GENERALIZED ORDERED SPACES WITH CAPACITIES

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We show that any GO-space having a capacity in the sense of Ščepin has a $G_\delta$-diagonal and is perfect. In addition, such a space has a $\sigma$-discrete dense subset and a dense metrizable subspace, and any GO-space having a capacity and a point-countable base (or having a $\sigma$-discrete dense subset and a point-countable base) is metrizable.

1. Introduction. In [14] Ščepin defined a capacity for a space $X$ to be a family of functions $\{\epsilon_x | x \in X\}$ such that, for each closed $F \subset X$,
   
   (C_1) $\epsilon_x(F)$ is a non-negative real number with $\epsilon_x(F) > 0$ iff $x \in \text{Int}(F)$,
   
   (C_2) if $F_1 \subset F_2$ are closed then $\epsilon_x(F_1) \leq \epsilon_x(F_2)$,
   
   (C_3) for a fixed closed $F$, the function $x \to \epsilon_x(F)$ is continuous,
   
   (C_4) for a fixed $x$, if $\{F_\alpha | \alpha < \kappa\}$ is a family of closed sets satisfying $F_\alpha \supset F_\beta$ whenever $\alpha < \beta < \kappa$, then $\epsilon_x(\bigcap_\alpha F_\alpha) = \inf_\alpha \epsilon_x(F_\alpha)$.

   In that same paper Ščepin announced without proof that a linearly ordered topological space (LOTS) having a capacity is metrizable. The purpose of this note is to prove a more general result from which Ščepin's result follows immediately, namely, that any GO-space (= suborderable space) with a capacity has a $G_\delta$-diagonal. (Recall that the class of GO-spaces is precisely the class of subspaces of LOTS.) Along the way to that result, we show that any GO-space with a capacity is perfect (i.e., closed sets are $G_\delta$). In §4 we will discuss two old questions about perfect GO-spaces in the context of GO-spaces having a capacity, proving that a GO-space with a capacity has a $\sigma$-discrete dense subset and a GO-space with a capacity and a point-countable base must be metrizable. Finally, examples in §5 show that our results are sharp.

   Terminology and notation not defined in this paper will follow [8, 11, 12].

2. Preliminary results and perfect normality. We proceed via a sequence of lemmas.

2.1. Lemma. Any GO-space having a capacity is a first-countable space.
Proof. Fix a non-isolated point $p$ of $X$. If $[p, \rightarrow)$ is not open then $\varepsilon_p([p, \rightarrow]) = 0$ and there is a well-ordered, strictly increasing net $\{x_\alpha | \alpha < \kappa\}$ whose supremum is $p$. Let $F_\alpha = [x_\alpha, \rightarrow)$. According to $(C_4)$, $0 = \varepsilon_p([p, \rightarrow)) = \inf\{\varepsilon_p(F_\alpha) | \alpha < \kappa\}$. For each $n$, choose $\alpha_n < \kappa$ such that $\alpha_{n+1} < \alpha_n$ and $\varepsilon_p(F_{\alpha_n}) < 1/n$. If some point $y$ of $X$ has $x_{\alpha_n} \leq y < p$ for each $n$, then for each $n$ we have $0 < \varepsilon_p([y, \rightarrow)) < \varepsilon_p([x_{\alpha_n}, \rightarrow)) < 1/n$, which is impossible. Hence $p$ is the limit of a sequence $z_n = x_{\alpha_n}$ from $(\leftarrow, p)$. If $(\leftarrow, p]$ is open, then $\{(z_n, p) | n \geq 1\}$ is a neighborhood base at $p$. If $(\leftarrow, p]$ is not open, we can obtain a sequence $w_1 > w_2 > \cdots$ having $p$ as its limit, and then $\{(z_n, w_n) | n \geq 1\}$ is a local base at $p$. Other cases are handled analogously. □

