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GENERALIZED ORDERED SPACES WITH CAPACITIES

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We show that any *GO*-space having a capacity in the sense of Ščepin has a G_δ-diagonal and is perfect. In addition, such a space has a σ-discrete dense subset and a dense metrizable subspace, and any *GO*-space having a capacity and a point-countable base (or having a σ-discrete dense subset and a point-countable base) is metrizable.

1. Introduction. In **[14]** Ščepin defined a *capacity* for a space X to be a family of functions $\{\varepsilon_x \mid x \in X\}$ such that, for each closed $F \subset X$,

(C_1) $\varepsilon_x(F)$ is a non-negative real number with $\varepsilon_x(F) > 0$ iff $x \in \mathrm{Int}(F)$,

(C_2) if $F_1 \subset F_2$ are closed then $\varepsilon_x(F_1) \leq \varepsilon_x(F_2)$,

(C_3) for a fixed closed F, the function $x \to \varepsilon_x(F)$ is continuous,

(C_4) for a fixed x, if $\{F_\alpha \mid \alpha < \kappa\}$ is a family of closed sets satisfying $F_\alpha \supset F_\beta$ whenever $\alpha < \beta < \kappa$, then $\varepsilon_x(\bigcap_\alpha F_\alpha) = \inf_\alpha \varepsilon_x(F_\alpha)$.

In that same paper Ščepin announced without proof that a linearly ordered topological space (LOTS) having a capacity is metrizable. The purpose of this note is to prove a more general result from which Ščepin's result follows immediately, namely, that any *GO*-space (= suborderable space) with a capacity has a G_δ-diagonal. (Recall that the class of *GO*-spaces is precisely the class of subspaces of LOTS.) Along the way to that result, we show that any *GO*-space with a capacity is *perfect* (i.e., closed sets are G_δ). In §4 we will discuss two old questions about perfect *GO*-spaces in the context of *GO*-spaces having a capacity, proving that a *GO*-space with a capacity has a σ-discrete dense subset and a *GO*-space with a capacity and a point-countable base must be metrizable. Finally, examples in §5 show that our results are sharp.

Terminology and notation not defined in this paper will follow **[8, 11, 12]**.

2. Preliminary results and perfect normality. We proceed via a sequence of lemmas.

2.1. LEMMA. *Any GO-space having a capacity is a first-countable space.*

Proof. Fix a non-isolated point p of X. If $[p, \rightarrow)$ is not open then $\varepsilon_p[p, \rightarrow) = 0$ and there is a well-ordered, strictly increasing net $\{x_\alpha \mid \alpha < \kappa\}$ whose supremum is p. Let $F_\alpha = [x_\alpha, \rightarrow)$. According to ($C_4$), $0 = \varepsilon_p([p, \rightarrow)) = \inf\{\varepsilon_p(F_\alpha) \mid \alpha < \kappa\}$. For each n, choose $\alpha_n < \kappa$ such that $\alpha_{n-1} < \alpha_n$ and $\varepsilon_p(F_{\alpha_n}) < 1/n$. If some point y of X has $x_{\alpha_n} \leq y < p$ for each n, then for each n we have $0 < \varepsilon_p([y, \rightarrow)) < \varepsilon_p([x_{\alpha_n}, \rightarrow)) < 1/n$, which is impossible. Hence p is the limit of a sequence $z_n = x_{\alpha_n}$ from $(\leftarrow, p)$. If $(\leftarrow, p]$ is open, then $\{(z_n, p] \mid n \geq 1\}$ is a neighborhood base at p. If $(\leftarrow, p]$ is not open, we can obtain a sequence $w_1 > w_2 > \cdots$ having p as its limit, and then $\{(z_n, w_n) \mid n \geq 1\}$ is a local base at p. Other cases are handled analogously. □

