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Let X be a semi-locally contractible metrizable space. We show that X is locally equi-connected (LEC) if and only if X has a local mixer introduced by van Mill and van de Vel [$\mathbf{MV}_{1,2}$].

Throughout this paper, all spaces are metrizable and maps are continuous. Let X be a space. We will use the same symbol ΔX to denote the diagonals of X^2 and X^3, that is,

$$\Delta X = \{(x, x): x \in X\} \quad \text{or} \quad = \{(x, x, x): x \in X\},$$

and we will let

$$\begin{aligned}\Delta^* X &= \{(x, y, z) \in X^3: x = y \text{ or } y = z \text{ or } z = x\} \\ &= \bigcup_{x \in X} (X \times \{x\} \times \{x\} \cup \{x\} \times X \times \{x\} \cup \{x\} \times \{x\} \times X)\end{aligned}$$

A *local mixer* for X is a map $\mu: U \to X$ of a neighborhood U of $\Delta^* X$ in X^3 to X which satisfies the following condition:

(*) if $((x_n, y_n, z_n))_{n=1}^{\infty}$ is a sequence of points in X^3 such that the sequences $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ both converge to $a \in X$, then the sequences $(\mu(x_n, y_n, z_n))_{n=m}^{\infty}$, $(\mu(x_n, z_n, y_n))_{n=m}^{\infty}$ and $(\mu(z_n, x_n, y_n))_{n=m}^{\infty}$ converge to a for some m;

or, equivalently, (see [$\mathbf{MV}_2$, Lemma 2.3]):

(#) for each $x \in X$ and for each neighborhood V of x, there exists a neighborhood W of x such that
$$\begin{aligned}E(W) = (X \times W \times W) &\cup (W \times X \times W) \\ &\cup (W \times W \times X) \subset \mu^{-1}(V),\end{aligned}$$
that is, $E(W) \subset U = \operatorname{dom} \mu$ and $\mu(E(W)) \subset V$.

When $U = X^3 = \operatorname{dom} \mu$, we call μ a *mixer* for X. If X is compact, then (*) (or (#)) is equivalent to the condition that

$$\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x$$

for all $x, y \in X$. The concept of a (local) mixer for a (compact) metric space was introduced by van Mill and van de Vel [**MV**$_{1,2}$].

We say that X is *locally equi-connected* (*LEC*) provided there is a map $\lambda: U \times [0, 1] \to X$, where U is a neighborhood of ΔX in X^2, such that

$$\lambda(x, y, 0) = x, \quad \lambda(x, y, 1) = y \qquad \text{for all } (x, y) \in U,$$
$$\lambda(x, x, t) = x \quad \text{for all } x \in X, t \in [0, 1];$$

the map λ is called a (*local*) *equi-connecting function*. When $U = X^2$, we say that X is *equi-connected* (*EC*). These concepts were introduced by Fox [**F**] (cf. [**S**], [**H**], [**D**] and [**C**]).

A space X is said to be *semi-locally contractible* if each point of X has a neighborhood which is contractible in X; a space X is said to be *semi-locally path-connected* if each point of X has a neighborhood whose any two points can be joined by a path in X. In [**MV**$_{1,2}$], van Mill and van de Vel showed that

(i) each semi-locally path-connected space admitting a (local) mixer is (LC^∞) C^∞;

(ii) each A(N)R has a (local) mixer; and

(iii) each contractible space admitting a mixer is EC and each semi-locally contractible *separable* space with a local mixer is LEC.

In this paper, we will show that each (L)EC space has a (local) mixer (Theorem I), and each semi-locally contractible space admitting a local mixer is LEC (Theorem II). These results generalize (ii) and (iii). From these results we obtain the following characterization of (local) equi-connectedness.