2.2. PROPOSITION. Any GO-space with a capacity is perfect.

Proof. Let $U$ be any open set and let $\mathcal{V} = \{V_\alpha | \alpha \in A\}$ be the family of all convex components of $U$. For each $\alpha \in A$ choose $p_\alpha \in V_\alpha$. Then $\varepsilon_{p_\alpha}(\overline{V_\alpha}) > 0$. Let $P_n = \{p_\alpha | \varepsilon_{p_\alpha}(\overline{V_\alpha}) \geq 1/n\}$. We claim that $P_n$ is a closed, discrete subspace of $X$. Obviously $P_n$ is discrete-in-itself. We show $P_n$ is closed. Let $q$ be a limit point of $P_n$. Since $X$ is first-countable, there is a strictly monotonic sequence $(q_k)$ from $P_n$ whose limit is $q$, say $q_k = p_{\alpha_k}$. Let $F = \{q\} \cup (\bigcup \{\overline{V_{\alpha_k}} | k \geq 1\})$. Then $F$ is a closed set and, by $(C_3)$, $\varepsilon_q(F) = \lim_{k \to \infty} \varepsilon_{q_{2k}}(F) \geq 1/n$ because $\varepsilon_{q_{2k}}(F) \geq \varepsilon_{q_{2k}}(\overline{V_{\alpha_{2k}}}) \geq 1/n$. Hence $q \in \text{Int}(F)$. But the sequence $\{q_{2k+1} | k \geq 1\}$ also converges to $q$ and no term of that sequence lies in $F$, contradicting $q \in \text{Int}(F)$. Hence $P_n$ is closed and discrete.

Since $X$ is first countable, each set $V_\alpha \in \mathcal{V}$ is an $F_\sigma$-set so we may find closed convex sets $D(\alpha, k)$ having $p_\alpha \in D(\alpha, 1) \subset D(\alpha, 2) \subset \cdots$ and $\bigcup \{D(\alpha, k) | k \geq 1\} = V_\alpha$. Let $E(n, k) = \bigcup \{D(\alpha, k) | p_\alpha \in P_n\}$. Since $P_n$ is closed and discrete, each $E(n, k)$ is closed, and $U = \bigcup \mathcal{V} = \bigcup \{E(n, k) | n \geq 1, k \geq 1\}$. □

REMARK. Corollary 4.3 below provides an even stronger conclusion than does Proposition 2.2.

2.3. LEMMA. Suppose $(\leftarrow, p]$ is not open. Let $\delta > 0$. Then there is a point $q > p$ such that for each $t \in [p, q]$, $\varepsilon_t([p, q]) < \delta$.

Proof. Since $p$ is a limit point of $(p, \rightarrow)$ there is a sequence $b_1 > b_2 > \cdots$ whose limit is $p$. Then $0 = \varepsilon_p(\leftarrow, p]) = \inf\{\varepsilon_p(\leftarrow, b_n] | n \geq 1\}$ so that for some $n_0$, $\varepsilon_p(\leftarrow, b_{n_0}]) < \delta$. Then $\varepsilon_p([p, b_{n_0}]) < \delta$. Now assume no point $q$, as described in the Lemma, exists. Let $c_0 = b_{n_0}$. Then there is a
point \( t_0 \in [p, c_0] \) with \( \epsilon_{t_0}([p, c_0]) \geq \delta \). Necessarily, \( p < t_0 \). Let \( c_1 = \min\{b_{n_0+1}, t_0\} \) and find \( t_1 \in [p, c_1] \) with \( \epsilon_{t_1}([p, c_1]) \geq \delta \). In general, find a point \( t_{k+1} \in [p, c_{k+1}] \) with \( \epsilon_{t_{k+1}}([p, c_{k+1}]) \geq \delta \), where \( c_{k+1} = \min\{b_{n_0+k+1}, t_k\} \). If \( m \) is fixed and \( k > m \), \( p < c_k < c_m \) and so \( \epsilon_{t_k}([p, c_m]) \geq \epsilon_{t_k}([p, c_k]) \geq \delta \). Letting \( k \to \infty \), we obtain \( \epsilon_p([p, c_m]) = \lim_k \epsilon_{t_k}([p, c_m]) \geq \delta \). But \( c_m \leq b_{n_0+m} < b_{n_0} \) so we obtain \( \delta \leq \epsilon_p([p, c_m]) \leq \epsilon_p([p, b_{n_0}]) < \delta \), a contradiction. \( \square \)

