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In this paper we introduce the concept of a (d, i) -sequence $(d, i \in N)$ in a commutative ring A , noetherian and with identity (cf. Def. 1.1). Let $K(\underline{z}, A)$ be the Koszul complex on A , with respect to the sequence $\underline{z} = z_1, \dots, z_n$: the concept of a (d, i) -sequence is expressed in terms of the structure of $H_i(K(\underline{z}, A))$; in particular, it turns out that \underline{z} is an (n, i) -sequence iff $H_i(K(\underline{z}, A)) = 0$, and such a condition implies \underline{z} is a (d, i) -sequence for any $d \leq n$. If $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A} = A/(z_{h+1}, \dots, z_n)$, $d \leq h \leq n$, then \underline{z} is seen to be a (d, i) -sequence in A ; so, in particular, if $H_i(K(\bar{z}; {}_d\bar{A})) = 0$ in ${}_d\bar{A}$, then \underline{z} is a (d, i) -sequence. Moreover, for $i = 1$, the two conditions are equivalent, so that \underline{z} is a $(d, 1)$ -sequence means precisely that $\bar{z}_1, \dots, \bar{z}_d$ is regular in ${}_d\bar{A}$. For $i > 1$, examples show that \underline{z} is a (d, i) -sequence is a condition strictly weaker than $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$, and we investigate the relationship between those two properties. In fact, their equivalence allows us to read the depth of a quotient ring $A/(z_{h+1}, \dots, z_n)$ in terms of the Koszul complex $K(\underline{z}; A)$ and implies, for (d, i) -sequences, properties which are a natural generalization of *good properties* satisfied by regular sequences, such as the depth-sensitivity of the Koszul complex. A characteristic condition for their equivalence is a kind of weak surjectivity of a natural map acting between $\text{syz}^{i+1}(K(\underline{z}; A))$ and $\text{syz}^{i+1}(K(\bar{z}; {}_h\bar{A}))$.

From an algebraic form of that weak surjectivity we get some sufficient conditions, in terms of weak regularity of the sequence z_{h+1}, \dots, z_n . For instance, if z_{h+1}, \dots, z_n is a d -sequence, or a relative regular sequence, or less, if z_{h+1}, \dots, z_n is a relative regular A -sequence with respect to a convenient set of ideals, then \underline{z} is a (d, i) -sequence in A implies $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$.

Moreover, if \underline{z} is a (d, i) -sequence and z_{d+1}, \dots, z_n is a regular sequence, then $H_i(K(\underline{z}; A)) = 0$, while this vanishing implies that it is possible to find x_1, \dots, x_n in $I = (z_1, \dots, z_n)$ such that $z_1, \dots, z_{i-1}, x_1, \dots, x_n$ is a (d, i) -sequence and x_{d+1}, \dots, x_n is a regular sequence.

In the last section we give an interpretation of our results in terms of the behaviour of some systems of linear equations.

N. 1. Let A be a noetherian ring (with 1) and $\underline{z} = z_1, \dots, z_n$ a sequence of elements of A such that $(z_1, \dots, z_n)A \neq A$. We denote by $K(\underline{z}; A)$ the Koszul complex with respect to \underline{z} , i.e. the differential graded algebra (DGA for short) (cf. [G-L] cap. I for a definition)

$$0 \rightarrow \wedge^n A^n \xrightarrow{d_n} \wedge^{n-1} A^n \rightarrow \dots \rightarrow \wedge^2 A^n \xrightarrow{d_2} A^n \xrightarrow{d_1} A \xrightarrow{d_0} sA / (\underline{z})A \rightarrow 0$$

generated by $e_i, i = 1, \dots, n$, with differential

$$d_j(e_{j_1} \wedge \dots \wedge e_{j_r}) = \sum_{t=1, \dots, j} (-1)^{t+1} z_{i_t} \cdot e_{i_1} \wedge \dots \wedge \check{e}_{i_t} \wedge \dots \wedge e_{i_r}.$$

Also we write

$$\text{syz}^i(K(\underline{z}; A)) = \ker(d_{i-1}) \subseteq \wedge^{i-1} A^n$$

for $i = 1, \dots, n + 1$. As in [M-R], for every $1 \leq i \leq d \leq n$, $T_i^{(n,d)}$ will mean the free A -module generated by $e_{j_1 \dots j_i} = e_{j_1} \wedge \dots \wedge e_{j_i}$, with $1 \leq j_1 < \dots < j_i \leq n$ and $j_i > d$, which is a complementary module of $\wedge^i(Ae_1 \oplus \dots \oplus Ae_d)$, briefly $\wedge^i A_{1 \dots d}$, in $\wedge^i A^n$, so $\wedge^i A^n = \wedge^i A_{1 \dots d} \oplus T_i^{(n,d)}$.

Then

$$\begin{aligned} \pi_i &: \wedge^i A^n \rightarrow T_i^{(n,d)}, \\ \chi_i &: \wedge^i A^n \rightarrow \wedge^i A_{1 \dots d} \end{aligned}$$

will be the usual projections, i.e.

$$\begin{aligned} \pi_i \left(\sum_{1 \leq j_1 < \dots < j_i \leq n} a_{j_1 \dots j_i} e_{j_1 \dots j_i} \right) &= \sum_{1 \leq j_1 < \dots < j_i \leq n} a_{j_1 \dots j_i} e_{j_1 \dots j_i}, \\ \chi_i \left(\sum_{1 \leq j_1 < \dots < j_i \leq n} a_{j_1 \dots j_i} e_{j_1 \dots j_i} \right) &= \sum_{1 \leq j_1 < \dots < j_i \leq d} a_{j_1 \dots j_i} e_{j_1 \dots j_i} \end{aligned}$$

and, more generally,

$$\chi_i^h : \wedge^i A^n \rightarrow \wedge^i A_{1 \dots h} \quad (h \leq n)$$

will be like χ_i when we set $d = h$.

When z_1, \dots, z_n are fixed elements of A , we write

$${}_t \bar{A} = A / (z_{t+1}, \dots, z_n)A$$

for $t = 0, \dots, n$ (${}_n \bar{A} = A$), and $\bar{z}_i \in {}_t \bar{A}$ means the image of z_i by the natural map $\rho_t: A \rightarrow {}_t \bar{A}$.

For every two integers $s \geq r$ we can define a map of DGAs,

$$(1) \quad \Psi^{(s,r)}: K(\bar{z}_1, \dots, \bar{z}_s; {}_s \bar{A}) \rightarrow K(\bar{z}_1, \dots, \bar{z}_r; {}_r \bar{A}),$$

by

$$\Psi_0^{(s,r)} = \text{the natural map } {}_s \bar{A} \rightarrow {}_r \bar{A}$$

and

$$\Psi_1^{(s,r)}(e_i) = \begin{cases} f_i & \text{for } 1 \leq i \leq r, \\ 0 & \text{for } r < i \leq s, \end{cases}$$

where e_1, \dots, e_s and f_1, \dots, f_r are, respectively, the free generators of $K(\bar{z}_1, \dots, \bar{z}_s; {}_sA)$ and $K(\bar{z}_1, \dots, \bar{z}_r; {}_rA)$. $\Psi^{(n;h)}$ will be denoted simply by Ψ^h . Finally, we make the usual convention of setting

$$c_{j_1 \dots j_i} = (-1)^\delta c_{\tau(j_1) \dots \tau(j_i)},$$

where τ is a permutation on $\{j_1, \dots, j_i\}$ and δ is 0 or 1 according to whether τ is an even or an odd permutation.

DEFINITION 1.1. Let $\underline{z} = z_1, \dots, z_n$ be a system of non invertible elements of a ring A and i, d integers such that $1 \leq i \leq d \leq n$. We say that \underline{z} is a $(d, i)^*$ -sequence if $H_i(K(\underline{z}; A))$ can be generated by the image of $T_i^{(n,d)}$. We say that \underline{z} is a (d, i) -sequence if it is a $(d, i)^*$ -sequence and $(z_1, \dots, z_n)A \subseteq \text{rad } A$.

REMARK 1.2. (i) Obviously $\{z_1, \dots, z_d, z_{d+1}, \dots, z_n\}$ is a (d, i) -sequence if and only if $\{z_{\tau(1)}, \dots, z_{\tau(d)}, z_{\sigma(d+1)}, \dots, z_{\sigma(n)}\}$ is a (d, i) -sequence, where τ and σ are permutations on $\{1, \dots, d\}$ and $\{d + 1, \dots, n\}$, respectively.

(ii) Any (d, i) -sequence is a (d', i) -sequence, for $d' \leq d$.

(iii) $(d, i)^*$ -sequences go up and down by faithful flatness. In fact, if $f: A \rightarrow B$ is a morphism, then

$$K(\underline{z}; A) \otimes_A B = K(f(\underline{z}); B) \quad \text{and} \quad d_i(\wedge^i A^n) \otimes_A B = d_i(\wedge^i B^n).$$

Now, the $(d, i)^*$ condition says in A or, respectively, in B

1. $\text{syz}_A^{i+1}(A/\underline{z}A) = d_{i+1}(\wedge^{i+1} A^n) + [T_i^{(n,d)}(A) \cap \text{syz}_A^{i+1}(A/\underline{z}A)]$
2. $\text{syz}_B^{i+1}(B/f(\underline{z})B)$
 $= d_{i+1}(\wedge^{i+1} B^n) + [T_i^{(n,d)}(B) \cap \text{syz}_B^{i+1}(B/f(\underline{z})B)].$

If f is faithfully flat, then

$$\text{syz}^{i+1}(A/\underline{z}A) \otimes_A B \simeq \text{syz}^{i+1}(B/f(\underline{z})B),$$

and for every two A -modules M and N ,

$$(M \otimes_A B) \cap (N \otimes_A B) \simeq (M \cap N) \otimes_A B$$

so that 2. comes from 1. by tensoring with B ; again by faithful flatness the conclusion follows.