THEOREM. *A metrizable space is* (*L*)*EC if and only if it is* (*semi-locally*) *contractible and has a* (*local*) *mixer.*

First, we will prove the following:

THEOREM I. *If X is LEC, then X has a local mixer. If X is EC, then X has a mixer.*

Proof. Let U be a neighborhood of ΔX in X^2 and $\lambda: U \times [0, 1] \to X$ an equi-connecting function. For each $a \in X$, let U'_a be a neighborhood of a in X so that $U'^2_a \subset U$. There is an open neighborhood U''_a of a such that $U''_a \subset U'_a$ and $\lambda(U''^2_a \times [0, 1]) \subset U'_a$ [**D**, Lemma 2.3]. By [**BP**, Ch. II, Theorem 4.1], there exists a metric d on X compatible with the topology of

X and such that $\{\{x \in X: d(x, a) \leq 1\}: a \in X\}$ refines $\{U''_a: a \in X\}$. We define a metric d^* on X^3 by

$$d^*((x, y, z), (x', y', z')) = \max\{d(x, x'), d(y, y'), d(z, z')\}.$$

Then d^* induces the product topology of X^3.

For each $n \in \mathbf{N}$, define

$$V_0(n) = \{(x, y, z) \in X^3: d^*((x, y, z), \Delta X) \leq 1/n\}.$$

Observe that for each $(x, y, z) \in V_0(1)$ and for each s, $t \in [0, 1]$, $\lambda(\lambda(x, y, s), z, t)$ is well defined. Put

$$X_1 = \bigcup_{a \in X} X \times \{a\} \times \{a\},$$
$$X_2 = \bigcup_{a \in X} \{a\} \times X \times \{a\},$$
$$X_3 = \bigcup_{a \in X} \{a\} \times \{a\} \times X,$$

and for each $n \in \mathbf{N}$ and for $i = 1, 2, 3$, define

$$V_i(n) = \{(x, y, z) \in X^3: d^*((x, y, z), X_i) \leq 1/4n\}.$$

Then we have a neighborhood

$$V = V_0(1) \cup V_1(1) \cup V_2(1) \cup V_3(1)$$

of $\Delta^* X = X_1 \cup X_2 \cup X_3$. For $i = 1, 2, 3$, put

$$Y_i = \bigcup_{n \in \mathbf{N}} (V_i(n) \backslash \operatorname{int} V_0(n)).$$

Since $V_i(n) \backslash \operatorname{int} V_0(n-1) \subset V_i(n-1) \backslash \operatorname{int} V_0(n-1)$ for each $n > 1$, it follows that

$$Y_i = \bigcup_{n > 1} (V_i(n) \cap (V_0(n-1) \backslash \operatorname{int} V_0(n))) \cup (V_i(1) \backslash \operatorname{int} V_0(1)),$$

that is, Y_i is a union of closed sets which is locally finite in $V \backslash \Delta X$. Thus Y_i is a closed neighborhood of $X_i \backslash \Delta X$ in $V \backslash \Delta X$. Moreover, Y_1, Y_2 and Y_3 are mutually disjoint. Indeed, if not, then we can assume without loss of generality that there exists a point

$$(x, y, z) \in (V_1(n) \backslash \operatorname{int} V_0(n)) \cap (V_2(m) \backslash \operatorname{int} V_0(m))$$

for some $n \leq m$. Then

$$d^*((x, y, z), (b, a, a)) \leq 1/4n$$
$$d^*((x, y, z), (a', b', a')) \leq 1/4m \leq 1/4n$$

for some $a, a', b, b' \in X$. Since

$$d(x, a) \leq d(x, a') + d(a', z) + d(z, a) \leq 3/4n < 1/n,$$

we have $d^*((x, y, z), (a, a, a)) < 1/n$; hence $(x, y, z) \in \operatorname{int} V_0(n)$. This is a contradiction.