**Remark.** There is an obvious analogue of (2.3) in case \([p, \to)\) is not open.

2.4. **Lemma.** Suppose neither \((\leftarrow, p]\) nor \([p, \to)\) is open (i.e., \( p \) is a two-sided limit point of \( X \)). Let \( \delta > 0 \). Then there are points \( q \) and \( r \) with \( q < p < r \) having the property that for every \( t \in [q, r] \), \( \epsilon_t([q, r]) < \delta \).

**Proof.** The proof is analogous to the proof of (2.3). \( \square \)

2.5. **Notation.** Let \((X, \mathcal{T}, <)\) be a GO-space. Let

\[
\begin{align*}
R &= \{x \in X | [x, \to) \text{ is open}\}, \\
L &= \{x \in X | (\leftarrow, x] \text{ is open}\}, \\
I &= \{x \in X | \{x\} \text{ is open}\}, \\
R^* &= R - I, \text{ and} \\
L^* &= L - I.
\end{align*}
\]

2.6. **Lemma.** Assume \( X \) is a GO-space having a capacity. Each of the sets defined in (2.5) is an \( F_\sigma \)-set.

**Proof.** In the light of (2.2), \( I \) is an \( F_\sigma \)-set since \( I \) is open. If we can show that \( R \) is an \( F_\sigma \)-set, then so is \( R^* \) because \( R^* = R - I \).

To show that \( R \) is an \( F_\sigma \)-set, observe that for each \( x \in R \), \( \epsilon_x([x, \to)) > 0 \). Let \( R_n = \{x \in R | \epsilon_x([x, \to)) \geq 1/n \} \). Suppose \( p \) is a limit point of \( R_n \). Choose a strictly monotonic sequence \( \langle x_k \rangle \) from \( R_n \) whose limit is \( p \). There are two cases.

**Case 1.** Suppose \( x_1 < x_2 < \cdots \). Then \( [p, \to) = \cap \{[x_k, \to) | k \geq 1\} \) so that \( \epsilon_p([p, \to)) = \inf \{\epsilon_p([x_k, \to)) | k \geq 1\} \). If \( k \) is fixed and \( m > k \) then \( x_k < x_m \) so that \( \epsilon_{x_m}([x_k, \to)) \geq \epsilon_{x_m}([x_m, \to)) \geq 1/n \). Letting \( m \to \infty \), we obtain \( \epsilon_p([x_k, \to)) = \lim \epsilon_{x_m}([x_k, \to)) \geq 1/n \). Hence \( \epsilon_p([p, \to)) \geq 1/n \). But then \( p \) must be an interior point of \([p, \to)\) so that the increasing sequence \( \langle x_k \rangle \) could not have converged to \( p \).
Case 2. Suppose \( x_1 > x_2 > \cdots \). According to (C3), \( \epsilon_p([p, \to]) = \lim_k \epsilon_{x_k}([p, \to]) \). Since \( p < x_k, \epsilon_{x_k}([p, \to]) \geq \epsilon_{x_k}([x_k, \to]) \geq 1/n \). Hence \( \epsilon_p([p, \to]) \geq 1/n \). But then \( p \) must be an interior point of \([p, \to]\) so that \( p \in R \). Hence \( p \in R_n \) as required.

Analogously, \( L \) and \( L^* \) are \( F_o \) sets. \( \square \)

3. \( G_\delta \)-diagonals. Ceder [6] observed that the diagonal of space \( X \) is a \( G_\delta \)-subset of \( X \times X \) if there are open coverings \( \delta(n) \) of \( X \) (for \( n \geq 1 \)) such that given \( x \neq y \) in \( X \), \( \text{St}(x, \delta(n)) \subset X - \{y\} \) for some \( n \). In perfect spaces, a weaker condition suffices. The proof of the next lemma is easy.

