CHARACTERIZING THE DIVIDED DIFFERENCE WEIGHTS FOR EXTENDED COMPLETE TCHEBYCHEFF SYSTEMS

Richard Blaine Barrar and Henry Loeb
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Newman and Rivlin have shown that there is a 1-1 correspondence between the nodes and weights of the $n$th order divided difference of $n$th degree polynomials. Their method applies only to polynomials. In this paper we develop a new approach and apply it to extend their results to the setting of Extended Complete Tchebycheff Systems.

0. Introduction. In [7] Newman and Rivlin (see also the reference there to S. Karlin's results) were able to characterize the weights which appear in the $n$th order divided difference formula with respect to the base functions $\{u_j(x) = x^j\}_{j=0}^n$ and to establish a 1-1 correspondence between these weights and the corresponding set of nodes, $0 = x_0 < x_1 < \cdots < x_n$, used in the formula. We propose in this paper to extend this result to the setting where the family $\{u_j(x)\}_{j=0}^n$ forms an Extended Complete Tchebycheff System (E.C.T.S.) on $[0, \infty)$. This means for each $k$, where $0 \leq k \leq n$, any non-trivial linear combination of the functions $\{u_0, \ldots, u_k\}$ has at most $k$ zeros (including multiplicities) in $[0, \infty)$ where each $u_j \in C^n[0, \infty)$. We further assume that $u_0(x) \equiv 1$. For the remainder of this paper we shall postulate that these basic hypotheses concerning $\{u_j\}_{j=0}^n$ hold.

Among the E.C.T.S. for which these results are valid, we will highlight the families generated by the Cauchy Kernel and the Exponential Kernel.

1. Statement of problem. Let

(1) \[ S = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n: 0 < x_1 < \cdots < x_n \}, \quad x_0 \equiv 0. \]

$A$ is defined to be the set of all $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ such that the following properties are valid

\begin{align*}
(i) \quad & (-1)^{n-i} a_i > 0 \quad (i = 0, 1, \ldots, n); \\
(ii) \quad & \sum_{i=0}^n a_i = 0; \\
(iii) \quad & (-1)^{n-j} \sum_{i=j}^n a_i > 0, \quad j = 1, \ldots, n.
\end{align*}
The sets \( S \) and \( A \) are related through the classical concept of divided differences. For each \( x \in S \) and each real-valued function \( f \) defined on \([0, \infty)\), consider the \( n \)th order divided difference of \( f \) with respect to the points \((x_0, x_1, \ldots, x_n)\) defined as follows.

\[
\frac{f[x_0, \ldots, x_n]}{u_0, \ldots, u_n} = U \left[ \begin{array}{c} u_0, \ldots, u_n \\ x_0, \ldots, x_n \end{array} \right],
\]

where

\[
U \left[ q_0, \ldots, q_n \\ x_0, \ldots, x_n \right] = \det \{ q_i(x_j) ; i, j = 0, 1, \ldots, n \}.
\]

We then set

\[
a_i = (-1)^{n+i} \frac{U \left[ u_0, \ldots, u_{n-1} \right]}{u_0, \ldots, u_n} \frac{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n}{x_0, \ldots, x_n}, \quad i = 0, 1, \ldots, n.
\]

Clearly,

\[
f[x_0, \ldots, x_n] = \sum_{i=0}^{n} a_i f(x_i).
\]

The \( \{a_i\} \) are called the weights of the divided difference formula. Cramer's Rule, together with (3), (4), shows that for a given \( x \in S \), \( a = (a_0, \ldots, a_n) \) satisfies (4) iff

\[
\sum_{i=0}^{n} a_i u_j(x_i) = \delta_{nj}, \quad j = 0, 1, \ldots, n,
\]

where \( \delta_{nj} \) is the Kronecker delta symbol.

Thus for each \( x \in S \), we can associate an \( a \) via the relationship (4). Let \( g \) be the map defined by (4), that is \( g(x) = a \). The main purpose of this paper is to show that \( g \) is a 1-1 map of \( S \) onto \( A \). As we indicated in the introduction, Newman and Rivlin proved this result for the special case of polynomials; that is, where \( u_i = x^i \).