2.2. PROPOSITION. *Any GO-space with a capacity is perfect.*

Proof. Let U be any open set and let $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ be the family of all convex components of U. For each $\alpha \in A$ choose $p_\alpha \in V_\alpha$. Then $\varepsilon_{p_\alpha}(\overline{V}_\alpha) > 0$. Let $P_n = \{p_\alpha \mid \varepsilon_{p_\alpha}(\overline{V}_\alpha) \geq 1/n\}$. We claim that P_n is a closed, discrete subspace of X. Obviously P_n is discrete-in-itself. We show P_n is closed. Let q be a limit point of P_n. Since X is first-countable, there is a strictly monotonic sequence $\langle q_k \rangle$ from P_n whose limit is q, say $q_k = p_{\alpha_k}$. Let $F = \{q\} \cup (\bigcup\{\overline{V}_{\alpha_{2k}} \mid k \geq 1\})$. Then F is a closed set and, by (C_3), $\varepsilon_q(F) = \lim_{k\to\infty} \varepsilon_{q_{2k}}(F) \geq 1/n$ because $\varepsilon_{q_{2k}}(F) \geq \varepsilon_{q_{2k}}(\overline{V}_{\alpha_{2k}}) \geq 1/n$. Hence $q \in \mathrm{Int}(F)$. But the sequence $\{q_{2k+1} \mid k \geq 1\}$ also converges to q and no term of that sequence lies in F, contradicting $q \in \mathrm{Int}(F)$. Hence P_n is closed and discrete.

Since X is first countable, each set $V_\alpha \in \mathcal{V}$ is an F_σ-set so we may find closed convex sets $D(\alpha, k)$ having $p_\alpha \in D(\alpha, 1) \subset D(\alpha, 2) \subset \cdots$ and $\bigcup\{D(\alpha, k) \mid k \geq 1\} = V_\alpha$. Let $E(n, k) = \bigcup\{D(\alpha, k) \mid p_\alpha \in P_n\}$. Since P_n is closed and discrete, each $E(n, k)$ is closed, and $U = \bigcup \mathcal{V} = \bigcup\{E(n, k) \mid n \geq 1, k \geq 1\}$. □

REMARK. Corollary 4.3 below provides an even stronger conclusion than does Proposition 2.2.

2.3. LEMMA. *Suppose $(\leftarrow, p]$ is not open. Let $\delta > 0$. Then there is a point $q > p$ such that for each $t \in [p, q]$, $\varepsilon_t([p, q]) < \delta$.*

Proof. Since p is a limit point of $(p, \rightarrow)$ there is a sequence $b_1 > b_2 > \cdots$ whose limit is p. Then $0 = \varepsilon_p((\leftarrow, p]) = \inf\{\varepsilon_p((\leftarrow, b_n]) \mid n \geq 1\}$ so that for some n_0, $\varepsilon_p((\leftarrow, b_{n_0}]) < \delta$. Then $\varepsilon_p([p, b_{n_0}]) < \delta$. Now assume no point q, as described in the Lemma, exists. Let $c_0 = b_{n_0}$. Then there is a point $t_0 \in [p, c_0]$ with $\varepsilon_{t_0}([p, c_0]) \geq \delta$. Necessarily, $p < t_0$. Let $c_1 = \min\{b_{n_0+1}, t_0\}$ and find $t_1 \in [p, c_1]$ with $\varepsilon_{t_1}([p, c_1]) \geq \delta$. In general, find a point $t_{k+1} \in [p, c_{k+1}]$ with $\varepsilon_{t_{k+1}}([p, c_{k+1}]) \geq \delta$, where $c_{k+1} = \min\{b_{n_0+k+1}, t_k\}$. If m is fixed and $k > m$, $p < c_k < c_m$ and so $\varepsilon_{t_k}([p, c_m]) \geq \varepsilon_{t_k}([p, c_k]) \geq \delta$. Letting $k \to \infty$, we obtain $\varepsilon_p([p, c_m]) = \lim_k \varepsilon_{t_k}([p, c_m]) \geq \delta$. But $c_m \leq b_{n_0+m} < b_{n_0}$ so we obtain $\delta \leq \varepsilon_p([p, c_m]) \leq \varepsilon_p([p, b_{n_0}]) < \delta$, a contradiction. □

REMARK. There is an obvious analogue of (2.3) in case $[p, \to)$ is not open.