3.1. Lemma. Suppose \( X \) is perfect. Then \( X \) has a \( G_\delta \)-diagonal if there is a countable family \( \Psi \) such that

(a) each \( \delta \in \Psi \) is a collection of open subsets of \( X \), and,
(b) given \( x \neq y \) in \( X \), some \( \delta \in \Psi \) has \( x \in \text{St}(x, \delta) \subset X - \{y\} \).

3.2. Lemma. Suppose \( X \) is a GO-space with a capacity. Then there is a countable family \( \Psi_R \) such that

(a) each \( \delta \in \Psi_R \) is a collection of open subsets of \( X \), and,
(b) given \( x \in R \) and \( y \neq x \), some \( \delta \in \Psi_R \) has \( x \in \text{St}(x, \delta) \subset X - \{y\} \).

Proof. Let \( \delta_0 = \{\{x\} \mid x \in I\} \). For \( n \geq 1 \) and for \( p \in R^* \), use Lemma (2.3) to find a point \( q(p, n) > p \) such that for every \( t \in [p, q(p, n)] \), \( \epsilon([p, q(p, n)]) < 1/n \). Let \( \delta(n) = \{[p, q(p, n)] \mid p \in R^* \} \). Next, use Lemma (2.6) to write \( L = \cup \{L_k \mid k \geq 1\} \) where each \( L_k \) is closed in \( X \), and notice that \( R^* \cap L = \emptyset \). Now define, for \( n \geq 1 \), \( \delta(-n) = X - L_n \). We let \( \Psi_R = \{\delta(n) \mid n \) is any integer\}.

Fix \( x \in R \) and \( y \neq x \). If \( x \in I \), then \( \text{St}(x, \delta(0)) = \{x\} \subset X - \{y\} \) as required, so assume \( x \in R - I = R^* \). Let \( J \) be the convex hull of the two-point set \( \{x, y\} \). There are two cases.

Case 1. If there is some point \( t \) having \( \epsilon_t(J) > 0 \), find a positive integer \( n \) having \( \epsilon_t(J) > 1/n \). Since \( x \in R^* \), \( [x, q(x, n)] \in \delta(n) \) so that \( x \in \text{St}(x, \delta(n)) \). Suppose some member \([p, q(p, n)]\) of \( \delta(n) \) contains both \( x \) and \( y \). By convexity, \( J \subset [p, q(p, n)] \) so we have \( \epsilon_t(J) \leq \epsilon_t([p, q(p, n)]) \). But \( t \in [p, q(p, n)] \) so that \( 1/n > \epsilon_t([p, q(p, n)]) \geq \epsilon_t(J) > 1/n \), which is impossible. Hence \( y \notin \text{St}(x, \delta(n)) \).

Case 2. If there is no point \( t \) in \( X \) such that \( \epsilon_t(J) > 0 \), then \( y < x \), because if \( x < y \) we would have \([x, y] = [x, \to] \cap (\leftarrow, y) \), so \( x \) would be an interior point of \( J \), whence \( \epsilon_x(J) > 0 \). Since \( y < x \) and since no point \( t \)
of $X$ lies strictly between $x$ and $y$, we conclude that $(\leftarrow, y] = (\leftarrow, x)$ is open. Thus $y \in L$. Choose $n$ so that $y \in L_n$. Because $R^* \cap L_n = \emptyset$, $x \in \text{St}(x, \emptyset(-n)) = X - L_n \subset X - \{y\}$, as required. \hfill \Box

3.3. Remark. Suppose $X$ is a GO-space with a capacity. There is an analogue of (3.2) which constructs a countable family $\Psi_L$ of open collections such that if $x \in L$ and $y \in X - \{x\}$, then some $\emptyset \in \Psi_L$ has $x \in \text{St}(x, \emptyset) \subset X - \{y\}$.