Let $f, g\colon V \setminus \Delta X \to [0, 1]$ be Urysohn functions such that

$$f(Y_2 \cup Y_3) = 0, \quad f(Y_1) = 1; \quad g(Y_3) = 0, \quad g(Y_1 \cup Y_2) = 1.$$

We define $\mu\colon V \to X$ by

$$\mu(x, y, z) = \begin{cases} \lambda(\lambda(x, y, f(x, y, z)), z, g(x, y, z)) & \text{if } (x, y, z) \in V_0(1) \setminus \Delta X, \\ z & \text{if } (x, y, z) \in Y_1 \cup Y_2, \\ x & \text{if } (x, y, z) \in Y_3 \cup \Delta X. \end{cases}$$

Clearly, μ is well defined and continuous at each point of $V \setminus \Delta X$. We will show the condition (#), which implies μ is continuous at each point of ΔX and μ is a local mixer for X.

Let $a \in X$ and W be a neighborhood of a. By [**D**, Lemma 2.3] there exists a neighborhood W'' of a such that $W'' \subset \{x \in X\colon d(x, a) \leq 1\}$ and

$$\lambda(\lambda(W''^2 \times [0, 1]) \times W'' \times [0, 1]) \subset W.$$

Note that $W''^3 \subset V$ and $\mu(W''^3) \subset W$. Choose $n > 1$ so that

$$\{x \in X\colon d(x, a) \leq 1/n + 1/4n\} \subset W''$$

and put

$$W' = \{x \in X\colon d(x, a) \leq 1/4n\}.$$

Then it follows that

$$\begin{aligned} E(W') &= (X \times W' \times W') \cup (W' \times X \times W') \cup (W' \times W' \times X) \\ &\subset V_1(n) \cup V_2(n) \cup V_3(n) \subset V. \end{aligned}$$

Observe that $E(W') \cap V_0(n) \subset W''^3$. It follows that $\mu(E(W') \setminus V_0(n)) \subset W$. Since $(E(W') \setminus V_0(n)) \cap V_i(n) \subset Y_i$ for $i = 1, 2, 3$, it is easily seen that $\mu(E(W') \setminus V_0(n)) \subset W$. Therefore $\mu(E(W')) \subset W$.

In the case $U = X$, using a metric d on X such that the diameter of X with respect to d is less than 1, we have a mixer μ for X because $V = X$. □

Next, improving the technique of van Mill and van de Vel [**MV**$_2$, Theorem 3.1], we will prove the following without separability:

THEOREM II. *Let X be a semi-locally contractible space. If X has a local mixer, then X is LEC.*

Proof. Let μ be a local mixer for X. Using the semi-local contractibility of X and the A. H. Stone Theorem (e.g., see [**BP**, Ch. II Theorem 2.1]), X has a locally finite σ-discrete cover $\mathcal{U} = \bigcup_{n\in\mathbf{N}} \mathcal{U}_n$ by open sets contractible within X, where each $\mathcal{U}_n$ is discrete in X. For each $U \in \mathcal{U}$, let F_U: $U \times [0, 1] \to X$ be a contraction of U onto some $x_U \in X$, that is,

$$F_U(x, 0) = x \quad \text{and} \quad F_U(x, 1) = x_U \quad \text{for all } x \in U.$$

Take a closed cover $\{A(U): U \in \mathcal{U}\}$ and an open cover $\{B(U): U \in \mathcal{U}\}$ of X so that

$$A(U) \subset B(U) \subset \operatorname{cl} B(U) \subset U$$

for each $U \in \mathcal{U}$. For each $U \in \mathcal{U}$, let f_U: $X \to [0, 1]$ be a Urysohn function such that

$$f_U(\operatorname{cl} B(U)) = 1 \quad \text{and} \quad f_U(X \setminus U) = 0.$$

For each $n \in \mathbf{N}$, we define a homotopy F^n: $X \times [0, 1] \to X$ by

$$F^n(x, t) = \begin{cases} F_U(x, f_U(x) \cdot t) & \text{if } x \in U \in \mathcal{U}_n, \\ x & \text{otherwise.} \end{cases}$$