2.4. LEMMA. *Suppose neither* $(\leftarrow, p]$ *nor* $[p, \to)$ *is open* (*i.e.*, p *is a two-sided limit point of* X). *Let* $\delta > 0$. *Then there are points* q *and* r *with* $q < p < r$ *having the property that for every* $t \in [q, r]$, $\varepsilon_t([q, r]) < \delta$.

Proof. The proof is analogous to the proof of (2.3). □

2.5. NOTATION. Let $(X, \mathfrak{T}, <)$ be a *GO*-space. Let
$R = \{x \in X \mid [x, \to) \text{ is open}\}$,
$L = \{x \in X \mid (\leftarrow, x] \text{ is open}\}$,
$I = \{x \in X \mid \{x\} \text{ is open}\}$,
$R^* = R - I$, and
$L^* = L - I$.

2.6. LEMMA. *Assume* X *is a GO-space having a capacity. Each of the sets defined in* (2.5) *is an* F_σ-*set*.

Proof. In the light of (2.2), I is an F_σ-set since I is open. If we can show that R is an F_σ-set, then so is R^* because $R^* = R - I$.

To show that R is an F_σ-set, observe that for each $x \in R$, $\varepsilon_x([x, \to)) > 0$. Let $R_n = \{x \in R \mid \varepsilon_x([x, \to)) \geq 1/n\}$. Suppose p is a limit point of R_n. Choose a strictly monotonic sequence $\langle x_k \rangle$ from R_n whose limit is p. There are two cases.

Case 1. Suppose $x_1 < x_2 < \cdots$. Then $[p, \to) = \bigcap\{[x_k, \to) \mid k \geq 1\}$ so that $\varepsilon_p([p, \to)) = \inf\{\varepsilon_p([x_k, \to)) \mid k \geq 1\}$. If k is fixed and $m > k$ then $x_k < x_m$ so that $\varepsilon_{x_m}([x_k, \to)) \geq \varepsilon_{x_m}([x_m, \to)) \geq 1/n$. Letting $m \to \infty$, we obtain $\varepsilon_p([x_k, \to)) = \lim \varepsilon_{x_m}([x_k, \to)) \geq 1/n$. Hence $\varepsilon_p([p, \to)) \geq 1/n$. But then p must be an interior point of $[p, \to)$ so that the increasing sequence $\langle x_k \rangle$ could not have converged to p.

Case 2. Suppose $x_1 > x_2 > \cdots$. According to (C_3), $\varepsilon_p([p, \rightarrow)) = \lim_k \varepsilon_{x_k}([p, \rightarrow))$. Since $p < x_k$, $\varepsilon_{x_k}([p, \rightarrow)) \geq \varepsilon_{x_k}([x_k, \rightarrow)) \geq 1/n$. Hence $\varepsilon_p([p, \rightarrow)) \geq 1/n$. But then p must be an interior point of $[p, \rightarrow)$ so that $p \in R$. Hence $p \in R_n$ as required.

Analogously, L and L^* are F_σ sets. □

3. G_δ-diagonals. Ceder [6] observed that the diagonal of space X is a G_δ-subset of $X \times X$ if there are open coverings $\mathcal{G}(n)$ of X (for $n \geq 1$) such that given $x \neq y$ in X, $\mathrm{St}(x, \mathcal{G}(n)) \subset X - \{y\}$ for some n. In perfect spaces, a weaker condition suffices. The proof of the next lemma is easy.