3.4. Lemma. Suppose $X$ is a GO-space with a capacity. Let $E = X - (R \cup L \cup I)$. Then there is a countable family $\Psi_E$ such that
(a) each $\emptyset \in \Psi_E$ is a collection of open subsets of $X$, and,
(b) if $x \in E$ and if $y \in X - \{x\}$, then for some $\emptyset \in \Psi_E$, $x \in \text{St}(x, \emptyset) \subset X - \{y\}$.

Proof. For each $p \in E$, use Lemma (2.4) to select points $a(p, n) < p < b(p, n)$ such that for each $t \in [a(p, n), b(p, n)], e_t([a(p, n), b(p, n)]) < 1/n$. For $n \geq 1$, let $\emptyset(n) = \{[a(p, n), b(p, n)] | p \in E\}$, and let $\Psi_E = \{\emptyset(n) | n \geq 1\}$. The proof that $\Psi_E$ satisfies (b) above is similar to, but even easier than, the proof that $\Psi_R$ satisfies (b) of (3.2). \hfill \Box

3.5. Theorem. Any GO-space with a capacity has a $G_\delta$-diagonal.

Proof. Using the collections found in (3.2)–(3.4) let $\Psi = \Psi_R \cup \Psi_L \cup \Psi_E$. Then $\Psi$ satisfies the hypotheses of (3.1) so that, since $X$ is perfect in the light of (2.2), $X$ has a $G_\delta$-diagonal. \hfill \Box

3.6. Corollary (Ščepin). Any LOTS with a capacity is metrizable.

Proof. Any LOTS with a $G_\delta$-diagonal is metrizable [10]. \hfill \Box

4. Some results on perfect spaces. There are two old questions which concern perfect GO-spaces. The first is due to R. W. Heath, and the second was posed by M. Maurice and J. van Wouwe.

(H) Find a real example of a perfect GO-space which has a point-countable base and yet is not metrizable.

(MvW) Find a real example of a perfect GO-space which does not have a $\sigma$-discrete dense subset.

(These questions ask for "real examples", i.e., examples in ZFC, since if there is a Souslin line, then there is a counterexample to each [2], [13], [15].)
In this section we show that no counterexample to (H) or to (MvW) can have a capacity.

It is known that any GO-space having a σ-discrete dense subset is perfect [15]. We begin this section by proving the converse for GO-spaces having a capacity, thereby strengthening (2.2). We need the following result, due to Przymusiński [1].

4.1. Proposition. Let \((X, \mathcal{T}, <)\) be a GO-space having a \(G_δ\)-diagonal. Then there is a topology \(\mathcal{M}\) on \(X\) such that:
   
   (a) \((X, \mathcal{M})\) is metrizable;
   
   (b) \(\mathcal{M} \subseteq \mathcal{T}\);
   
   (c) \((X, \mathcal{M}, <)\) is a GO-space.

4.2. Theorem. Suppose \(X\) is a perfect GO-space having a \(G_δ\)-diagonal. Then \(X\) has a σ-discrete dense subset.

Proof. Let \(\mathcal{T}\) and \(<\) be, respectively, the topology and ordering of \(X\). Use (4.1) to find a metrizable GO-topology \(\mathcal{M} \subseteq \mathcal{T}\). Let \(D\) be a σ-discrete dense subset of the metric space \((X, \mathcal{M})\) and let \(I = \{x|\{x\} \in \mathcal{T} - \mathcal{M}\}\). Then \(D\) is also σ-discrete in \((X, \mathcal{T})\) and \(I\) is an \(F_δ\) in \((X, \mathcal{T})\), whence \(I\) is also σ-discrete in \((X, \mathcal{T})\). Let \(E = D \cup I\).

Now let \(W\) be any nonvoid open set. If \(W \cap I \neq \emptyset\) then \(W \cap E \neq \emptyset\), so assume \(W\) contains no isolated points. Then there are points \(a < b\) in \(W\) such that \(\emptyset \neq (a, b) \subseteq W\). But then \((a, b) \in \mathcal{M}\) so \((a, b) \cap D \neq \emptyset\). Hence \(W \cap E \neq \emptyset\), as required.