Since $\mathcal{U}$ is locally finite and each $\mathcal{U}_n$ is discrete, there exists an open cover $\mathcal{V}$ of X each element of which meets at most finitely many elements of $\mathcal{U}$ and at most one element of $\mathcal{U}_n$ for each $n \in \mathbf{N}$. For each $V \in \mathcal{V}$, let $k(V)$ be the number of $U \in \mathcal{U}$ with $V \cap U \neq \emptyset$. And for each $0 \leq i \leq k(V)$, let $\mathcal{W}(V, i)$ be an open cover of V such that

(1) $$\mathcal{W}(V, k(V)) \leq_\mu \mathcal{W}(V, k(V) - 1) \leq_\mu \cdots \leq_\mu \mathcal{W}(V, 1) \leq_\mu \mathcal{W}(V, 0) = \{V\};$$

(2) if $W \in \mathcal{W}(V, i)$, $U \in \mathcal{U}$ and $W \cap A(U) \neq \emptyset$, then $W \subset B(U)$,

where $\mathcal{W}' \leq_\mu \mathcal{W}$ means that for each $W' \in \mathcal{W}'$ there is some $W \in \mathcal{W}$ such that

$$E(W') = (X \times W' \times W') \cup (W' \times X \times W') \cup (W' \times W' \times X) \subset \mu^{-1}(W),$$

where this relation is denoted by $W' \subset_\mu W$. Now we have an open neighborhood,

$$W^* = \bigcup \{W^2: W \in \mathcal{W}(V, k(V)) \text{ for some } V \in \mathcal{V}\},$$

of ΔX in X^2.

We will construct two maps

$$G, H: W^* \times [1, \infty) \to X$$

as follows:

$$G(x, y, 1) = x; \qquad H(x, y, 1) = y,$$

and for $t \in [n, n+1]$, inductively,

$$G(x, y, t) = \mu(G(x, y, n), H(x, y, n), F^n(G(x, y, n), t-n));$$
$$H(x, y, t) = \mu(G(x, y, n), H(x, y, n), F^n(H(x, y, n), t-n)).$$

We will show that G and H are well defined and for each $W \in \mathcal{W}(V, k(V))$, $V \in \mathcal{V}$, there is some $n(W) \in \mathbf{N}$ such that

(3) if $x, y \in W$ and $t \geq n(W)$, then $G(x, y, t) = G(x, y, n(W)) = H(x, y, n(W)) = H(x, y, t)$.

To this end, let $V \in \mathcal{V}$ and $W \in \mathcal{W}(V, k(V))$. Set

$$\{U \in \mathcal{U}: V \cap U \neq \varnothing\} = \{U_i: i = 1, 2, \ldots, k(V)\},$$
$$U_i \in \mathcal{U}_{n(i)} \quad \text{for } i = 1, 2, \ldots, k(V),$$
$$n(1) < n(2) < \cdots < n(k(V)).$$

By (1), take $W_i \in \mathcal{W}(V, i)$, $i = 0, 1, \ldots, k(V)$, so that

(4) $$W = W_{k(V)} \subset_\mu W_{k(V)-1} \subset_\mu \cdots \subset_\mu W_1 \subset_\mu W_0 = V.$$

Since $W_{k(V)} \cap U = \varnothing$ for any $U \in \mathcal{U}_n$, $n < n(1)$, it follows that G and H are well defined on $W^2 \times [1, n(1)]$ and

$$G(x, y, t) = x, \quad H(x, y, t) = y \qquad \text{for each } (x, y, t) \in W^2 \times [1, n(1)].$$

Suppose G and H are well defined on $W^2 \times [1, n(i)]$ and

(5) $$G(W^2 \times [1, n(i)]) \cup H(W^2 \times [1, n(i)]) \subset W_{k(V)-i+1}.$$

From the definition and (4), it follows that G and H are well defined on $W^2 \times [n(i), n(i)+1]$ and

$$G(W^2 \times [n(i), n(i)+1]) \cup H(W^2 \times [n(i), n(i)+1]) \subset W_{k(V)-i}.$$