(iv) For $n = d$, \underline{z} is an (n, i) -sequence iff $H_i(K(\underline{z}; A)) = 0$; so, in particular, if $\text{depth}(z_1, \dots, z_n) \geq n - i + 1$, then \underline{z} is a (d, j) -sequence, for

every d, j such that $i \leq j \leq d \leq n$ (because of (ii) and the depth-sensitivity of the Koszul complex $[\mathbf{A}\text{-}\mathbf{B}]$).

(v) If d_i is the largest integer such that \underline{z} is a (d_i, i) -sequence, $n - d_i$ gives a *measure* of the obstruction to \underline{z} having depth bigger than or equal to $n - i + 1$; in particular, for $i = 1$, $n - d_1$ says how far z_1, \dots, z_n is from regularity.

THEOREM 1.3. *Let $\underline{z} = z_1, \dots, z_n$ be a system of elements of a ring A , with $(z_1, \dots, z_n)A \subseteq \text{rad } A$, and i, d, h integers such that $1 \leq i \leq d \leq h \leq n$. If $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$, then z_1, \dots, z_n is a (d, i) -sequence in A .*

Proof. Let

$$\alpha = \sum_{1 \leq j_1 < \dots < j_i \leq n} a_{j_1 \dots j_i} e_{j_1 \dots j_i}$$

be an element in $\text{syz}^{i+1}(K(\underline{z}; A))$. Since Ψ^h is a map of complexes,

$$\Psi_i^h(\alpha) = \sum_{1 \leq j_1 < \dots < j_i \leq h} \bar{a}_{j_1 \dots j_i} f_{j_1 \dots j_i}$$

will be in $\text{syz}^{i+1}(K(\bar{z}; \bar{A}))$. Then the hypothesis on ${}_h\bar{A}$ says

$$\sum_{1 \leq j_1 < \dots < j_i \leq d} \bar{a}_{j_1 \dots j_i} f_{j_1 \dots j_i} = d_{i+1} \bar{\beta} + \sum_{\substack{1 \leq j_1 < \dots < j_i \leq h \\ j_i > d}} \bar{b}_{j_1 \dots j_i} f_{j_1 \dots j_i}$$

for some $\bar{\beta} \in \wedge^{i+1} {}_h\bar{A}^h$ and $\bar{b}_{j_1 \dots j_i} \in {}_h\bar{A}$.

From this we get

$$\begin{aligned} \sum_{1 \leq j_1 < \dots < j_i \leq d} a_{j_1 \dots j_i} e_{j_1 \dots j_i} &= d_{i+1} \beta + \sum_{\substack{1 \leq j_1 < \dots < j_i \leq h \\ j_i > d}} b_{j_1 \dots j_i} e_{j_1 \dots j_i} \\ &+ \sum_{\substack{1 \leq j_1 < \dots < j_i \leq n \\ j_i > h}} c_{j_1 \dots j_i} e_{j_1 \dots j_i} \\ &+ \sum_{1 \leq j_1 < \dots < j_i \leq h} \left(\sum_{t=h+1}^n c_{j_1 \dots j_i} z_t e_{j_1 \dots j_i} \right) \end{aligned}$$

for some $\beta \in \wedge^{i+1} A^n$, $c_{j_1 \dots j_i} \in A$, for $1 \leq j_1 < \dots < j_i \leq n$ and for any lifting $b_{j_1 \dots j_i}$ of $\bar{b}_{j_1 \dots j_i}$.

Now we can conclude the proof just by remarking that

$$z_k e_{j_1 \dots j_i} = d_{i+1} (e_{j_1 \dots j_i}) + \alpha \wedge e_k$$

for every $k \neq j_1, \dots, j_i$, for some $\alpha \in \wedge^{i+1} A^n$.

COROLLARY 1.4. *If $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - i + 1$ in ${}_d\bar{A}$, then z_1, \dots, z_n is a (d, i) -sequence in A .*

Proof. Use Remark 1.2 and then apply Theorem 1.3 for $h = d$.

Theorem 1.3 allows us to lift (d, i) -sequences from ${}_h\bar{A}$ to A , but, conversely, a (d, i) -sequence in A does not necessarily remain a (d, i) -sequence in ${}_h\bar{A}$, as we can see from the following

EXAMPLE 1.5. Let $A = k[[X, Y, Z]]/I$, $I = (X^2 - Z^2, XY, XZ)$, k a field, and denote by x, y, z the images of X, Y, Z in A . We show that x, y, z is a $(2, 2)$ -sequence in A , but, in $\bar{A} = A/(z)$, \bar{x}, \bar{y} is not, i.e. $\text{depth}(\bar{x}, \bar{y}) = 0$ in \bar{A} . The second fact is trivial since $A/(z) = k[[X, Y]]/X(X, Y)$ so $\text{depth}(A/(z)) = 0$.

Now let $\beta = a_{12}e_1 \wedge e_2 + a_{13}e_1 \wedge e_3 + a_{23}e_2 \wedge e_3$ be an element of $\text{syz}^3(K(x, y, z; A))$; this says

$$(2) \quad a_{12}y + a_{13}z = 0, \quad -a_{12}x + a_{23}z = 0, \quad a_{13}x + a_{23}y = 0,$$

and we have to show $a_{12} \in (z)A$. Since \bar{y} is a regular element in $A/(x)$, from the third equation in (2) we get $a_{23} = \lambda x$, for some $\lambda \in A$. On the other hand, in $A/(z)A$ we have $\text{ann}(\bar{x}, \bar{y}) = (\bar{x})\bar{A}$, so from the first and second equations of (2) we have $a_{12} = \mu x + \nu z$ for some $\mu, \nu \in A$. Now the second equation in (2) becomes $\mu x^2 = 0$, so $\mu x \in \text{ann}(x)$, but $\mu x \in \text{ann}(y)$ and in A we have $\text{ann}(x) \cap \text{ann}(y) = (y, z)A \cap (x)A = (z^2)A$. Thus $a_{12} \in (z)A$.

Corollary 1.4 suggests investigating the condition we need in order to have the relationship

$$(3) \quad \begin{aligned} & z_1, \dots, z_n \text{ is a } (d, i)\text{-sequence in } A \\ & \Rightarrow \text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - i + 1 \text{ in } {}_d\bar{A}. \end{aligned}$$

We know this is always verified, for $i = 1$, for a local ring (A, \mathfrak{m}) and $(z_1, \dots, z_n)A = \mathfrak{m}$ (see [M-R] Theorem 2.5); here we prove it without any assumption on A and z_1, \dots, z_n (cf. Corollary 2.4).

For $i \geq 2$ we certainly need some conditions (Example 1.5 says (3) is not generally true). We will see that (3) holds whenever z_{d+1}, \dots, z_n is a regular sequence; nevertheless there are weaker conditions which imply our relation. We will dedicate the next two sections to investigating such a question and some related ones. For this, most of our technique depends on the solutions of some systems of linear equations and their subsystems, and we will devote the last part of the paper to explaining this idea.

N. 2. The equality $\Psi_i^h(\text{syz}^{i+1}(K(\underline{z}; A))) = \text{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$ (cf. (1) N. 1) clearly implies that if z_1, \dots, z_h is a (d, i) -sequence in A , then $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence of ${}_h\bar{A}$. However, such a condition is not a necessary one, because the (d, i) -condition gives a link only between those components of the elements of $\text{syz}^{i+1}(K(\underline{z}; A))$ which lie in $\wedge^i A_{1\dots d}$. It is easy to check that (d, i) -sequences descend from A to ${}_h\bar{A}$ if the following condition is verified:

Weak Lifting Condition (W.L.C.)_{h,i}. Let

$$\bar{\alpha} = \bar{a} + \bar{b} \in \text{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A})),$$

where $\bar{a} \in \wedge^i {}_h\bar{A}_{1\dots d}$, $\bar{b} \in T_i^{(h,d)}$ (as ${}_h\bar{A}$ -module). Then there is $\alpha = a + b \in \text{syz}^{i+1}(K(\underline{z}; A))$, where $a \in \wedge^i A_{1\dots d}$, $b \in T_i^{(n,d)}$ (as A -module), such that $\Psi_i^h(a) = \bar{a}$. We will call α a weak lifting of $\bar{\alpha}$.

REMARK 2.1. $(\text{W.L.C.})_{d,i}$ is equivalent to $\Psi_i^d(\text{syz}^{i+1}(K(\underline{z}; A))) = \text{syz}^{i+1}(K(\underline{\bar{z}}; {}_d\bar{A}))$, because in such a situation we have $\bar{b} = 0$; in other words, $(\text{W.L.C.})_{d,i}$ is exactly the surjectivity of the restriction of Ψ_i^d to the syzygies.

Now, let us prove that $(\text{W.L.C.})_{h,i}$ is also necessary to pass (d, i) -sequences from A to ${}_h\bar{A}$.