**Lemma 1.** \( g \) maps \( S \) into \( A \).
Proof. Since \((u_0, \ldots, u_n)\) form an Extended Complete Tchebycheff System (E.C.T.S.), it is clear from the definition of the weights \(a_i\) in (4) that \(a = g(x)\) satisfies (i) and (ii). (In this regard recall that \(u_0 \equiv 1\.)

To prove (iii), for \(0 \leq j \leq n - 1\) pick \(u^{(j)}\) in the linear subspace \(U\) spanned by \((u_0, \ldots, u_n)\) with the properties

(a) \(u^{(j)}(x_i) = 1, i = 0, 1, \ldots, j,\)
(b) \(u^{(j)}(x_i) = 0, i = j + 1, \ldots, n.\)

Using (5) and the above it follows that

\[
\sum_{i=0}^{j} a_i = \sum_{i=0}^{n} a_i u^{(j)}(x_i) = b_n,
\]

where \(b_n\) is the coefficient of \(u_n\) in the expansion of \(u^{(j)}\). From [5, p. 379] we infer that \(\{(d/dx)u_j(x)\}_{j=1}^n\) forms an E.C.T.S. Thus by Rolle’s Theorem \((d/dx)u^{(j)}(x)\) has a maximum set of \(n - 1\) simple zeros consisting of \(j\) zeros in \((x_0, x_j)\) and \((n - j - 1)\) zeros in \((x_{j+1}, x_n)\). Further, since \(u^{(j)}(x_j) = 1\) and \(u^{(j)}(x_{j+1}) = 0,\) \(du^{(j)}/dx < 0\) in \([x_j, x_{j+1}]\) and thus \((-1)^{n-j} du^{(j)}/dx(x_n) > 0\). Using as data these \(n - 1\) zeros of \((d/dx)u^{(j)}(x)\) and \(x_n\), we conclude by Cramer’s Rule that \(\text{sgn}(d/dx)u^{(j)}(x_n) = \text{sgn} b_n;\) that is,

\[
(-1)^{n-j} \sum_{i=0}^{j} a_i > 0.
\]

By (2)(ii),

\[
\sum_{i=0}^{j} a_i = \left(\sum_{i=0}^{n} a_i - \sum_{i=j+1}^{n} a_i\right) = -\sum_{i=j+1}^{n} a_i.
\]

Finally, then

\[
(-1)^{n-(j+1)} \sum_{i=j+1}^{n} a_i > 0. \quad \square
\]

Lemma 2. Let \(\{x^{(e)}\}_{e=1}^{\infty} \subset S\) be a sequence with the property that the corresponding sequence \(\{a^{(e)}\} \subset A\) (where \(a^{(e)} = g(x^{(e)})\)) has the feature that \(a^{(e)} \to a \in A\). Then if \(x^{(e)} \to x\), we can conclude that \(x \in S\).

Proof. Assume the result is false. We treat two cases. Case (1): \(x^{(e)} \to x_0 \equiv 0\) for all \(e\). Thus using (5) for \(j = n\) we find the limit function satisfies

\[
\sum_{i=0}^{n} a_i u_n(0) = 1,
\]
which contradicts (2)(ii). Case (2): For some $i$ where $1 \leq i \leq n - 1$, $x_0 < x_i = x_{i+1}$. Thus by exploiting the fact that $a$ satisfies (2)(iii) and (5), we can find a set of numbers $\{b_j\}_{j=0}^k$, where $b_k \not\equiv 0$ with $0 \leq k \leq n - 1$ so that for the $k + 1$ distinct components of the limit vector $x$, say $(x_{i_0}, \ldots, x_{i_k})$, we have
\[
\sum_{i=0}^k b_j u_j(x_{i_j}) = 0 \quad (j = 0, 1, \ldots, n - 1).
\]
This contradicts the fact that $\{u_j\}_{j=0}^{n-1}$ form an E.C.T.S. Thus the proof is complete. 

