}$  we have, since  $xw = v^2 - 4uv - u^2$ , that  $32 \equiv 16 - 0 - 16 \pmod{64}$ , a clear impossibility. To prove  $\Leftarrow$  in (a) we have only to note that  $16 \mid w \Rightarrow 2 \mid x$  and that  $a \equiv -d \pmod{2} \Rightarrow x \equiv 4 \pmod{8}$ . We omit the proof of (b) as it is entirely similar.

To prove (c) and (e) we note first that  $x$  odd and  $x \equiv \pm 3w \pmod{8} \Leftrightarrow p - x^2 - 125w^2 \equiv 10 \pmod{16} \Leftrightarrow u^2 + v^2 \equiv 5 \pmod{8} \Leftrightarrow uv \equiv 2 \pmod{4}$ , proving (c). Then (e) follows easily from  $xw = v^2 - 4uv - u^2$ . Finally (c)  $\Rightarrow$  (d) ( $u$  and  $v$  are of opposite parity as  $x$  is odd), completing the proof.

EXAMPLE. Let  $p = 101$  so that  $(-29, 3, 2, 1)$  is a solution of (3.1). Since  $uv \equiv 2 \pmod{4}$  and  $2 \mid v$  we have  $5^{(p-1)/4} \equiv 1 \pmod{p}$ ,  $2^{(p-1)/5} \not\equiv 1 \pmod{p}$ , and  $x \equiv 3w \pmod{8}$ .

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Received December 9, 1981. Research supported by Natural Sciences and Engineering Research Council Canada grant A-7233.

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