PROPOSITION 2.2. *Let z_1, \dots, z_n be a (d, i) -sequence in A . Then $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$ if and only if $(\text{W.L.C.})_{h,i}$ holds.*

Proof. We already observed that $(\text{W.L.C.})_{h,i}$ is sufficient. Let us prove its necessity. With the same notation as in $(\text{W.L.C.})_{h,i}$, let $\bar{\alpha} = \bar{a} + \bar{b} \in \text{syz}^{i+1}(K(\underline{\bar{z}}; {}_h\bar{A}))$, where

$$\bar{a} = \sum_{1 \leq j_1 < \dots < j_i \leq d} \bar{a}_{j_1 \dots j_i} f_{j_1 \dots j_i}.$$

The (d, i) -condition on the sequence says that

$$\bar{a}_{j_1 \dots j_i} = \sum_{t \in \{1, \dots, h\} - \{j_1, \dots, j_i\}} \bar{c}_{j_1 \dots t \dots j_i} \cdot \bar{z}_t, \quad 1 \leq j_1 < \dots < j_i \leq d.$$

We lift these relations to A , defining

$$a_{j_1 \dots j_i} = \sum_{t \in \{1, \dots, h\} - \{j_1, \dots, j_i\}} c_{j_1 \dots t \dots j_i} \cdot z_t, \quad 1 \leq j_1 < \dots < j_i \leq d,$$

with $\rho_h(c_{j_1 \dots t \dots j_i}) = \bar{c}_{j_1 \dots t \dots j_i}$, and finally

$$a = \sum_{1 \leq j_1 < \dots < j_i \leq d} a_{j_1 \dots j_i} e_{j_1 \dots j_i}.$$

Moreover, we define $b \in T^{(n,h)}$ as follows:

$$b = \sum_{\substack{1 \leq j_1 < \dots < j_{i-1} \leq d \\ d < j_i \leq h}} b_{j_1 \dots j_i} e_{j_1 \dots j_i},$$

where

$$b_{j_1 \dots j_i} = \sum_{t \in \{1, \dots, d\} - \{j_1, \dots, j_i\}} c_{j_1 \dots t \dots j_i} \cdot z_t, \\ 1 \leq j_1 < \dots < j_{i-1} \leq d, \quad d < j_i \leq h.$$

Since trivially $\Psi_i^h(a) = \bar{a}$, it is enough to prove that

$$a + b \in \text{syz}^{i+1}(K(z; A)).$$

The coefficient of $e_{j_1 \dots j_{i-1}}$, $1 \leq j_1 < \dots < j_{i-1} \leq d$, in $d_i(a + b)$, is

$$\sum_{u \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\}} a_{j_1 \dots u \dots j_{i-1}} \cdot z_u + \sum_{r=d+1}^h b_{j_1 \dots j_{i-1} r} \cdot z_r \\ = \sum_{u \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\}} \left(\sum_{t \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}, u\}} c_{j_1 \dots u \dots t \dots j_{i-1}} \cdot z_t + \sum_{t=d+1}^h c_{j_1 \dots u \dots j_{i-1} t} \cdot z_t \right) z_u \\ + \sum_{r=d+1}^h \left(\sum_{t \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\}} c_{j_1 \dots t \dots j_{i-1} r} \cdot z_t \right) z_r \\ = \sum_{\substack{u, t \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\} \\ u \neq t}} c_{j_1 \dots u \dots t \dots j_{i-1}} \cdot z_t z_u \\ + \sum_{u \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\}} \sum_{t=d+1}^h c_{j_1 \dots u \dots j_{i-1} t} \cdot z_t z_u \\ + \sum_{r=d+1}^h \sum_{t \in \{1, \dots, d\} - \{j_1, \dots, j_{i-1}\}} c_{j_1 \dots t \dots j_{i-1} r} \cdot z_t z_r,$$

which is zero, because for $p \neq m$, $p, m = 1, \dots, d$, the coefficient of $z_p z_m$ is

$$(\dagger) \quad c_{j_1 \dots p \dots m \dots j_{i-1}} + c_{j_1 \dots m \dots p \dots j_{i-1}} = 0,$$

coming from the first Σ , and for $1 \leq p \leq i$ and $d < m \leq h$, the coefficient of $z_p z_m$ is again (\dagger) , coming from the second and the third Σ . Similarly, the coefficient of $e_{j_1 \dots j_{i-1}}$, $1 \leq j_1 < \dots < j_{i-2} \leq d$, $d < j_{i-1} \leq h$, in $d_i(a + b)$ is

$$\begin{aligned} & \sum_{u \in \{1, \dots, d\} - \{j_1, \dots, j_{i-2}\}} b_{j_1 \dots u \dots j_{i-1}} \cdot z_u = \\ & = \sum_{u \in \{1, \dots, d\} - \{j_1, \dots, j_{i-2}\}} \sum_{t \in \{1, \dots, d\} - \{j_1, \dots, j_{i-2}, u\}} c_{j_1 \dots u \dots t \dots j_{i-2}} \cdot z_u z_t = 0 \end{aligned}$$

with the same computation.

Theorem 1.3 and Proposition 2.2 can be restated as follows.

THEOREM 2.3. *The following conditions are equivalent*

- (1) z_1, \dots, z_n is a (d, i) -sequence in A and $(\text{W.L.C.})_{h,i}$ holds.
- (2) $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$.

In particular, for $h = d$, Theorem 2.3 becomes

COROLLARY 2.4. *The following conditions are equivalent:*

- (1) z_1, \dots, z_n is a (d, i) -sequence in A and $(\text{W.L.C.})_{d,i}$ holds.
- (2) $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - i + 1$ in ${}_d\bar{A}$.

The case $i = 1$ looks much simpler because of

PROPOSITION 2.5. *Condition $(\text{W.L.C.})_{h,1}$ is always verified, as $\Psi_1^h(\text{syz}^2(K(\underline{z}; A))) = \text{syz}^2(K(\underline{\bar{z}}; {}_h\bar{A}))$.*

Proof. It is clearly enough to show the equality for $h = n - 1$ because then we use induction. If

$$\sum_{i=1}^{n-1} \bar{a}_i f_i \in \text{syz}^2(K(\underline{\bar{z}}; {}_{n-1}\bar{A})),$$

we have $\sum_{i=1}^{n-1} \bar{a}_i \bar{z}_i = 0$; so there exist a_1, \dots, a_n such that $\sum_{i=1}^n a_i z_i = 0$, which implies $\sum_{i=1}^n a_i e_i \in \text{syz}^2(K(\underline{z}; A))$ and $\Psi_1^{(n-1)}(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^{n-1} \bar{a}_i f_i$.

As a consequence of Proposition 2.5, we get

PROPOSITION 2.6. *The following are equivalent:*

- (1) z_1, \dots, z_n is a $(d, 1)$ -sequence in A .
- (2) $\bar{z}_1, \dots, \bar{z}_d$ is a $(d, 1)$ -sequence in ${}_d\bar{A}$.

(3) $\bar{z}_1, \dots, \bar{z}_d$ is a regular sequence of ${}_d\bar{A}$.

(4) $(z_1, \dots, z_{i-1}, z_{d+1}, \dots, z_n)$: $z_i = (z_1, \dots, z_{i-1}, z_{d+1}, \dots, z_n)$, $i = 1, \dots, d, (z_0 = 0)$.

Proof. The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) follow from Proposition 2.5, Corollary 2.4 and Remark 1.2. The equivalence between (3) and (4) is just an easy computation, translating the definition of regularity for the sequence $\bar{z}_1, \dots, \bar{z}_d$.

From condition (3) of Proposition 2.6 and Remark 1.2(i), we immediately get

COROLLARY 2.7. *If $z_1, \dots, z_d, z_{d+1}, \dots, z_n$ is a $(d, 1)$ -sequence, then $x_1, \dots, x_{d-s}, z_{d+1}, \dots, z_n$ is a $(d - s, 1)$ -sequence, where x_1, \dots, x_{d-s} is any non empty subset of z_1, \dots, z_d .*

REMARK 2.8. If $(W.L.C.)_{d,i}$ holds, Corollary 2.7 can easily be generalized to the i case; so, in this case, if z_1, \dots, z_n is a (d, i) -sequence, then $x_1, \dots, x_{d-s}, z_{d+1}, \dots, z_n$ is a $(d - s, i)$ -sequence for x_1, \dots, x_{d-s} any subset of z_1, \dots, z_d , $0 \leq s \leq d - i$.

REMARK 2.9. Let (A, \mathfrak{m}) be a local ring and z_1, \dots, z_n a set of generators of \mathfrak{m} ; then $\bar{z}_1, \dots, \bar{z}_d$ is a set of generators of $\bar{\mathfrak{m}}$ in ${}_d\bar{A}$, so condition (3) of Proposition 2.6 says that ${}_d\bar{A}$ is a regular ring. Moreover, the $(d, 1)$ -condition on \underline{z} is the same as condition \mathbf{R}_2^d defined for a projective resolution of A/\mathfrak{m} in $[\mathbf{M-R}]$; so, Proposition 2.6 implies Theorems 2.5 and 2.7 of loc. cit.

Now we can generalize Corollary 2.12 of $[\mathbf{M-R}]$.

PROPOSITION 2.10. *If z_1, \dots, z_n is a $(d, 1)$ -sequence of A and $I = (z_{d+1}, \dots, z_n)$ has co-height $\leq d$ (i.e. $\dim(A/I) \leq d$), then A/I is Cohen-Macaulay. In particular, if (A, \mathfrak{m}) is local and $(z_1, \dots, z_n) = \mathfrak{m}$, then A/I is regular.*

Proof. Applying Remark 1.2(iii) and Proposition 2.6 we get $\text{depth}(A/I)_{\mathfrak{p}} \geq d$ for every $\mathfrak{p} \in \text{Max}(A/I)$; since, by hypothesis, $\dim(A/I) \leq d$, the conclusion follows.