2. Main results. In this section we will develop the topological tools which we will use to prove our principal result; that is, $g$ is a 1-1 map of $S$ onto $A$. We will employ a differential equation approach which has been exploited by Fitzgerald and Schumaker [4]; Barrar, Loeb and Werner [2]; Barrar and Loeb [1, 3].

Our approach, in contrast to other attacks on these types of problems, has the important property that it does not require any type of a priori uniqueness. In this regard see Fitzgerald, Schumaker [4] or Newman, Rivlin [7] where such information is used.

Consider a fixed $z^* \in A$. We want to demonstrate that there is exactly one $x^* \in S$ which satisfies
\[
\sum_{i=0}^n a_i^* u_j(x_i) = \delta_{nj} \quad (j = 0, 1, \ldots, n).
\]
Since $\sum_{i=0}^n a_i^* = 0$ and $u_0 \equiv 1$, this is equivalent to demonstrating it for the system
\[
\sum_{i=1}^n a_i^*(u_j(x_i) - u_j(x_0)) = \delta_{nj}, \quad j = 1, \ldots, n.
\]
For each $x \in S$, consider the system of $n$ ordinary differential equations
\[
\frac{d}{d\tau} \left[ \sum_{i=1}^n ((1 - \tau)a_i + \tau a_i^*)(u_j(x_i(\tau)) - u_j(x_0(\tau))) \right] = 0,
\]
where $a = g(x)$ and the initial conditions are $x(0) = x = (x_1, \ldots, x_n)$. Here $\tau$ is the independent variable, $x(\tau) = (x_1(\tau), \ldots, x_n(\tau))$, and $a = (a_0, \ldots, a_n)$. Integrating (7) we find that
\[
\sum_{j=1}^n ((1 - \tau)a_i + \tau a_i^*)(u_j(x_i(\tau)) - u_j(x_0(\tau))) = c_j, \quad j = 1, \ldots, n.
\]
We evaluate the constants $c_j$ by setting $\tau = 0$. One finds using (6) that
\[
\delta_{nj} = \sum_{i=1}^{n} a_i (u_j(x_i) - u_j(x_0)) = c_j, \quad j = 1, \ldots, n,
\]
and indeed at $\tau = 1$,
\[
\sum_{i=1}^{n} a_i^* (u_j(x_i(1)) - u_j(x_0)) = \delta_{nj} \quad (j = 1, \ldots, n).
\]
Thus, one notes that $a^* = g(x(1))$ and $x(1)$ is a desired solution for $a^*$. We see then that our main problem is to show that the system of differential equations has a solution in the interval $[0, 1]$. We proceed toward this goal.

For many important families of functions we will be able to verify the following assumption.

**Assumption A.** If \( \{x^{(v)}\}_{v=1}^{\infty} \subseteq S \) has the characteristic that \( a^{(v)} = g(x^{(v)}) \to a \in A \) as \( v \to \infty \), then \( \{x^{(v)}\}_{v=1}^{\infty} \) are bounded.

For the remainder of this section we shall postulate that Assumption A is valid for the E.C.T.S. \( \{u_i\}_{i=0}^{n} \) on \([0, \infty)\) where \( u_0 \equiv 1 \).

Expanding (7) we obtain
\[
\sum_{i=1}^{n} \left[ \tau a_i^* + (1 - \tau) a_i \right] u'_i(x_i(\tau)) \frac{dx_i(\tau)}{d\tau} = \sum_{i=1}^{n} \left( a_i - a_i^* \right) \left[ u_j(x_i(\tau)) - u_j(x_0) \right] \quad (i = 1, \ldots, n)
\]
with \( u'_i(x) = \frac{d}{dx} u_j(x) \).