Since $W_{k(V)-i} \cap U = \varnothing$ for any $U \in \mathcal{U}_n$, $n(i) < n < n(i+1)$, it follows that G and H are well defined on $W^2 \times [n(i)+1, n(i+1)]$ and

$$G(x, y, t) = G(x, y, n(i)+1), \quad H(x, y, t) = H(x, y, n(i)+1)$$
$$\text{for each } (x, y, t) \in W^2 \times [n(i)+1, n(i+1)],$$

where we consider $n(k(V)+1) = \infty$ and $[1, \infty] = [1, \infty)$. Thus, by induction, we conclude that G and H are well defined on $W^2 \times [1, \infty)$ and satisfy condition (5) for all $i = 1, 2, \ldots, k(V)$. Next, we will find $n(W) \in \mathbf{N}$. Since $\{A(U)\colon U \in \mathcal{U}\}$ covers X, there is a $U \in \mathcal{U}$ such that $W \cap A(U) \neq \varnothing$, however $U = U_{i_0}$ for some $i_0 = 1, 2, \ldots, k(V)$ because $V \cap U \neq \varnothing$. Thus we have some $i_0 = 1, 2, \ldots, k(V)$ such that

$$W_{k(V)-i_0+1} \cap A(U_{i_0}) \neq \varnothing.$$

From (2) and (5), it follows that

$$G(W^2 \times \{n(i_0)\}) \cup H(W^2 \times \{n(i_0)\}) \subset B(U_{i_0}) \subset U_{i_0} \in \mathcal{U}_{n(i_0)}.$$

Recall that $f_{U_{i_0}}(\mathrm{cl}\, B(U_{i_0})) = 1$. This implies

$$F^{n(i_0)}(B(U_{i_0}) \times \{1\}) = F_{U_{i_0}}(B(U_{i_0}) \times \{1\}) = x_{U_{i_0}}.$$

Hence from the definition, we have

$$\begin{aligned} G(x, y, n(i_0) + 1) &= H(x, y, n(i_0) + 1) \\ &= \mu(G(x, y, n(i_0)), H(x, y, n(i_0)), x_{U_{i_0}}) \end{aligned}$$

for each $x, y \in W$. Put $n(W) = n(i_0) + 1$. Then (3) follows from the definition and the property of a (local) mixer.

Now we define $G', H'\colon W^* \times [0, 1] \to X$ by

$$G'(x, y, t) = \begin{cases} G(x, y, 1/t) & \text{if } t \neq 0, \\ G(x, y, n(W)) & \text{if } t = 0 \text{ and } x, y \in W; \end{cases}$$

$$H'(x, y, t) = \begin{cases} H(x, y, 1/t) & \text{if } t \neq 0, \\ H(x, y, n(W)) & \text{if } t = 0 \text{ and } x, y \in W. \end{cases}$$

From (3) these are obviously continuous and

$$G' \mid W^* \times \{0\} = H' \mid W^* \times \{0\}.$$

Thus we have an equi-connecting function $\lambda\colon W^* \times [0, 1] \to X$ defined by

$$\lambda(x, y, t) = \begin{cases} G'(x, y, 1 - 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ H'(x, y, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$ □

In the following, we will consider after J. Dugundji [**D**] a condition that a space with a local mixer is an ANR. Let μ be a (local) mixer for a space X. For $A \subset X$ define $A^{(\mu,1)} = \mu(A^3)$ when $A^3 \subset \mathrm{dom}\, \mu$, and inductively, define $A^{(\mu,n+1)} = \mu((A^{(\mu,n)})^3)$ when $(A^{(\mu,n)})^3 \subset \mathrm{dom}\, \mu$. We define $A^{(\mu,\infty)} = \bigcup_{n \in \mathbf{N}} A^{(\mu,n)}$ if each $A^{(\mu,n)}$ is well defined. For $A \subset B$ ($\subset X$), we say that A is *μ-stable* in B provided $A^{(\mu,\infty)}$ is well defined and $A^{(\mu,\infty)} \subset B$.