By Remark 1.2(ii) if z_1, \dots, z_n is a (d, i) -sequence, it is also a (d', i) -sequence for every $d' \leq d$ ($i \leq d'$). It seems meaningful to ask whether every (d, i) -sequence is also a (d, j) -sequence for $j \geq i$ ($j \leq d$). For $d = n$

this becomes the well-known rigidity of the Koszul complex (see for instance [G-L]).

Another partial answer to this question is given by

PROPOSITION 2.11. *If z_1, \dots, z_n is a $(d, 1)$ -sequence, then it is a (d, i) -sequence for every $i \geq 1$.*

Proof. By Proposition 2.6 and the $(d, 1)$ -condition on \underline{z} , we have $\bar{z}_1, \dots, \bar{z}_d$ is a $(d, 1)$ -sequence in ${}_d\bar{A}$, which means $H_1(K(\bar{z}; {}_d\bar{A})) = 0$. Now, the mentioned rigidity of the Koszul complex implies $H_i(K(\bar{z}; {}_d\bar{A})) = 0$ for every $i > 1$. Then Theorem 1.3 says z_1, \dots, z_n is a (d, i) -sequence in A .

The well-known depth-sensitivity of the Koszul complex says, in particular, that if $H_i(K(\underline{z}; A)) = 0$ then there exist $n - i + 1$ elements x_1, \dots, x_{n-i+1} in (z_1, \dots, z_n) which form a regular sequence, i.e. $H_1(K(x_1, \dots, x_{n-i+1}; A)) = 0$. Now we prove a sort of $(d, 1)$ -sensitivity of the Koszul complex. Namely, we have

THEOREM 2.12. *If z_1, \dots, z_n is a (d, i) -sequence and $(\text{W.L.C.})_{d,i}$ holds, then for every s , $0 \leq s < i$, there exist $x_1, \dots, x_{d-s} \in (z_1, \dots, z_d)$ such that $x_1, \dots, x_{d-s}, z_{d+1}, \dots, z_n$ is a $(d, i - s)$ -sequence. In particular, we can find $d - i + 1$ elements in (z_1, \dots, z_d) such that $x_1, \dots, x_{d-i+1}, z_{d+1}, \dots, z_n$ is a $(d, 1)$ -sequence.*

Proof. By Corollary 2.4 $H_i(K(\bar{z}; {}_d\bar{A})) = 0$, so, for $0 \leq s < i$, we can find $\bar{x}_1, \dots, \bar{x}_{d-s} \in (\bar{z}_1, \dots, \bar{z}_d)$ such that $H_{i-s}(K(\bar{x}_1, \dots, \bar{x}_{d-s}; {}_d\bar{A})) = 0$. Now from Corollary 1.4 we get the desired result.

The $(\text{W.L.C.})_{d,i}$ condition in the previous theorem seems to be necessary (in some sense) to the above $(d, 1)$ -sensitivity of the Koszul complex. In fact, if $(\text{W.L.C.})_{d,i}$ does not hold in ${}_d\bar{A} = A/(z_{d+1}, \dots, z_n)$, then Proposition 2.2 says we can find a (d, i) -sequence, $z_1, \dots, z_d, z_{d+1}, \dots, z_n$, such that $\bar{z}_1, \dots, \bar{z}_d$ is not a (d, i) -sequence in ${}_d\bar{A}$. Now, let $1 \leq j < i$ be such that $(\text{W.L.C.})_{d,j}$ holds in ${}_d\bar{A}$; then we cannot find $d - i + j$ elements, say $x_1, \dots, x_{d-i+j} \in (z_1, \dots, z_d)$, such that $x_1, \dots, x_{d-i+j}, z_{d+1}, \dots, z_n$ is a $(d - i + j, j)$ -sequence. Namely, otherwise it should be $H_j(K(\bar{x}_1, \dots, \bar{x}_{d-i+j}; {}_d\bar{A})) = 0$, which implies $H_i(K(\bar{x}_1, \dots, \bar{x}_{d-i+j}; {}_d\bar{A})) = 0$, and also $H_i(K(\bar{z}_1, \dots, \bar{z}_d; {}_d\bar{A})) = 0$, which means, by Corollary 2.4, that $\bar{z}_1, \dots, \bar{z}_d$ should be a (d, i) -sequence.

Therefore, for $j = 1$, using Proposition 2.5, we get

PROPOSITION 2.13. *If for any (d, i) -sequence $z_1, \dots, z_d, z_{d+1}, \dots, z_n$, with fixed tail z_{d+1}, \dots, z_n , we can find a $(d - i + 1, 1)$ -sequence*

$x_1, \dots, x_{d-i+1}, z_{d+1}, \dots, z_n$ with $x_1, \dots, x_{d-i+1} \in (z_1, \dots, z_d)$, then $(W.L.C.)_{d,i}$ must hold for ${}_d\bar{A} = A/(z_{d+1}, \dots, z_n)$.

In order to investigate the behavior of (d, i) -sequences when we pass to a quotient with respect to elements of its *head* (the first d elements), let us denote by

$$\phi: K(z_1, \dots, z_n; A) \rightarrow K(\bar{z}_2, \dots, \bar{z}_n; \bar{A} = A/(z_1))$$

the usual map of DGA, defined by

$$\begin{aligned} \phi_0: A &\xrightarrow{\text{nat}} A/(z_1), \\ \phi_1(e_i) &= \begin{cases} f_i & \text{for } i \geq 2, \\ 0 & \text{for } i = 1, \end{cases} \end{aligned}$$

where $\{e_i\}_{i=1, \dots, n}$ and $\{f_i\}_{i=2, \dots, n}$ are free generators of $K_1(\underline{z}; A)$ and $K_1(\bar{\underline{z}}; \bar{A})$, respectively, and denote by

$$\tilde{\phi}_i: H_i \rightarrow \tilde{H}_i \quad \text{and} \quad \phi_i^*: H_i/T_i^{(n,d)}H_i \rightarrow \tilde{H}_i/\tilde{T}_i^{(n-1,d-1)}\tilde{H}_i$$

the induced maps, where $H_i = H_i(K(\underline{z}; A))$ and $\tilde{H}_i = H_i(K(\bar{\underline{z}}; \bar{A}))$.

The crucial fact for our purpose is

LEMMA 2.14. *With the above notation,*

- (i) *if z_1 is regular in A , then ϕ_i^* is surjective,*
- (ii) *if \bar{z}_1 is regular in ${}_d\bar{A}$, then ϕ_i^* is injective.*

Proof. (i) is trivial since the regularity of z_1 in A implies the surjectivity on the induced map

$$\tilde{\phi}_i: \text{syz}^{i+1}(K(\underline{z}; A)) \rightarrow \text{syz}^{i+1}(K(\bar{\underline{z}}; \bar{A})).$$

(ii) Let $[\alpha]$ be an element of $H_i(K(\underline{z}; A))$ and suppose $\tilde{\phi}_i([\alpha]) \in \tilde{T}_i^{(n-1,d-1)}\tilde{H}_i$; then

$$[\alpha] = \left[\sum_{\substack{2 \leq j_1 < \dots < j_i \leq n \\ j_i > d}} a_{j_1 \dots j_i} e_{j_1 \dots j_i} + \sum_{2 \leq j_2 < \dots < j_i \leq n} a_{1j_2 \dots j_i} e_{1j_2 \dots j_i} \right].$$

Since α is a cycle, for every $2 \leq j_2 < \dots < j_i \leq d$, we have

$$a_{1j_2 \dots j_i} \cdot z_1 + \sum_{k=d+1}^n a_{j_2 \dots j_i k} \cdot z_k = 0.$$

So the regularity of \bar{z}_i in ${}_d\bar{A}$ implies

$$a_{1j_2 \cdots j_i} = \sum_{k=d+1}^n c_{j_2 \cdots j_i k} z_k$$

for every $2 \leq j_2 < \cdots < j_i \leq d$. Then

$$[\alpha] = \left[\sum_{\substack{2 \leq j_1 < \cdots < j_i \leq n \\ j_i > d}} b_{j_1 \cdots j_i} e_{j_1 \cdots j_i} + \sum_{\substack{2 \leq j_2 < \cdots < j_i \leq n \\ j_i > d}} a_{1j_2 \cdots j_i} e_{1j_2 \cdots j_i} \right]$$

for some $b_{j_1 \cdots j_i}$. Thus $[\alpha] \in T_i^{(n,d)} H_i$.

PROPOSITION 2.15. *If z_1, \dots, z_n is a (d, i) -sequence and z_1, \dots, z_s , $1 \leq s \leq d - i$, is a regular A -sequence, then $\tilde{z}_{s+1}, \dots, \tilde{z}_n$ is a $(d - s, i)$ -sequence in $\tilde{A} = A/(z_1, \dots, z_s)$.*

Proof. By induction reduce to the case $s = 1$, then apply Lemma 2.14(i) to conclude $\tilde{H}_i / \tilde{T}_i^{(n-1, d-1)} \tilde{H}_i = 0$, so $\tilde{z}_2, \dots, \tilde{z}_n$ is a $(d - 1, i)$ -sequence in $\tilde{A} = A/(z_1)$.

Conversely, we have

PROPOSITION 2.16. *If z_1, \dots, z_n is a sequence (in $\text{rad } A$) such that $\tilde{z}_{s+1}, \dots, \tilde{z}_n$ is a $(d - s, i)$ -sequence in $\tilde{A} = A/(z_1, \dots, z_s)$ and $\bar{z}_1, \dots, \bar{z}_s$ is a regular sequence in ${}_d\bar{A}$, then it is a (d, i) -sequence.*

Proof. Again by induction reduce to $s = 1$, then apply Lemma 2.14 (ii), so $H_i / T_i^{(n,d)} H_i = 0$, i.e. z_1, \dots, z_n is a (d, i) -sequence.