3.1. LEMMA. *Suppose X is perfect. Then X has a G_δ-diagonal if there is a countable family Ψ such that*

(a) *each $\mathcal{G} \in \Psi$ is a collection of open subsets of X, and,*

(b) *given $x \neq y$ in X, some $\mathcal{G} \in \Psi$ has $x \in \mathrm{St}(x, \mathcal{G}) \subset X - \{y\}$.*

3.2. LEMMA. *Suppose X is a GO-space with a capacity. Then there is a countable family Ψ_R such that*

(a) *each $\mathcal{G} \in \Psi_R$ is a collection of open subsets of X, and,*

(b) *given $x \in R$ and $y \neq x$, some $\mathcal{G} \in \Psi_R$ has $x \in \mathrm{St}(x, \mathcal{G}) \subset X - \{y\}$.*

Proof. Let $\mathcal{G}_0 = \{\{x\} \mid x \in I\}$. For $n \geq 1$ and for $p \in R^*$, use Lemma (2.3) to find a point $q(p, n) > p$ such that for every $t \in [p, q(p, n)]$, $\varepsilon_t([p, q(p, n)]) < 1/n$. Let $\mathcal{G}(n) = \{[p, q(p, n)) \mid p \in R^*\}$. Next, use Lemma (2.6) to write $L = \bigcup\{L_k \mid k \geq 1\}$ where each L_k is closed in X, and notice that $R^* \cap L = \varnothing$. Now define, for $n \geq 1$, $\mathcal{G}(-n) = \{X - L_n\}$. We let $\Psi_R = \{\mathcal{G}(n) \mid n \text{ is any integer}\}$.

Fix $x \in R$ and $y \neq x$. If $x \in I$, then $\mathrm{St}(x, \mathcal{G}(0)) = \{x\} \subset X - \{y\}$ as required, so assume $x \in R - I = R^*$. Let J be the convex hull of the two-point set $\{x, y\}$. There are two cases.

Case 1. If there is some point t having $\varepsilon_t(J) > 0$, find a positive integer n having $\varepsilon_t(J) > 1/n$. Since $x \in R^*$, $[x, q(x, n)) \in \mathcal{G}(n)$ so that $x \in \mathrm{St}(x, \mathcal{G}(n))$. Suppose some member $[p, q(p, n))$ of $\mathcal{G}(n)$ contains both x and y. By convexity, $J \subset [p, q(p, n)]$ so we have $\varepsilon_t(J) \leq \varepsilon_t([p, q(p, n)])$. But $t \in [p, q(p, n)]$ so that $1/n > \varepsilon_t([p, q(p, n)]) \geq \varepsilon_t(J) > 1/n$, which is impossible. Hence $y \notin \mathrm{St}(x, \mathcal{G}(n))$.

Case 2. If there is no point t in X such that $\varepsilon_t(J) > 0$, then $y < x$, because if $x < y$ we would have $[x, y) = [x, \rightarrow) \cap (\leftarrow, y)$, so x would be an interior point of J, whence $\varepsilon_x(J) > 0$. Since $y < x$ and since no point t

of X lies strictly between x and y, we conclude that $(\leftarrow, y] = (\leftarrow, x)$ is open. Thus $y \in L$. Choose n so that $y \in L_n$. Because $R^* \cap L_n = \varnothing$, $x \in \mathrm{St}(x, \mathcal{G}(-n)) = X - L_n \subset X - \{y\}$, as required. □

3.3. Remark. Suppose X is a GO-space with a capacity. There is an analogue of (3.2) which constructs a countable family Ψ_L of open collections such that if $x \in L$ and $y \in X - \{x\}$, then some $\mathcal{G} \in \Psi_L$ has $x \in \mathrm{St}(x, \mathcal{G}) \subset X - \{y\}$.