4.3. Corollary. Any GO-space with a capacity has a σ-discrete dense set.

Proof. Combine (2.2), (3.5) and (4.2).

4.4. Corollary. Any GO-space with a capacity has a dense metrizable subspace.

Proof. The σ-discrete dense set \(D\) found in (4.3) is, in its relative topology, semistratifiable in the sense of Creede [7] and any semistratifiable GO-space is metrizable [11].
To show that no counterexample to (MvW) can have a capacity we prove a bit more, namely:

4.5. **Theorem.** Let $X$ be a GO-space having a $\sigma$-discrete dense set and a point-countable base. Then $X$ is metrizable.

**Proof.** Since any GO-space having a $\sigma$-discrete dense set is perfect and paracompact [15], it will be enough to show that a space $X$ which satisfies the hypotheses of (4.5) has a $\sigma$-disjoint base. Then $X$ is quasi-developable [3] and perfect, so $X$ is developable [3]. But a developable paracompact space is metrizable.

Let $D = \bigcup \{D(n) \mid n \geq 1\}$ be a $\sigma$-discrete dense subset of $X$. A standard argument [Prop. 3.4, 5] provides a $\sigma$-disjoint base for points of $D$. Let $I$ be the set of isolated points of $X$ (so $I \subset D$). Let $R^*$ and $L^*$ be as in (2.5) and let $E = X - (R^* \cup L^* \cup I)$. A standard argument shows that the collection $\mathcal{V} = \bigcup \{\mathcal{V}_n \mid n \geq 1\}$, where $\mathcal{V}_n$ is the collection of convex components of $X - D(n)$, contains a $\sigma$-disjoint base for all points of $E$. Therefore it suffices to find $\sigma$-disjoint collections $\mathcal{C}$ and $\mathcal{C}'$ which contain neighborhood bases for all points of $R^* - D$ and $L^* - D$, respectively. We show how to find $\mathcal{C}$.

Let $\mathcal{B}$ be a point-countable base for $X$, and let $\mathcal{V} = \bigcup \{\mathcal{V}_n \mid n \geq 1\}$ be as above. For $n \geq 1$ and $V \in \mathcal{V}_n$, let $\mathcal{B}_n(V) = \{B \cap V \mid B \in \mathcal{B}\}$ and for some $p \in R^* \cap V$, $(\{p, \to\} \cap V) \subset B \subset \{p, \to\}$). Let $\mathcal{B}_n = \bigcup \{\mathcal{B}_n(V) \mid V \in \mathcal{V}_n\}$ and $\mathcal{B} = \bigcup \{\mathcal{B}_n \mid n \geq 1\}$. Then we have

1. $\mathcal{B}$ is point-countable, and
2. $\mathcal{B}$ contains a neighborhood base at each point of $R^* - D$.

Fix $n$ and $V \in \mathcal{V}_n$. For each $P \in \mathcal{B}_n(V)$ there is a unique $y_P \in P \cap V$ having $P = \{y_P, \to\} \cap V$. Let $C(n, V) = \{y_P \mid P \in \mathcal{B}_n(V)\}$ and choose $S(n, V) = \{x(V, \alpha) \mid \alpha < \kappa(V)\}$, a cofinal strictly increasing subset of $C(n, V)$. Because $\mathcal{B}_n(V)$ is point-countable, we have

3. If $\alpha < \kappa(V)$ then $|C(n, V) \cap (\leftarrow, x(V, \alpha))] \leq \omega_0$.

For each $y \in C(n, V)$, let $\alpha(n, V, y)$ be the first index $\beta < \kappa(V)$ such that $y < x(V, \beta)$ and define

$$\mathcal{C}(n, V, \alpha) = \{[y, x(V, \alpha)] \mid y \in C(n, V) \text{ and } \alpha(n, V, y) = \alpha\}.$$  