N. 3. Now, our aim is to translate $(\text{W.L.C.})_{h,i}$ into an algebraic form. The next proposition will be helpful; it says, roughly, that, if $(\text{W.L.C.})_{h,i}$ holds, we can build a weak lifting of $\bar{\alpha} = \bar{a} + \bar{b}$ starting from any lifting a of \bar{a} .

PROPOSITION 3.1. *Let*

$$\bar{a} + \bar{b} \in \text{syz}^{i+1}(K(\bar{z}; {}_h\bar{A})), \quad a + b \in \text{syz}^{i+1}(K(\underline{z}; A)),$$

where $a \in \wedge^i A_{1 \cdots d}$, $b \in T_i^{(n,d)}$, $\Psi_i^h(a) = \bar{a}$. Then, for any $a' \in (\Psi_i^h)^{-1}(\bar{a})$, there exists $b' \in T_i^{(n,d)}$ such that $a' + b' \in \text{syz}^{i+1}(K(\underline{z}; A))$.

Proof. If

$$a = \sum_{1 \leq j_1 < \cdots < j_i \leq d} a_{j_1 \cdots j_i} e_{j_1 \cdots j_i}$$

and

$$a' = \sum_{1 \leq j_1 < \dots < j_t \leq d} a'_{j_1 \dots j_t} e_{j_1 \dots j_t},$$

then $\Psi_t^h(a) = \Psi_t^h(a')$ is equivalent to

$$a_{j_1 \dots j_t} = a'_{j_1 \dots j_t} + \sum_{t=h+1}^n \alpha_{j_1 \dots j_t} \cdot z_t, \quad 1 \leq j_1 < \dots < j_t \leq d.$$

The element we are looking for is

$$b' = \sum_{\substack{d \leq j_1 < \dots < j_t \leq h \\ j_{t-1} > d}} a_{j_1 \dots j_t} e_{j_1 \dots j_t} + \\ + \sum_{\substack{1 \leq j_1 < \dots < j_{t-1} \leq d \\ j_t > h}} \left(a_{j_1 \dots j_t} + \sum_{k=1}^d \alpha_{j_1 \dots k \dots j_t} \cdot z_k \right) e_{j_1 \dots j_t}$$

because it is a matter of computation to see that $d_t(a' + b') = 0$.

We recall the notation (cf. N. 1): $\Psi_t^h: \wedge^t A^n \rightarrow \wedge^t A_{1 \dots h}$ defined by

$$\Psi_t^h \left(\sum_{1 \leq j_1 < \dots < j_t \leq n} a_{j_1 \dots j_t} e_{j_1 \dots j_t} \right) = \sum_{1 \leq j_1 < \dots < j_t \leq h} a_{j_1 \dots j_t} f_{j_1 \dots j_t}.$$

LEMMA 3.2. *Let $\bar{a} + \bar{b} \in \text{syz}^{t+1}(K(\bar{z}; {}_h\bar{A}))$, where $\bar{a} \in \wedge^t {}_h\bar{A}_{1 \dots d}$, $\bar{b} \in T_t^{(h,d)}$, and let $a \in (\Psi_t^h)^{-1}(\bar{a})$, $b \in (\Psi_t^h)^{-1}(\bar{b})$. Then there exists $c \in T_t^{(n,h)}$, whose components with at least two indices bigger than h are zero, such that $\chi_{i-1}^h d_i(a + b + c) = 0$.*

Proof. The hypothesis $d_i(\bar{a} + \bar{b}) = 0$ implies $d_i(a + b) \in (z_{h+1}, \dots, z_n) \wedge^{i-1} A_{1 \dots h}$, that is

$$d_i(a + b) = \sum_{1 \leq j_1 < \dots < j_{i-1} \leq h} \sum_{t=h+1}^n c_{j_1 \dots j_{i-1} t} \cdot z_t e_{j_1 \dots j_{i-1}}.$$

It is easy to verify that we can choose

$$c = \sum_{\substack{1 \leq j_1 < \dots < j_{i-1} \leq h \\ t=h+1, \dots, n}} -c_{j_1 \dots j_{i-1} t} e_{j_1 \dots j_{i-1} t}.$$

According to Lemma 3.2, we can give the following

DEFINITION 3.3. Let $\bar{a} + \bar{b} \in \text{syz}^{i+1}(K(\bar{z}; {}_h\bar{A}))$, where $\bar{a} \in \bigwedge^i {}_h\bar{A}_{1\dots d}$, $\bar{b} \in T_i^{(h,d)}$, and let $a \in (\Psi_i^h)^{-1}(\bar{a})$. We set $\phi(a) = a + b'$, where $b' \in T_i^{(n,d)}$ is chosen such that:

(α) $\chi_{i-1}^h d_i(a + b') = 0$;

(β) $\Psi_i^h(b') = \bar{b}$;

(γ) the components of $\phi(a)$ with at least two indices bigger than h are zero.

PROPOSITION 3.4. *The following conditions are equivalent:*

(i) (W.L.C.) $_{h,i}$.

(ii) Let $\bar{a} + \bar{b} \in \text{syz}^{i+1}(K(\bar{z}; {}_h\bar{A}))$, $\bar{a} \in \bigwedge^i {}_h\bar{A}_{1\dots d}$, $\bar{b} \in T_i^{(n,d)}$, $a \in (\Psi_i^h)^{-1}(\bar{a})$. Then there exists $\lambda \in T_i^{(n,d)}$ such that

(1) $\pi_{i-1} d_i(\phi(a)) = \pi_{i-1} d_i(\lambda)$;

(2) $\chi_{i-1}^h d_i(\lambda) = 0$.

Proof. (i) \Rightarrow (ii). Let $a + b$ be a weak lifting of $\bar{a} + \bar{b}$ (cf. Proposition 3.1) and $\phi(a) = a + b'$; then $\lambda = b' - b$ is the required element. In fact $\lambda \in T_i^{(n,d)}$, as b and b' belong to that module; moreover from $d_i(a + b) = 0$ we get

$$\begin{aligned} \pi_{i-1} d_i(\phi(a)) &= \pi_{i-1} d_i(a) + \pi_{i-1} d_i(b') \\ &= -\pi_{i-1} d_i(b) + \pi_{i-1} d_i(b') = \pi_{i-1} d_i(b - b'); \end{aligned}$$

so condition (1) is verified. Now, since we have $\chi_{i-1}^h d_i(a + b') = 0$ and $\chi_{i-1}^h d_i(a + b) = 0$, then $\chi_{i-1}^h d_i(b' - b) = 0$ and (2) follows.

(ii) \Rightarrow (i). From $\phi(a) = a + b'$, we obtain a weak lifting $a + b$ of $\bar{a} + \bar{b}$ by choosing $b = b' - \lambda$. In fact $b' - \lambda \in T_i^{(n,d)}$ and, moreover,

$$\begin{aligned} \pi_{i-1} d_i(a + b' - \lambda) &= \pi_{i-1} d_i(\phi(a)) - \pi_{i-1} d_i(\lambda) = 0, \\ \chi_{i-1}^d d_i(a + b' - \lambda) &= \chi_{i-1}^d d_i(\phi(a)) - \chi_{i-1}^d d_i(\lambda) = 0. \end{aligned}$$

REMARK 3.5. Condition (1) of (ii) in Proposition 3.4 can be replaced by $\pi_{i-1}^h d_i(\phi(a)) = \pi_{i-1}^h d_i(\lambda)$, because of condition (2).

Now, let us look for conditions stronger than those in Proposition 3.4 which are easier to formulate and verify. First, we point out that $d_i(\phi(a))$ is a boundary with some zero components, so that it is of the form

$$\beta = \sum_{1 \leq k_1 < \dots < k_{i-2} \leq h} \beta_{k_1 \dots k_{i-1}} e_{k_1 \dots k_{i-1}},$$

where

$$I. \quad \beta_{k_1 \cdots k_{i-1}} \in I_{k_1 \cdots k_{i-2}}^h = (z_1, \dots, \check{z}_{k_1}, \dots, \check{z}_{k_{i-2}}, \dots, z_h)A,$$

$$II. \quad \sum_{t \in \{1, \dots, n\} - \{j_1, \dots, j_{i-2}\}} \beta_{j_1 \cdots t \cdots j_{i-2}} \cdot z_t = 0$$

(the β 's with more than one index bigger than h are zero). From this, taking into account only relations II, corresponding to $1 \leq j_1 < \cdots < j_{i-2} \leq h$, we easily get

COROLLARY 3.6. (W.L.C.) $_{h,i}$ is verified if $\sum_{t=h+1}^n \beta_{k_1 \cdots k_{i-2}t} \cdot z_t = 0$, with $\beta_{k_1 \cdots k_{i-2}t} \in I_{k_1 \cdots k_{i-2}}^h$, implies

$$\beta_{k_1 \cdots k_{i-2}t} = \sum_{u \in \{1, \dots, h\} - \{k_1, \dots, k_{i-2}\}} \lambda_{k_1 \cdots u \cdots k_{i-2}t} \cdot z_u$$

for $1 \leq k_1 < \cdots < k_{i-2} \leq h$, where, for every $1 \leq k_1 < \cdots < k_{i-1} \leq h$,

$$(4) \quad \sum_{v=h+1, \dots, n} \lambda_{k_1 \cdots k_{i-1}v} z_v = 0.$$

Proof. As we just observed,

$$d_i(\phi(a)) = \sum_{\substack{1 \leq k_1 < \cdots < k_{i-2} \leq h \\ k_{i-1} > h}} \beta_{k_1 \cdots k_{i-1}} e_{k_1 \cdots k_{i-1}},$$

where

$$\beta_{k_1 \cdots k_{i-1}} \in I_{k_1 \cdots k_{i-2}}^h \quad \text{and} \quad \sum_{t=h+1, \dots, n} \beta_{k_1 \cdots k_{i-2}t} \cdot z_t = 0,$$

so

$$\beta_{k_1 \cdots k_{i-1}} = \sum_{u \in \{1, \dots, h\} - \{k_1, \dots, k_{i-2}\}} \lambda_{k_1 \cdots u \cdots k_{i-1}} \cdot z_u$$

and this shows condition (1) of (ii) in Proposition 3.4 holds. Condition (2) is implied by (4) if we choose $\lambda_{k_1 \cdots k_{i-1}r} = 0$ for $r \leq h$.