3.4. Lemma. *Suppose X is a GO-space with a capacity. Let $E = X - (R \cup L \cup I)$. Then there is a countable family Ψ_E such that*

(a) *each $\mathcal{G} \in \Psi_E$ is a collection of open subsets of X, and,*

(b) *if $x \in E$ and if $y \in X - \{x\}$, then for some $\mathcal{G} \in \Psi_E$, $x \in \mathrm{St}(x, \mathcal{G}) \subset X - \{y\}$.*

Proof. For each $p \in E$, use Lemma (2.4) to select points $a(p, n) < p < b(p, n)$ such that for each $t \in [a(p, n), b(p, n)]$, $\varepsilon_t([a(p, n), b(p, n)]) < 1/n$. For $n \geq 1$, let $\mathcal{G}(n) = \{(a(p, n), b(p, n)) \mid p \in E\}$, and let $\Psi_E = \{\mathcal{G}(n) \mid n \geq 1\}$. The proof that Ψ_E satisfies (b) above is similar to, but even easier than, the proof that Ψ_R satisfies (b) of (3.2). □

3.5. Theorem. *Any GO-space with a capacity has a G_δ-diagonal.*

Proof. Using the collections found in (3.2)–(3.4) let $\Psi = \Psi_R \cup \Psi_L \cup \Psi_E$. Then Ψ satisfies the hypotheses of (3.1) so that, since X is perfect in the light of (2.2), X has a G_δ-diagonal. □

3.6. Corollary (*Ščepin*). *Any LOTS with a capacity is metrizable.*

Proof. Any LOTS with a G_δ-diagonal is metrizable [**10**]. □

4. Some results on perfect spaces. There are two old questions which concern perfect GO-spaces. The first is due to R. W. Heath, and the second was posed by M. Maurice and J. van Wouwe.

(H) Find a real example of a perfect GO-space which has a point-countable base and yet is not metrizable.

(MvW) Find a real example of a perfect GO-space which does not have a σ-discrete dense subset.

(These questions ask for "real examples", i.e., examples in ZFC, since if there is a Souslin line, then there is a counterexample to each [**2**], [**13**], [**15**].)

In this section we show that no counterexample to (H) or to (MvW) can have a capacity.

It is known that any *GO*-space having a σ-discrete dense subset is perfect [**15**]. We begin this section by proving the converse for *GO*-spaces having a capacity, thereby strengthening (2.2). We need the following result, due to Przymusiński [**1**].

4.1. PROPOSITION. *Let* $(X, \mathfrak{T}, <)$ *be a GO-space having a* G_δ*-diagonal. Then there is a topology* $\mathfrak{M}$ *on* X *such that*:

(a) $(X, \mathfrak{M})$ *is metrizable*;

(b) $\mathfrak{M} \subset \mathfrak{T}$;

(c) $(X, \mathfrak{M}, <)$ *is a GO-space.*

4.2. THEOREM. *Suppose* X *is a perfect GO-space having a* G_δ*-diagonal. Then* X *has a* σ*-discrete dense subset.*

Proof. Let $\mathfrak{T}$ and $<$ be, respectively, the topology and ordering of X. Use (4.1) to find a metrizable *GO*-topology $\mathfrak{M} \subset \mathfrak{T}$. Let D be a σ-discrete dense subset of the metric space $(X, \mathfrak{M})$ and let $I = \{x \mid \{x\} \in \mathfrak{T} - \mathfrak{M}\}$. Then D is also σ-discrete in $(X, \mathfrak{T})$ and I is an F_σ in $(X, \mathfrak{T})$, whence I is also σ-discrete in $(X, \mathfrak{T})$. Let $E = D \cup I$.