If $V \neq W$ belong to $\mathcal{V}(n)$ or if $V = W$ and $\alpha \neq \beta$, then $\mathcal{C}(n, V, \alpha) \cap \mathcal{C}(n, W, \beta) = \emptyset$. Furthermore,

4. each $\mathcal{C}(n, V, \alpha)$ is countable.
Index $C(n, V, \alpha)$ as \{${C(n, V, \alpha, k) \mid k \geq 1}$\} and let $C'(n, k) = \{C(n, V, \alpha, k) \mid V \in \mathcal{V}_n, \alpha < \kappa(V)\}$. Then we have

5. the family $\mathcal{C} = \bigcup \{C(n, V, \alpha) \mid n \geq 1, V \in \mathcal{V}_n, \text{ and } \alpha < \kappa(V)\}$ has $\mathcal{C} = \bigcup \{C'(n, k) \mid n \geq 1, k \geq 1\}$, so that $\mathcal{C}$ is $\sigma$-disjoint.

It remains only to show that $\mathcal{C}$ contains a neighborhood base at each point of $R^* - D$. Fix $p \in R^* - D$ and $r > p$. Find $B \in \mathfrak{B}$ with $p \in B \subset [p, r]$. Because $p \notin I$ we may find $q > p$ with $(p, q) \subset B \subset [p, r)$ and $(p, q) \neq \varnothing$. Choose $n$ so that $(p, q) \cap D(n) \neq \varnothing$ and choose $d \in (p, q) \cap D(n)$. Because $p \in R - D$, some convex component $V \in \mathcal{V}_n$ contains $p$. Then $V \subset \langle \leftarrow, d \rangle$ and so

$p \in [p, \rightarrow) \cap V \subset [p, \rightarrow) \cap \langle \leftarrow, d \rangle \subset [p, d) \subset [p, q) \subset B \subset [p, \rightarrow),$ i.e., the set $Q = B \cap V$ belongs to $\mathfrak{B}_n(V)$. The unique point $y_Q$ with $Q = \{y_Q, \rightarrow) \cap V$ is $y_Q = p$, so $p \in C(n, V)$. Compute $\alpha = \alpha(n, V, p)$. Then $[p, x(V, \alpha)) \in C(n, V, \alpha) \in \mathcal{C}$ and $[p, x(V, \alpha)) \subset Q \subset B \subset [p, r)$. Hence $\mathcal{C}$ contains a neighborhood base at each point of $R^* - D$, as required.

4.6. COROLLARY. Any GO-space having a capacity and a point-countable base is metrizable.

Theorem 2.1 of [4] shows that a perfect GO-space with a $\delta\theta$-base has a point-countable base. Hence we have:

4.7. COROLLARY. Any GO-space having a capacity and a $\delta\theta$-base is metrizable.

We conclude this section by pointing out that, in the light of (4.5), any counterexample for (H) is also a counterexample of the type required in (MvW).

5. Examples.

5.1 It is easy to see that the Sorgenfrey line [3] has a capacity. Thus, Theorem (3.5) cannot be strengthened to assert that a GO-space with a capacity is metrizable.

5.2 No uncountable subspace of the Michael line [3, 11] can have a capacity unless it is metrizable. For if $X$ is an uncountable subspace of the Michael line, then $X$ is quasi-developable since it has a $\sigma$-disjoint base [11]. If $X$ had a capacity then $X$ would be perfect (2.2) and perfect quasi-developable space is developable [3]. But a developable GO-space is metrizable. (We remark that, under (MA + \neg CH), there are uncountable
5.3 It is not true that a perfect GO-space with a $G_δ$-diagonal and a $σ$-discrete dense set must have a capacity. Let $X$ be the GO-space obtained from the usual real line $ℝ$ by making the half-line $(x, +∞)$ open whenever $x$ is irrational and using the usual open interval neighborhoods for rational numbers. Then $X$ is separable and has a $G_δ$-diagonal. However the set $R = \{ x \in X \mid (x, +∞) \text{ is open} \}$ is not an $F_σ$-subset of $X$, so $X$ does not have a capacity.

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