Two weaker versions of Corollary 3.6 are the following:

COROLLARY 3.7. (W.L.C.) $_{h,i}$ is verified if

$$\sum_{t=h+1}^n \beta_{k_1 \cdots k_{i-2}t} \cdot z_t = 0,$$

$$\beta_{k_1 \cdots k_{i-2}t} \in I_{k_1 \cdots k_{i-2}}^h \quad (1 \leq k_1 < \cdots < k_{i-2} \leq h)$$

implies

$$\beta_{k_1 \cdots k_{i-2} t} = \sum_{u=h+1}^n \beta_{k_1 \cdots k_{i-2} u t} \cdot z_u + \varepsilon_{k_1 \cdots k_{i-2} t}$$

for some $\varepsilon_{k_1 \cdots k_{i-2} t} \in I_{k_1 \cdots k_{i-2}}^h \cdot (0 : z_t)$.

Proof. Just check that (4) holds.

According to the following definition (Fiorentini [F]), if $N \subseteq M$ are A -modules, x_1, \dots, x_n is said to be a relative regular M -sequence with respect to N if

$$((x_1, \dots, x_i)N : x_{i+1}) \cap N = (x_1, \dots, x_i)M, \quad i = 0, \dots, n - 1,$$

we have

COROLLARY 3.8. $(W.L.C.)_{h,i}$ is verified if z_{h+1}, \dots, z_n is a relative A -sequence with respect to $I_{k_1 \cdots k_{i-2}}^h$ for every $1 \leq k_1 < \cdots < k_{i-2} \leq h$. In particular, $(W.L.C.)_{h,i}$ is verified if z_{h+1}, \dots, z_n is a relative regular sequence (cf. [F]) or a d -sequence (see [H]).

Proof. Just apply Corollary 3.7, since from the hypothesis we get the required implication for $\varepsilon = 0$.

Finally we have the result quoted in N. 1.

COROLLARY 3.9. If z_{d+1}, \dots, z_n is a regular sequence, the following are equivalent:

- (i) z_1, \dots, z_n is a (d, i) -sequence;
- (ii) $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - i + 1$ in ${}_d \bar{A}$.

REMARK 3.10. The interesting particular case $h = n - 1$ gives rises to an easy way of expressing the conditions of both Corollaries 3.6 and 3.7 (which in such a case coincide); precisely, with the above notation,

$$(0 : z_n) \cap I_{k_1 \cdots k_{i-2}}^{n-1} = (0 : z_n) \cdot I_{k_1 \cdots k_{i-2}}^{n-1}.$$

Moreover the condition of Corollary 3.8 simply becomes

$$(0 : z_n) \cap I_{k_1 \cdots k_{i-2}}^{n-1} = 0.$$

As an easy application of the previous remark, we have

PROPOSITION 3.11. Let a, b, c be a $(2, 2)$ -sequence of a local ring (A, \mathfrak{m}) , and suppose $\text{Tor}_1(A/(a, b), A/\text{ann}(c)) = 0$. Then $\text{depth}(\bar{a}, \bar{b}) \geq 1$ in $\bar{A} = A/(c)$.

Proof. The Tor condition says that $(a, b) \cap \text{ann}(c) = (a, b) \cdot \text{ann}(c)$. From the previous remark this implies (\bar{a}, \bar{b}) is a $(2, 2)$ -sequence, and this just means $\text{depth}(\bar{a}, \bar{b}) \geq 1$.

Now we are able to give an application which is a sort of generalization of what we proved in Proposition 2.10.

PROPOSITION 3.12. *Let $I = (z_1, \dots, z_m)$ be an ideal of A generated by a relative regular sequence (in particular a d -sequence or a regular sequence), with $\dim(A/I) \leq d$. If, for some $i > 0$, there exist $d + i - 1$ elements x_1, \dots, x_{d+i-1} such that $x_1, \dots, x_{d+i-1}, z_1, \dots, z_m$ is a $(d + i - 1, i)$ -sequence, then A/I is Cohen-Macaulay. Moreover, if (A, \mathfrak{m}) is local and $x_1, \dots, x_{d+i-1}, z_1, \dots, z_m$ is a system of generators of \mathfrak{m} , then A/I is regular.*

Proof. By Corollary 3.8 (or Corollary 3.9) and Proposition 2.2, we have, in A/I , $\text{depth}(\bar{x}_1, \dots, \bar{x}_{d+i-1}) \geq d + i - 1 - i + 1 = d$, so, for every $\mathfrak{p} \in \text{Max}(A/I)$, $\text{depth}(A/I)_{\mathfrak{p}} \geq d$ and the conclusion follows.

We can realize how much the condition z_{d+1}, \dots, z_n is a regular sequence is stronger than $(\text{W.L.C.})_{d,i}$ by looking at the next proposition, which points out a strict relation between (d, i) -sequences, the regularity of the tails of sequences and the vanishing of the Koszul homology.

PROPOSITION 3.13. *For a sequence z_1, \dots, z_n (in $\text{rad } A$) we have:*

(1) *If z_1, \dots, z_n is a (d, i) -sequence and z_{d+1}, \dots, z_n is a regular sequence, then $H_i(K(\underline{z}; A)) = 0$.*

(2) *if $H_i(K(\underline{z}; A)) = 0$, then there exist $x_1, \dots, x_{n-i+1} \in (z_1, \dots, z_n)$ such that $z_1, \dots, z_{i-1}, x_1, \dots, x_{n-i+1}$ is a (d, i) -sequence and $x_{d-i+2}, \dots, x_{n-i+1}$ is a regular sequence.*

Proof. (1) From Corollary 3.9, the hypothesis implies $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - i + 1$; as z_{d+1}, \dots, z_n is a regular sequence, we get $\text{depth}(z_1, \dots, z_n) \geq n - i + 1$, which implies $H_i(K(\underline{z}; A)) = 0$.

(2) The hypothesis is equivalent to $\text{depth}(z_1, \dots, z_n) \geq n - i + 1$, so we can take a regular sequence x_1, \dots, x_{n-i+1} in (z_1, \dots, z_n) . So we have

$$\text{depth}(z_1, \dots, z_{i-1}, x_1, \dots, x_{n-i+1}) \geq n - i + 1,$$

with $x_{d-i+2}, \dots, x_{n-i+1}$ a regular sequence. This implies

$$\text{depth}(\bar{z}_1, \dots, \bar{z}_{i-1}, \bar{x}_1, \dots, \bar{x}_{d-i+1}) \geq d - i + 1,$$

so, by Corollary 3.9 and the regularity of $x_{d-i+2}, \dots, x_{n-i+1}$, we have $z_1, \dots, z_{i-1}, x_1, \dots, x_{n-i+1}$ is a (d, i) -sequence.

Let us remark that, for $i = 1$, part (1) of the previous proposition becomes the well-known result

$$H_1(K(\bar{z}; {}_d\bar{A})) = 0 \text{ and } z_{d+1}, \dots, z_n \text{ a regular sequence} \\ \Rightarrow H_1(K(z; A)) = 0.$$

If (W.L.C.) $_{d,i}$ holds for every i , the smallest i for which z_1, \dots, z_n is a (d, i) -sequence says exactly that $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) = d - i + 1$ in ${}_d\bar{A}$ (cf. Corollary 2.4); if, moreover, z_{d+1}, \dots, z_n is a regular sequence, such an i says that $\text{depth}(z_1, \dots, z_n) = n - i + 1$ (cf. Proposition 3.13). If we do not assume (W.L.C.) $_{d,i}$ holds, we can only say that $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) = s$ implies z_1, \dots, z_n is a (d, i) -sequence for $i \geq d + 1 - s$; however it may be a (d, i) -sequence for smaller i 's, as we saw in Example 1.5, where $\text{depth}(\bar{x}, \bar{y}) = 0$, and x, y, z is a $(2, 2)$ -sequence.

Let us give more examples of (d, i) -sequences z_1, \dots, z_n which do not pass to the quotient ${}_d\bar{A}$, that is where (W.L.C.) $_{d,i}$ does not hold.

EXAMPLE 3.14. We consider again the local ring of Example 1.5:

$$A = k[[X, Y, Z]] / (X^2 - Z^2, XY, XZ) = k[[x, y, z]].$$

The element z is not a d -sequence (according to [H]): in fact $(0 : z) = (x)$ and $(0 : z^2) = (x, z)$. This remark agrees with what we proved in Example 1.5, that is, (x, y, z) is a $(2, 2)$ -sequence and (\bar{x}, \bar{y}) is not a $(2, 2)$ -sequence in $A/(z)$ (cf. Corollary 3.8).