It is important to note that for \( \tau \in [0, 1] \) and \( x(\tau) \in S \), the Jacobian matrix of the system (9),
\[
J(\tau) = \left\{ \left( \tau a_i^* + (1 - \tau) a_i \right) u'_i(x_i(\tau)) ; i, j = 1, \ldots, n \right\}
\]
is non-singular. This follows from the fact that \( \{u'_i\}_{i=1}^{n} \) form a E.C.T.S. and that \( (\tau a^* + (1 - \tau) a) \) satisfies (2)(i) when \( \tau \in [0, 1] \).

Further, it is easy to check using Assumption A and Lemma 2 that \( \{x(\tau) ; \tau \in [0, 1] \} \) is bounded, and if \( \{ \tau_v \}_{v=1}^{\infty} \subseteq [0, 1] \) has the property that \( x(\tau_v) \to x \), then \( x \in S \). These facts can be used to show that the system of differential equations has a solution over \([0, 1]\). The basic ingredients of such an existence proof are enunciated in [1, 2].

For each \( x \in S \), let \( \Phi \) be the map from \( S \to B \) defined by \( \Phi(x) = x(1) \) for \( x \in S \) where \( B = \{ x \in S : g(x) = a^* \} \). If \( x \in B \), it is easy to verify
that \( x(\tau) \equiv x \) is a solution of (9) and, indeed, by the uniqueness of the solution of the system of differential equations, the only one. Thus \( \Phi \) maps \( S \) onto \( B \) and since by the theory of differential equations \( \Phi \) is continuous, \( \Phi \) maps the connected set \( S \) onto the connected set \( B \).

Let \( x^* \in B \). Then \( x^* \) is a solution of the non-linear system (6). Further, the Jacobian matrix of the system is

\[
\{ a^*_i u'_j(x^*_i); i, j = 1, \ldots, n \}.
\]

Since \( a^* \) satisfies (2)(i) and \( \{ u'_j(x) \}^n_{j=1} \) form a E.C.T.S., the matrix is non-singular. We can conclude by the \textit{implicit function theorem} that \( x^* \) is an isolated point of \( B \). Since \( x^* \) is an arbitrary point of the connected set \( B \), it follows that \( B \) consists of exactly one point. Summarizing,

\textbf{MAIN THEOREM.} \textit{For each} \( a^* \in A \), \textit{there is exactly one} \( x^* \) \textit{in} \( S \) \textit{which satisfies}

\[
\sum_{i=0}^{n} a^*_i u(x^*_i) = \delta_{jn} \quad (i = 0, 1, \ldots, n),
\]

\textit{and the map} \( g \) \textit{defined by} (4) \textit{is a 1-1 map which takes} \( S \) \textit{onto} \( A \).

3. Applications. In this section we present some examples of E.C.T.S. which satisfy Assumption A and thus satisfy the hypothesis of the Main Theorem.

Consider the exponential kernel \( K(\lambda, x) = e^{\lambda x} \) and any set of \( n \) positive numbers \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \) with \( \lambda_0 = 0 \). Then we set

\[
u_i(x) = K(\lambda_i, x), \quad i = 0, 1, \ldots, n.
\]

\textbf{Lemma 3.} \textit{The exponential family of functions defined in} (11) \textit{has the property that if a sequence} \( \{x^{(v)}\}_{v=1}^{\infty} \subset S \) \textit{yields a sequence} \( \{a^{(v)} = g(x^{(v)})\}_{v=1}^{\infty} \) \textit{with the characteristic that} \( a^{(v)} \to a \in A \), \textit{then the} \( \{x^{(v)}\}_{v=0}^{\infty} \) \textit{are bounded.}

