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Harold George Diamond

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HAROLD G. DIAMOND

The series

$$S(x)=1+\sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{n_1n_2\cdots n_k\leq x\\ n_1,n_2,\ldots,n_k>1}}\frac{1}{\log n_1\log n_2\cdots\log n_k}$$

is interpreted as a statement about Beurling generalized prime numbers and is estimated by means of Beurling theory.

This series was considered by I. Kasara in [5], in which he asserted that

$$\text{(1)}\qquad \text{“}S(x)=x+O(x/\log x).\text{”}$$

This assertion is not correct as it stands. We shall show that

$$\text{(2)}\qquad S(x)=cx+O\{x\exp(-(\log x)^{1/2-\varepsilon})\},$$

where $c \doteq 1.24292$.

We begin by giving the heuristic argument. Each integer in $(1,x]$ is uniquely expressible as a product of a certain number of primes. Thus we have

$$\text{(3)}\qquad [x]=1+\pi_1(x)+\pi_2(x)+\cdots$$

for $x\geq 1$, where

$$\pi_k(x)=\#\{n\leq x\colon n\text{ has exactly }k\text{ prime factors}\}$$

with repetitions allowed.

An estimate from prime number theory [4, §22.18] and a small calculation give, for each fixed k,

$$\text{(4)}\qquad \pi_k(x)\sim x(\log\log x)^{k-1}/\{(k-1)!\log x\}$$
$$\sim\frac{1}{k!}\sum_{\substack{n_1n_2\cdots n_k\leq x\\ n_1,n_2,\ldots,n_k>1}}\frac{1}{\log n_1\log n_2\cdots\log n_k}.$$

This relation and (3) suggest formula (1). However, (4) does not hold uniformly in k, so this argument does not even show that $S(x)\sim cx$.

Define arithmetic functions

$$f(n) = \begin{cases} 1/\log n, & n \geq 2, \\ 0, & n < 2, \end{cases}$$

and

$$e(n) = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

For g and h arithmetic functions, define the convolution $g * h$ by

$$g * h(n) = \sum_{ij=n} g(i)h(j).$$

Finally, define an arithmetic function s by setting $s(n) = S(n) - S(n-1)$. The formula defining S can now be rewritten as

$$s = e + f + f * f/2! + f * f * f/3! + \cdots = \exp f. \tag{5}$$

The last formula is of the type that appears in the theory of Beurling generalized prime numbers, with

$$\sum_{n \leq x} f(n) \leftrightarrow \Pi(x), \qquad S(x) \leftrightarrow [x].$$

Viewed from this perspective, (1) is suspicious, because special conditions are required in order that a Beurling generalized number system should have density exactly 1.

We prove (2) with the aid of Theorem 3.3b of [**3**]: *Suppose f and s satisfy* (5) *and*

$$\sum_{n \leq x} \frac{f(n)}{n} = \int_1^x \frac{1 - t^{-1}}{t \log t} dt + \log c + O\{\exp(-\log^a x)\}$$

for some $c > 0$ and $a \in (0, 1)$. Then

$$S(x) = cx + O\left\{x \exp\left(-[\log x \log\log x]^{a'}\right)\right\},$$

where $a' = a/(1 + a)$.

Here we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \sum_{2 \leq n \leq x} \frac{1}{n \log n} = \int_2^x \frac{dt}{t \log t} + \gamma' + O\left(\frac{1}{x \log x}\right),$$

where $\gamma' \doteq .428166$ [**2**, p. 244, Table 2], and

$$\int_1^x \frac{1 - t^{-1}}{t \log t} dt = \log\log x + \gamma + O\left(\frac{1}{x \log x}\right),$$

where $\gamma \doteq .577216$ [**1**, p. 228, footnote 3].

It follows that

$$\sum_{n\le x}\frac{f(n)}{n}-\int_1^x\frac{1-t^{-1}}{t\log t}\,dt=\gamma'-\gamma-\log\log 2+O\left(\frac{1}{x\log x}\right).$$

Thus, $c=\exp(\gamma'-\gamma-\log\log 2)\doteq 1.24292$, and we can take a to be any number less than 1 in Theorem 3.3b of **[3]**. This proves (2).

We note in conclusion that if it is assumed that $S(x)\sim cx$, then the constant c can be evaluated by an Abelian argument. We use the formula

$$\int_1^\infty x^{-\sigma}\,dS(x)=\exp\sum_{n\ge 2}\frac{1}{n^\sigma\log n},$$

valid for $\sigma>1$, and evaluate each side. On the one hand,

$$\int_1^\infty x^{-\sigma}\,dS(x)=\sigma\int_1^\infty x^{-\sigma-1}S(x)\,dx\sim\frac{c}{\sigma-1}$$

as $\sigma\to 1+$. On the other hand, as $\sigma\to 1+$,

$$\begin{aligned}\sum_{n\ge 2}\frac{1}{n^\sigma\log n}&\\ &=\log\frac{\sigma}{\sigma-1}+(\sigma-1)\int_1^\infty t^{-\sigma}\left\{\sum_{n\le t}\frac{f(n)}{n}-\int_1^t\frac{1-u^{-1}}{u\log u}\,du\right\}dt\\ &=\log\frac{\sigma}{\sigma-1}+(\sigma-1)\int_1^\infty t^{-\sigma}\{\gamma'-\gamma-\log\log 2+o(1)\}\,dt\\ &=\log\frac{1}{\sigma-1}+\gamma'-\gamma-\log\log 2+o(1).\end{aligned}$$

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UNIVERSITY OF ILLINOIS
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