As in $A/(z)$ every ideal can be generated by two elements. Let us examine all the sequences (α, β, z) in A . They cannot be $(2, 1)$ -sequences, because, in that case, they should pass to $A/(z)$ (cf. Proposition 2.5), but $\text{depth } A/(z) = 0$. So, the only meaningful question is whether or not they are $(2, 2)$ -sequences. It is a matter of computation to show that they are essentially of the following three types:

$$s_1 = (x + y^m \cdot u, y^n, z), \quad 1 \leq m < n, u \text{ invertible in } A; \\ s_2 = (y^m, y^n, z), \quad m, n \geq 1; \\ s_3 = (x, y^m, z), \quad m \geq 1;$$

now s_1 and s_2 are not $(2, 2)$ -sequences (for instance the cycle

$$xe_{12} - u^{-1}y^{n-m}ze_{13} + ze_{23} \notin \wedge^3 A^3 + T_2^{(3,2)}$$

in $K(x + y^m u, y^n, z; A)$ and, respectively, the cycle

$$x(e_{12} + e_{13} + e_{23}) \notin \wedge^3 A^3 + T_2^{(3,2)}$$

in $K(y^m, y^n, z; A)$, and s_3 is a $(2, 2)$ -sequence which does not pass to the quotient $A/(z)$.

By using Proposition 1.3, the previous example gives rise to the following one: in the ring $B = k[[X, Y, Z]]$, the sequences

$$(X, Y^h, Z, X^2 - Z^2, XY, XZ), \quad h \geq 1,$$

are $(2, 2)$ -sequences which $\text{mod}(X^2 - Z^2, XY, XZ)$ remain $(2, 2)$ -sequences (note that $X^2 - Z^2, XY, XZ$ is a d -sequence); however they do not give rise to $(2, 2)$ -sequences in $B/(Z, X^2 - Y^2, XY, XZ) = A/(z)$, so, in particular, $(Z, X^2 - Z^2, XY, XZ)$ is not a d -sequence in B .

EXAMPLE 3.15. Let $B = k[[X, Y]]$, $n = (X, Y)$, $A = B/n^3 = k[[x, y]]$. Then:

(a) $(x, y; x^2, xy, y^2)$ is a $(2, 2)$ -sequence in A .

(b) $(\bar{x}, \bar{y}; \bar{x}^2, \bar{x}\bar{y})$ is a $(2, 2)$ -sequence in $A/(y^2)$, though y^2 is not a d -sequence. This fact shows that *the condition to be generated by a d -sequence is strictly stronger than $(W.L.C.)_{d,t}$* (cf. Corollary 3.8).

(c) (\bar{x}, \bar{y}) is not a $(2, 2)$ -sequence in $A/(x^2, xy, y^2)$.

Let us prove (a). It is equivalent to show that

$$(5) \quad \begin{cases} a_{12}y + a_{13}x^2 + a_{14}xy + a_{15}y^2 = 0, \\ a_{12}x - a_{23}x^2 - a_{24}xy - a_{25}y^2 = 0, \\ a_{13}x + a_{23}y - a_{34}xy - a_{35}y^2 = 0, \\ a_{14}x + a_{24}y + a_{34}x^2 - a_{45}y^2 = 0, \\ a_{15}x + a_{25}y + a_{35}x^2 + a_{45}xy = 0 \end{cases}$$

implies $a_{12} \in (x^2, xy, y^2)$. Now, working $\text{mod}(x^2, xy, y^2)$ we can see that $a_{1i}, a_{2i}, i > 2$, are not invertible, so (5) becomes $a_{12}y = a_{12}x = 0$. From this the conclusion follows easily.

The proof of (b) is similar; (c) is trivial, as $\dim A/(x^2, xy, y^2) = 0$.

N. 4. In this last section we just want to give a new version of the results we obtained in the previous ones. Here we use essentially the idea of looking at the syzygies of the Koszul complex as particular systems of linear equations, so that conditions on syzygies can be seen as conditions on the solutions of these systems. For a better understanding, we introduce some general notation and definitions.

Let (F) be a system of linear equations with coefficients in a ring A and with indeterminates $\underline{X} = \{X_1, \dots, X_n\}$; let $g: A \rightarrow B$ be any ring homomorphism and denote by $(g(F))$ the system we get from (F) when

we apply g to the coefficients of (F) . So $(g(F))$ is a system of linear equations with coefficients in B and with indeterminates $\{X_{i_1}, \dots, X_{i_r}\}$ where $\{X_{i_1}, \dots, X_{i_r}\} \subseteq \{X_1, \dots, X_n\}$ (we delete all indeterminates with coefficient zero).

DEFINITION 4.1. (F) is said to be admissible with respect to $g: A \rightarrow B$ (or $(g(F))$) if for every solution $\beta = \{\beta_{i_1}, \dots, \beta_{i_r}\}$ of $(g(F))$, $\beta_{i_j} \in B$, $j = 1, \dots, r$, there exists a solution $\alpha = \{\alpha_1, \dots, \alpha_n\}$, $\alpha_k \in A$, $k = 1, \dots, n$, of (F) such that $g(\alpha_{i_j}) = \beta_{i_j}$, $j = 1, \dots, r$.

An easy consequence of the previous definition is

PROPOSITION 4.2. Let $g: A \rightarrow B$ a surjective morphism and $\text{Ker}(g) = (u_1, \dots, u_r)$; consider a system of the form

$$(F) \quad \sum_{j=1}^{n_i} a_j^{(i)} X_j^{(i)} + \sum_{t=1}^r u_t Y_t^{(i)} = 0, \quad i = 1, \dots, h.$$

Then (F) is admissible with respect to g .

Proof. It is almost trivial since every solution β of $(g(F))$,

$$\sum_{j=1}^{n_i} g(a_j^{(i)}) X_j^{(i)} = 0, \quad i = 1, \dots, h,$$

can be lifted to $\alpha = \{\alpha_j^{(i)}\}_{j=1, \dots, n_i; i=1, \dots, h}$ in A , so $\sum_{j=1}^{n_i} a_j^{(i)} \alpha_j^{(i)} \in \text{Ker}(g)$. Then we can find elements in A , $\gamma = \{\gamma_t^{(i)}\}_{t=1, \dots, r; i=1, \dots, h}$, with

$$\sum_{j=1}^{n_i} a_j^{(i)} \alpha_j^{(i)} + \sum_{t=1}^r \gamma_t^{(i)} u_t = 0.$$

Now (α, γ) is a solution of (F) and $g(\alpha) = \beta$.

We point out that Proposition 4.2 can be easily generalized by letting g be surjective only on the solutions of $(g(F))$. We now introduce a similar terminology to deal with a subsystem of a system of linear equations.

DEFINITION 4.3. Let (F) be a system of linear equations with coefficients in A and indeterminates \underline{X} , \underline{Y} , where $\underline{X} = \{X_1, \dots, X_n\}$ and $\underline{Y} = \{Y_1, \dots, Y_m\}$, and let (F') be a subsystem of (F) with indeterminates \underline{X} . We say that (F) is admissible with respect to (F') if, for every solution $\underline{X} = \alpha$ of (F') , there exists a solution of (F) of the form $\underline{X} = \alpha$, $\underline{Y} = \beta$.

EXAMPLE. In \mathbf{Z} it is easy to check that

$$(F) \quad \begin{cases} x + y + z = 0, \\ 3x - y + z = 0, \\ 3x + 4y + z + 5t = 0 \end{cases}$$

is admissible with respect to

$$(F') \quad \begin{cases} x + y + z = 0, \\ 3x - y + z = 0. \end{cases}$$

Now let us return to our subject and let $z = \{z_1, \dots, z_n\}$ be elements of A with $(z_1, \dots, z_n) \subseteq \text{rad}(A)$.

For a fixed i consider the system

$$(S) \quad \sum_{\substack{t=1 \\ t \neq j_1, \dots, j_{i-1}}}^n (-1)^{s_t} z_t X_{j_1 \dots t \dots j_{i-1}} = 0, \quad 1 \leq j_1 < \dots < j_{i-1} \leq n,$$

with $\binom{n}{i-1}$ linear equations and $\binom{n}{i}$ indeterminates, and $s_t =$ number of j 's preceding t . There is a natural bijection between the set of solutions of (S) and $\text{syz}^{i+1}(K(\underline{z}; A))$, so the definition of (d, i) -sequence can be restated as follows:

Every solution of (S) must have the form

$$\langle d, i \rangle \quad X_{j_1 \dots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \dots j_i}}^n \alpha_{j_1 \dots j_i s} z_s \quad \text{for every } 1 \leq j_1 < \dots < j_i \leq d,$$

with $\alpha_{j_1 \dots j_i s} \in A$ and the usual convention on the α 's.

We remark that the condition $\langle d, i \rangle$ concerns only some components of every solution of (S) .

Now let us fix an integer $h, d \leq h \leq n$, and denote by

$$(S_h) \quad \sum_{\substack{t=1 \\ t \neq j_1 \dots j_{i-1}}}^n (-1)^{s_t} z_t X_{j_1 \dots t \dots j_{i-1}} = 0, \quad 1 \leq j_1 < \dots < j_{i-1} \leq h,$$

the subsystem of (S) corresponding to the indices $1, 2, \dots, h$.

Proposition 4.2 implies (S_h) is admissible with respect to the natural map $\phi_h: A \rightarrow_h \bar{A}$, i.e. with respect to the system

$$(\bar{S}_h) \quad \sum_{\substack{t=1 \\ t \neq j_1 \dots j_{i-1}}}^h (-1)^{s_t} \bar{z}_t X_{j_1 \dots t \dots j_{i-1}} = 0, \quad 1 \leq j_1 < \dots < j_{i-1} \leq h,$$

as we already knew by Proposition 3.2.