\textit{Proof.} Let us assume that the components of \( x^{(v)} \) are not bounded. Then by going to a subsequence if necessary we can develop the following situation:

\[
u \to \infty
\]

\[
\lim_{v \to \infty} x^{(v)}_n = \infty;
\]

\[
(b) \quad \lim_{v \to \infty} \left( x^{(v)}_n - x^{(v)}_i \right) = c_i, \quad i = l, \ldots, n, \text{ where } l \geq 1 \text{ and } c_i \geq c_{i+1}, \quad i = l, \ldots, n - 1, \text{ with } c_i \text{ finite};
\]

\[
(c) \quad \lim_{v \to \infty} \left( x^{(v)}_n - x^{(v)}_i \right) = \infty, \quad i = 1, \ldots, l - 1.
\]
Dividing each of the relationships
\[ \sum_{i=0}^{n} a_i^{(v)} e^{\lambda_j x_i^{(v)}} = \delta_{nj} \]
by \( e^{\lambda_j x_i^{(v)}} \) and letting \( v \to \infty \), we find that the limits satisfy
\[ \sum_{i=1}^{n} a_i e^{-\lambda_j c_i} = 0 \quad (j = 1, \ldots, n). \]

Let \( c_1 > c_2 > \cdots > c_k = 0 \) be the distinct values of \( \{c_i\}_{i=1}^{n} \) where \( k \leq n - l + 1 \leq n \). Then we can find numbers \( b_1, \ldots, b_k \) so that
\[ f(\lambda) \equiv \sum_{i=1}^{n} a_i e^{-\lambda c_i} = \sum_{m=1}^{k} b_m e^{-\lambda c_m}, \]
where by property (2)(iii), \( b_k \neq 0 \). Thus since \( f(\lambda_i) = 0 \), \( i = 1, \ldots, n \) and \( \{e^{-\lambda c_m}\}_{m=1}^{k} \) form an E.C.T.S., we have reached a contradiction. This completes the proof. \( \square \)

We claim that Lemma 3 is also valid for the Cauchy kernel, \( K(\lambda, x) = \frac{1}{1 + \lambda x} \).

**Lemma 4.** Let \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \) be given and set \( u_j(x) = \frac{1}{1 + \lambda_j x} \) (\( j = 0, 1, \ldots, n \)). Then Lemma 3 is valid for the \( \{u_j\}_{j=0}^{n} \).

**Proof.** Again assuming that \( x_i^{(v)} \to \infty \), we can, by going to a subsequence if necessary, achieve the situation:

(a) \( x_i^{(v)} \to \infty \), \( i = l, \ldots, n \), where \( l \geq 1 \);
(b) \( x_i^{(v)} \to c_i \), \( i = 0, \ldots, l - 1 \), \( c_i \) finite with \( c_i \leq c_{i+1} \) and \( c_0 = 0 \).

For each relationship
\[ \sum_{i=0}^{n} \frac{a_i^{(v)}}{1 + \lambda_j x_i^{(v)}} = 0, \]
letting \( v \to \infty \), we find
\[ \sum_{i=0}^{l-1} \frac{a_i}{1 + \lambda_j c_i} = 0 \quad (j = 0, \ldots, n - 1). \]

Pick out the distinct elements \( 0 = c_0 < \cdots < c_{k-1} \) of the set \( \{c_i\}_{i=0}^{l-1} \) where \( k \leq l \leq n \). Then there are \( k \) distinct numbers \( b_0, \ldots, b_{k-1} \) so that
\[ f(\lambda) \equiv \sum_{i=0}^{l-1} \frac{a_i}{1 + c_i \lambda} = \sum_{m=0}^{k-1} \frac{b_m}{1 + c_m \lambda}. \]
and where by properties (2)(i), (ii), (iii), \( b_0 \neq 0 \). Since \( f(\lambda_j) = 0 \) \( (j = 0, 1, \ldots, n - 1) \) we have contradicted the fact that the family \( \{1/(1 + c_{im})\}_{m=0}^{k-1} \) forms an E.C.T.S. 

Our results can be extended to treat multiple knots also.

As an example, we have the following result, which includes the results of [7].

**Lemma 5.** Let \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_r \) be given and consider the functions \( \{x^q e^{\lambda_p x}; \ q = 0, 1, \ldots, m_p - 1; \ p = 0, 1, \ldots, r\} \). If \( n + 1 = \sum_{p=0}^{r} m_p \) and if we set \( u_j(x) = x^q e^{\lambda_p x} \) with \( j = \sum_{i=1}^{p-1} m_i + q \) and \( m_{-1} = 0 \), then Lemma 3 is valid for the functions \( \{u_j\}_{j=0}^{n} \). (The \( \lambda_p \) are called the knots and the \( m_p \) are designated as the multiplicities of the knots of the kernel \( K(x, \lambda) = e^{\lambda x} \). It is well known that this set of functions is a E.C.T.S., see [5, p. 9].)