Now let W be any nonvoid open set. If $W \cap I \neq \varnothing$ then $W \cap E \neq \varnothing$, so assume W contains no isolated points. Then there are points $a < b$ in W such that $\varnothing \neq (a, b) \subset W$. But then $(a, b) \in \mathfrak{M}$ so $(a, b) \cap D \neq \varnothing$. Hence $W \cap E \neq \varnothing$, as required. □

4.3. COROLLARY. *Any GO-space with a capacity has a* σ*-discrete dense set.*

Proof. Combine (2.2), (3.5) and (4.2). □

4.4. COROLLARY. *Any GO-space with a capacity has a dense metrizable subspace.*

Proof. The σ-discrete dense set D found in (4.3) is, in its relative topology, semistratifiable in the sense of Creede [7] and any semistratifiable *GO*-space is metrizable [**11**]. □

To show that no counterexample to (MvW) can have a capacity we prove a bit more, namely:

4.5. THEOREM. *Let X be a GO-space having a σ-discrete dense set and a point-countable base. Then X is metrizable.*

Proof. Since any *GO*-space having a σ-discrete dense set is perfect and paracompact [**15**], it will be enough to show that a space X which satisfies the hypotheses of (4.5) has a σ-disjoint base. Then X is quasi-developable [**3**] and perfect, so X is developable [**3**]. But a developable paracompact space is metrizable.

Let $D = \bigcup\{D(n) \mid n \geq 1\}$ be a σ-discrete dense subset of X. A standard argument [Prop. 3.4, **5**] provides a σ-disjoint base for points of D. Let I be the set of isolated points of X (so $I \subset D$). Let R^* and L^* be as in (2.5) and let $E = X - (R^* \cup L^* \cup I)$. A standard argument shows that the collection $\mathcal{V} = \bigcup\{\mathcal{V}_n \mid n \geq 1\}$, where $\mathcal{V}_n$ is the collection of convex components of $X - D(n)$, contains a σ-disjoint base for all points of E. Therefore it suffices to find σ-disjoint collections $\mathcal{C}$ and $\mathcal{C}'$ which contain neighborhood bases for all points of $R^* - D$ and $L^* - D$, respectively. We show how to find $\mathcal{C}$.

Let $\mathcal{B}$ be a point-countable base for X, and let $\mathcal{V} = \bigcup\{\mathcal{V}_n \mid n \geq 1\}$ be as above. For $n \geq 1$ and $V \in \mathcal{V}_n$, let $\mathcal{P}_n(V) = \{B \cap V \mid B \in \mathcal{B}$ and for some $p \in R^* \cap V$, $([p, \rightarrow) \cap V) \subset B \subset [p, \rightarrow)\}$. Let $\mathcal{P}_n = \bigcup\{\mathcal{P}_n(V) \mid V \in \mathcal{V}_n\}$ and $\mathcal{P} = \bigcup\{\mathcal{P}_n \mid n \geq 1\}$. Then we have

1. $\mathcal{P}$ is point-countable, and
2. $\mathcal{P}$ contains a neighborhood base at each point of $R^* - D$.

Fix n and $V \in \mathcal{V}_n$. For each $P \in \mathcal{P}_n(V)$ there is a unique $y_P \in P \cap V$ having $P = [y_P, \rightarrow) \cap V$. Let $C(n, V) = \{y_P \mid P \in \mathcal{P}_n(V)\}$ and choose $S(n, V) = \{x(V, \alpha) \mid \alpha < \kappa(V)\}$, a cofinal strictly increasing subset of $C(n, V)$. Because $\mathcal{P}_n(V)$ is point-countable, we have

3. If $\alpha < \kappa(V)$ then $|C(n, V) \cap (\leftarrow, x(V, \alpha))| \leq \omega_0$.

For each $y \in C(n, V)$, let $\alpha(n, V, y)$ be the first index $\beta < \kappa(V)$ such that $y < x(V, \beta)$ and define

$$\mathcal{C}(n, V, \alpha) = \{[y, x(V, \alpha)) \mid y \in C(n, V) \text{ and } \alpha(n, V, y) = \alpha\}.$$

If $V \neq W$ belong to $\mathcal{V}(n)$ or if $V = W$ and $\alpha \neq \beta$, then $\mathcal{C}(n, V, \alpha) \cap \mathcal{C}(n, W, \beta) = \varnothing$. Furthermore,

4. each $\mathcal{C}(n, V, \alpha)$ is countable.