COROLLARY. *Let X be semi-locally contractible. If X has a local mixer μ with the property*:

(**) *for each $x \in X$ and each neighborhood W of x, there is neighborhood V of x which is μ-stable in W,*

then X is an ANR.

Proof. From Theorem II, X is LEC. By [**D**, Theorem 3.2], we may show that each open cover $\mathcal{W}$ of X has an open refinement $\mathcal{V}$ such that every partial realization $f\colon K^0 \to X$ in $\mathcal{V}$ of the 0-skeleton of any polytope K extends to a full realization of K in $\mathcal{W}$. This follows from (**) and the following lemma:

LEMMA. *Let X be semi-locally path-connected and have a local mixer μ. Assume that an open cover $\mathcal{W}$ of X has an open refinement $\mathcal{V}$ such that each $V \in \mathcal{V}$ is μ-stable in some $W \in \mathcal{W}$. Then every partial realization f: $K^0 \to X$ in $\mathcal{V}$ of the* 0*-skeleton of any polytope K extends to a full realization of K in $\mathcal{W}$.*

Proof. We define an extension of f over K by induction on the skeletons of K. Assume f has been extended to a map $f_n\colon K^n \to X$ so that

$$f_n(\sigma) \subset \bigcap \left\{V^{(\mu,n)} : f(\sigma \cap K^0) \subset V \in \mathcal{V}\right\}$$

for each closed simplex σ of K^n, where K^n denotes the n-skeleton of K. We denote the closed unit $(n+1)$-ball and unit n-sphere in $\mathbf{R}^{n+1}$ by $\mathbf{B}^{n+1}$ and $\mathbf{S}^n$, respectively. Let τ be any $(n+1)$-simplex and $h_\tau\colon \mathbf{B}^{n+1} \to \tau$ a fixed homeomorphism. Note that

$$f_n h_\tau(\mathbf{S}^n) = f_n(\partial\tau) \subset \bigcap \left\{V^{(\mu,n)} : f(\tau \cap K^0) \subset V \in \mathcal{V}\right\}.$$

Using the technique of [**MV**$_1$, Theorem 1.3], we have an extension g_τ: $\mathbf{B}^{n+1} \to X$ of $f_n h_\tau | \mathbf{S}^n$ such that

$$g_\tau(\mathbf{B}^{n+1}) \subset \bigcap \left\{V^{(\mu,n+1)} : f(\tau \cap K^0) \subset V \in \mathcal{V}\right\}.$$

Define a map $f_{n+1}\colon K^{n+1} \to X$ by

$$f_{n+1} | \tau = g_\tau h_\tau^{-1} \colon \tau \to X$$

on each $(n+1)$-simplex τ of K. Then f_{n+1} is an extension of f_n such that

$$f_{n+1}(\tau) \subset \bigcap \left\{V^{(\mu,n+1)} : f(\tau \cap K^0) \subset V \in \mathcal{V}\right\}$$

for each closed simplex τ of K^{n+1}. Thus we have an extension $\tilde{f}$: $K \to X$ defined by $\tilde{f} | K^n = f_n$ on each K^n. This extension $\tilde{f}$ is obviously a full realization of K in $\mathfrak{W}$. □

REMARK. Let μ: $U \to X$ be a map of a neighborhood U of ΔX in X^3 to X. We will call μ a *local weak mixer* provided μ satisfies the following condition:

(w) $$\mu(x, x, y) = \mu(x, y, x) = \mu(y, x, x) = x \text{ if } (x, x, y), (x, y, x), (y, x, x) \in U.$$

When $U = X^3$, we call μ a *weak mixer*. The properties of a local mixer used in the proof of Theorem II are (w) and:

(#)′ for each $x \in X$ and for each neighborhood V of x in X, there exists a neighborhood W of x such that $W \times W \times X \subset \mu^{-1}(V)$,

and then dom μ is a neighborhood of $\Delta X \times X$ in X^3 rather than of $\Delta^* X$. And moreover, if we assume X is locally contractible then it suffices that dom μ is a neighborhood of ΔX in X^3 and (#)′ can be replaced by:

(#)″ each $x \in X$ has a neighborhood W_x in X such that for any neighborhood V of x there is some neighborhood W of x with $W \times W \times W_x \subset \mu^{-1}(\mathrm{V})$.

If X is locally compact, then a (local) weak mixer satisfies (#)″. Thus we have

THEOREM. *A locally compact metrizable space is LEC if and only if it is locally contractible and has a local weak mixer.*

From this theorem, it follows that:

COROLLARY. *Let X be locally compact and locally contractible. If X has a local weak mixer then it has a local mixer. And, moreover, if X is contractible then it has a mixer.*

Supplement: In [$\mathbf{MV}_2$] it is a question whether every Banach space has a "natural" mixer. In Euclidean space let $\mu(x, y, z)$ be the inner center of the triangle with vertices x, y and z. Then μ is clearly the mixer. T.

Yagasaki gave a "natural" mixer for each *convex set* X in a *normed space* as follows:

$$\mu(x, y, z) = \begin{cases} \dfrac{1}{\|x-y\|+\|y-z\|+\|z-x\|} \\ \quad \cdot \{\|y-z\|\cdot x + \|z-x\|\cdot y + \|x-y\|\cdot z\} & \text{if } (x, y, z) \notin \Delta X, \\ x \ (= y = z) \quad \text{if } (x, y, z) \in \Delta X, \end{cases}$$

where $\| \ \|$ denotes the norm. In fact, if $(x, y, z) \notin \Delta X$ and $\|x-a\|$, $\|y-a\| < \varepsilon$, then

$$\begin{aligned} &\|\mu(x, y, z) - a\| \\ &\leq \frac{1}{\|x-y\|+\|y-z\|+\|z-x\|} \\ &\quad \times \{\|y-z\|\cdot\|x-a\| + \|z-x\|\cdot\|y-a\| + \|z-a\|\} \\ &< \frac{1}{\|y-z\|+\|z-x\|}\{\|y-z\|\cdot\varepsilon + \|z-x\|\cdot\varepsilon + \|x-y\|\cdot\|z-a\|\} \\ &= \varepsilon + \frac{\|x-y\|}{\|y-z\|+\|z-x\|}\|z-a\|. \end{aligned}$$

If $\|z-a\| < 2\varepsilon$, then

$$\frac{\|x-y\|}{\|y-z\|+\|z-x\|}\|z-a\| \leq \|z-a\| < 2\varepsilon.$$

If $\|z-a\| \geq 2\varepsilon$, then

$$\begin{aligned} \frac{\|x-y\|}{\|y-z\|+\|z-x\|}\|z-a\| &< \frac{\varepsilon}{\|z-a\|-\varepsilon}\|z-a\| \\ &= \frac{\varepsilon}{1-\varepsilon/\|z-a\|} \leq \frac{\varepsilon}{1-1/2} = 2\varepsilon, \end{aligned}$$

because

$$\begin{aligned} &\|x-y\| \leq \|x-a\| + \|y-a\| < 2\varepsilon, \\ &\|z-x\| \geq \|z-a\| - \|x-a\| > \|z-a\| - \varepsilon \end{aligned}$$

and, similarly, $\|y-z\| < \|z-a\| - \varepsilon$. Therefore $\|\mu(x, y, z) - a\| < 3\varepsilon$. Since μ is symmetric, this implies μ is continuous at each point of ΔX (hence at any point of X^3) and μ satisfies (#) (equivalently, (*)). Therefore μ is a mixer for$X$.

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