**Proof.** Letting

\[
f(\lambda, v) = \sum_{i=0}^{n} a_i^{(v)} e^{\lambda x_i^{(v)}},
\]

we have

\[
\frac{\partial^q f}{\partial \lambda^q}(\lambda, v) = \sum_{i=0}^{n} a_i^{(v)} (x_i^{(v)})^q e^{\lambda x_i^{(v)}}.
\]

The set of equations corresponding to (5) for \( a_i = a_i^{(v)}, x_i = x_i^{(v)} \) can be written as

\[
(13) \quad \delta_p r \delta_{q(m_p - 1)} = 0, 1, \ldots, m_p - 1; \qquad p = 0, 1, \ldots, r.
\]

Assuming \( x_n^{(v)} \to \infty \), if \( r \geq 1 \), we divide \( f(\lambda, v) \) by \( e^{\lambda x_n^{(v)}} \), and apply Leibnitz’s rule for differentiation of a product to find, using the notation of (12)(a), (b), (c), that in the limit as \( v \to \infty \), (13), for \( p \geq 1 \), becomes

\[
(14) \quad \sum_{i=1}^{n} a_i c_i^q e^{\lambda_p c_i} = 0, \quad q = 0, 1, \ldots, m_p - 1; p = 1, \ldots, r.
\]

Combining equal \( c_i \)'s as in Lemma 3, this becomes

\[
(15) \quad \sum_{s=1}^{w} b_s (c_i_s)^q e^{\lambda_p c_i_s} = 0, \quad q = 0, 1, \ldots, m_p - 1; p = 1, \ldots, r,
\]

where \( w \leq n + 1 - l \), \( b_w \neq 0 \) by (2)(iii), and \( l \geq 1 \). In (15) we are dealing with an E.C.T.S. of dimension \( \leq n + 1 - l \) with typical term \( x^q e^{\lambda_p x} \).
Further, the function in (15) has at least \( n + 1 - m_0 \) zeros. Thus \( n + 1 - m_0 < n + 1 - l \), that is,

\begin{equation}
(16) \quad m_0 > l \quad \text{if} \quad r \geq 1.
\end{equation}

For any \( r \), we divide the equations in (13) for \( \lambda = \lambda_0 \) by \( (x^{(e)}_n)^q \) for each \( q = 0, 1, \ldots, m_0 - 1 \), and take the limit as \( v \to \infty \). Using the notation of (12)(a), (b), (c) the result is a set of equations

\[ \sum a_i(d_i)^q = 0, \quad d_i \leq d_{i+1}, \quad q = 0, 1, \ldots, m_0 - 1. \]

Combining equal \( d_i \)'s we obtain a set

\begin{equation}
(17) \quad \sum_{s=1}^{g} b_{i_s}(d_{i_s})^q = 0, \quad q = 0, 1, \ldots, m_0 - 1.
\end{equation}

Note that \( x_i^{(e)} - x_n^{(e)} \to c_i \) (finite) implies \( x_i^{(e)}/x_n^{(e)} \to d_i = 1 \). Thus \( d_i = 1 \) \( (i = l, \ldots, n) \) with \( g \leq l \) and \( b_{i_s} \neq 0 \). In (17) we are dealing with a non-zero function with \( m_0 \) zeros generated from a E.C.T.S. of dimension at most \( l \). Therefore we must have

\begin{equation}
(18) \quad m_0 < l.
\end{equation}

If \( r = 0 \), (18) is a contradiction since \( m_0 = n + 1 \) and \( l < n + 1 \). If \( r \geq 1 \) both (16) and (18) must hold, which again is a contradiction. \( \square \)

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