Index $\mathcal{C}(n, V, \alpha)$ as $\{C(n, V, \alpha, k) \mid k \geq 1\}$ and let $\mathcal{C}'(n, k) = \{C(n, V, \alpha, k) \mid V \in \mathcal{V}_n, \alpha < \kappa(V)\}$. Then we have

5. the family $\mathcal{C} = \bigcup\{\mathcal{C}(n, V, \alpha) \mid n \geq 1,\ V \in \mathcal{V}_n$, and $\alpha < \kappa(V)\}$ has $\mathcal{C} = \bigcup\{\mathcal{C}'(n, k) \mid n \geq 1, k \geq 1\}$, so that $\mathcal{C}$ is σ-disjoint.

It remains only to show that $\mathcal{C}$ contains a neighborhood base at each point of $R^* - D$. Fix $p \in R^* - D$ and $r > p$. Find $B \in \mathcal{B}$ with $p \in B \subset [p, r[$. Because $p \notin I$ we may find $q > p$ with $[p, q) \subset B \subset [p, r)$ and $(p, q) \neq \varnothing$. Choose n so that $(p, q) \cap D(n) \neq \varnothing$ and choose $d \in (p, q) \cap D(n)$. Because $p \in R - D$, some convex component $V \in \mathcal{V}_n$ contains p. Then $V \subset (\leftarrow, d)$ and so

$$p \in [p, \rightarrow) \cap V \subset [p, \rightarrow) \cap (\leftarrow, d) \subset [p, d) \subset [p, q) \subset B \subset [p, \rightarrow),$$

i.e., the set $Q = B \cap V$ belongs to $\mathcal{P}_n(V)$. The unique point y_Q with $Q = [y_Q, \rightarrow) \cap V$ is $y_Q = p$, so $p \in C(n, V)$. Compute $\alpha = \alpha(n, V, p)$. Then $[p, x(V, \alpha)) \in \mathcal{C}(n, V, \alpha) \in \mathcal{C}$ and $[p, x(V, \alpha)) \subset Q \subset B \subset [p, r)$. Hence $\mathcal{C}$ contains a neighborhood base at each point of $R^* - D$, as required. □

4.6. COROLLARY. *Any GO-space having a capacity and a point-countable base is metrizable.* □

Theorem 2.1 of [**4**] shows that a perfect GO-space with a $\delta\theta$-base has a point-countable base. Hence we have:

4.7. COROLLARY. *Any GO-space having a capacity and a $\delta\theta$-base is metrizable.* □

We conclude this section by pointing out that, in the light of (4.5), any counterexample for (H) is also a counterexample of the type required in (MvW).

5. Examples.

5.1 It is easy to see that the Sorgenfrey line [**3**] has a capacity. Thus, Theorem (3.5) cannot be strengthened to assert that a GO-space with a capacity is metrizable. □

5.2 No uncountable subspace of the Michael line [**3, 11**] can have a capacity unless it is metrizable. For if X is an uncountable subspace of the Michael line, then X is quasi-developable since it has a σ-disjoint base [**11**]. If X had a capacity then X would be perfect (2.2) and perfect quasi-developable space is developable [**3**]. But a developable GO-space is metrizable. (We remark that, under (MA + $\neg$CH), there are uncountable subsets of the Michael line M which are metrizable; indeed Theorem (4.1) of [9] shows that any subspace X of M with $|X| < c$ is metrizable.) □

5.3 It is not true that a perfect *GO*-space with a G_δ-diagonal and a σ-discrete dense set must have a capacity. Let X be the *GO*-space obtained from the usual real line $\mathbf{R}$ by making the half-line $[x, \rightarrow)$ open whenever x is irrational and using the usual open interval neighborhoods for rational numbers. Then X is separable and has a G_δ-diagonal. However the set $R = \{x \in X \mid [x, \rightarrow) \text{ is open}\}$ is not an F_σ-subset of X, so X does not have a capacity. □

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