Clearly every solution of (S) gives a solution of (S_h) and then a solution of (\bar{S}_h) ; so, if there is an integer h , $d \leq h \leq n$, such that the solution of (\bar{S}_h) has the form $\langle d, i \rangle$ in ${}_h\bar{A}$, i.e.

$$X_{j_1 \cdots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \cdots j_i}}^h \bar{\alpha}_{j_1 \cdots j_i s} \bar{z}_s, \quad 1 \leq j_1 < \cdots < j_i \leq d,$$

then every solution of (S) in A will be in the form $\langle d, i \rangle$, i.e.

$$X_{j_1 \cdots j_i} = \sum_{\substack{s=1 \\ s \neq j_1 \cdots j_i}}^n \alpha_{j_1 \cdots j_i s} z_s, \quad 1 \leq j_1 < \cdots < j_i \leq d.$$

This simply says that if there exists h such that $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$, then z_1, \dots, z_n is a (d, i) -sequence in A , and that is Theorem 1.3.

When $n = d$, condition $\langle d, i \rangle$ concerns the whole solution, so in this case to say that every solution of (S) has the form $\langle n, i \rangle$ is equivalent to $\text{depth}(z_1, \dots, z_n) \geq n - i + 1$. The Corollary 1.4 becomes: if the solutions of (\bar{S}_d) have the form $\langle d, i \rangle$, then the same is true for the solutions of (S) .

Now we want to study how a property $\langle \mathcal{P} \rangle$, in particular $\langle d, i \rangle$, passes from the solutions of a system (F) to the solutions of a subsystem (F') . We have this first easy result.

LEMMA 4.4. *If (F) is a system of linear equations with two sets of indeterminates \underline{X} , \underline{Y} , and if the solutions of (F) satisfy a property $\langle \mathcal{P} \rangle$ related to the part concerning the \underline{X} indeterminates, then every admissible subsystem (F') , with indeterminates \underline{X} , has all the solutions satisfying $\langle \mathcal{P} \rangle$.*

Proof. It is trivial; just take a solution α for (F') and (α, β) the solution of (F) arising from the admissibility; then since (α, β) has $\langle \mathcal{P} \rangle$, which is related to the \underline{X} 's, α has $\langle \mathcal{P} \rangle$.

COROLLARY 4.5. *If z_1, \dots, z_n is a (d, i) -sequence and our system (S) is admissible with respect to (S_d) , then $\text{depth}(\bar{z}_1, \dots, \bar{z}_d) \geq d - 1 + i$.*

REMARK 4.6. For $i = 1$, $(S) \equiv (S_h)$ for every $h \leq n$, so Corollary 4.5 gives again: $z = \{z_1, \dots, z_n\}$ is a $(d, 1)$ -sequence implies $\bar{z}_1, \dots, \bar{z}_d$ is a regular sequence in ${}_d\bar{A}$ (cf. Proposition 2.6).

Corollary 4.5 can be easily generalized to

COROLLARY 4.7. *If the composition of system maps*

$$(S) \xrightarrow{\mu} (S_h) \xrightarrow{\nu} (\bar{S}_h),$$

where μ means to pass to a subsystem and ν is the induced map of the natural one $A \rightarrow_h \bar{A}$, is admissible (i.e. every solution of (S_h) in ${}_h\bar{A}$ can be lifted to a solution of (S) in A), then the condition $\langle d, i \rangle$ descends from A to ${}_h\bar{A}$.

We observe that the hypothesis of Corollary 4.7 is really weaker than in Corollary 4.5 (also if we use there h instead of d), since compositions of admissible maps (of systems) are admissible, but conversely if the composition is admissible (and the second map is too) the first map is not necessarily admissible. In fact, the admissibility of (S) with respect to (\bar{S}_h) simply means the surjectivity of $\psi_i^h: \text{syz}^{i+1}(K(\underline{z}; A)) \rightarrow \text{syz}^{i+1}(K(\underline{z}; {}_h\bar{A}))$, while the admissibility of (S) with respect to (S_h) means the strongest relation:

$$(\psi_i^h)^{-1}(\text{syz}^{i+1}(K(\underline{z}, {}_h\bar{A}))) = \text{syz}^{i+1}(K(z; A)).$$

Nevertheless the hypothesis of Corollary 4.7 is still not necessary to pass the $\langle d, i \rangle$ -condition from A to ${}_h\bar{A}$.

From now on, in our system (S) we denote by \underline{X} the set of indeterminates $\{X_{j_1 \dots j_i}\}_{1 \leq j_1 < \dots < j_i \leq d}$ and by \underline{Y} all the remaining indeterminates, i.e. $\{X_{j_1 \dots j_i}\}_{1 \leq j_1 < \dots < j_i \leq n; j_i > d}$; h is always an integer such that $d \leq h \leq n$.

We need a weak version of admissibility.

DEFINITION 4.8. Let (F) be a system of linear equations with coefficients in A and with two sets of indeterminates $\underline{X}, \underline{Y}$; let $g: A \rightarrow B$ be a ring homomorphism and (F') the system induced from (F) by g with indeterminates \underline{X} and \underline{Y}' , where $\underline{Y}' \subseteq \underline{Y}$. We say that (F) and (F') are admissible with respect to \underline{X} (or weakly admissible when there is no chance of confusion) if for every solution (\bar{a}, \bar{b}) of (F') in B , with $\underline{X} = \bar{a}$, $\underline{Y}' = \bar{b}$, there is a solution (a, c) of (F) in A , with $\underline{X} = a$, $\underline{Y} = c$, such that $g(a) = \bar{a}$ (more precisely, putting $a = \{a_1, \dots, a_i\}$ and $\bar{a} = \{\bar{a}_1, \dots, \bar{a}_i\}$, $g(a_i) = \bar{a}_i$).

EXAMPLE. Take $\mathbf{Z} \xrightarrow{\text{nat}} \mathbf{Z}/6\mathbf{Z}$ and

$$(F) \quad \begin{cases} x + y + 2z + 6t = 0, \\ 3x - y + 2z + 6t = 0, \\ 6x + 6y + 6z + 24t = 0, \end{cases}$$

so

$$(F') \quad \begin{cases} x + y + 2z = 0, \\ 3x - y + 2z = 0 \end{cases} \quad \text{in } \mathbf{Z}/6\mathbf{Z}.$$

As is easy to see, they are not admissible (for instance, we cannot lift the solution of (F') $x = y = \bar{0}, z = \bar{3}$), but with respect to the set of indeterminates (x, y) they are; namely, the only solutions of (F') in $\mathbf{Z}/6\mathbf{Z}$ have the form $(\bar{\lambda}, \bar{\lambda}, -\bar{\lambda})$ or $(\bar{\lambda}, \bar{\lambda}, -\bar{\lambda} + \bar{3})$, with $\bar{\lambda} \in \mathbf{Z}/6\mathbf{Z}$; so they can be lifted to a solution of (F) in \mathbf{Z} , for instance $(\lambda, \lambda, 2\lambda, -\lambda)$, for some $\lambda \in \mathbf{Z}$ whose image in $\mathbf{Z}/6\mathbf{Z}$ is $\bar{\lambda}$.

Of course, when $\underline{Y}' = \emptyset$, admissibility coincides with weak admissibility; in particular, this happens for (S) and (\bar{S}_d) .

Let us go back to our system (S) ; now Proposition 3.1 can be restated as follows.

LEMMA 4.9. *If $(S) \xrightarrow{\mu} (\bar{S}_h)$ (the usual composition $(S) \rightarrow (S_h) \rightarrow (\bar{S}_h)$) is weakly admissible and (\bar{a}, \bar{b}) is a solution of (\bar{S}_h) in ${}_h\bar{A}$, for every $a' \in \mu^{-1}(\bar{a})$, we can find c' , set of elements in A , such that (a', c') is a solution of (S) .*

Finally, the new version of Proposition 2.2 is

THEOREM 4.10. *For z_1, \dots, z_n in A , with $(z_1, \dots, z_n) \subseteq \text{rad}(A)$, the following are equivalent:*

- (i) z_1, \dots, z_n is a (d, i) -sequence in A and $(S) \rightarrow (\bar{S}_h)$ is weakly admissible.
- (ii) $\bar{z}_1, \dots, \bar{z}_h$ is a (d, i) -sequence in ${}_h\bar{A}$.

REMARK 4.11. The conditions in Corollaries 3.6–3.9 are all sufficient in order to have $(S) \rightarrow (\bar{S}_d)$ admissible.

Just to show how one can deal with these problems in terms of linear systems, let us rewrite the proof of Proposition 3.11.

The admissibility of the systems

$$(S_2) \quad \begin{cases} bx + xy = 0, \\ -ax + cz = 0, \end{cases} \quad (\bar{S}_2) \quad \begin{cases} \bar{b}x = 0, \\ \bar{a}x = 0, \end{cases}$$

says that, for some lifting α of a solution of $\bar{\alpha}$ of (\bar{S}_2) , there exists a solution (α, β, γ) of (S_2) ; since $\beta a + \gamma b \in (a, b) \cap (0 : c)$, for the Tor-condition, $\beta a + \gamma b \in (a, b) \cdot (0 : c)$, so we have elements $\beta', \gamma' \in (0 : c)$

such that $\beta a + \gamma b = \beta' a + \gamma' b$. Now $(\alpha, \beta - \beta', \gamma - \gamma')$ is a solution of

$$(S) \quad \begin{cases} bx + cy = 0, \\ -ax + cz = 0, \\ ay + bz = 0, \end{cases}$$

that is, (S) and (\bar{S}_2) are admissible. The $(2, 2)$ -condition implies $\alpha = \lambda c$, for some $\lambda \in A$, i.e. $\bar{\alpha} = 0$.

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