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Wolfgang Wollny

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WOLFGANG WOLLNY

In the second part of his eighteenth problem Hilbert formulated: "A fundamental region of each group of Euclidean motions together with all its congruent copies evidently gives rise to a covering of the space without gaps. The question arises as to the existence of such polyhedra which cannot be fundamental regions of any group of motions, but nevertheless furnish such a covering of the total space by congruent reiteration." Following ideas of Heesch this question of Hilbert's will be analysed in detail in this article, restricting to the case of two dimensions — the Euclidean plane E^2.

A. Introduction. After some necessary terminology has been introduced the above question is subdivided into two special questions: (1) Do there exist tiles B_t forming homogeneous tilings but having only non-trivial stabilizers? (2) Do there exist tiles B_t, which tile the E^2 but cannot form a homogeneous tiling? Positive answers to both special questions are obtained for the case of coverings by non-compact tiles, in terms of construction rules for special types of regions in the form of strips. These complete the list of previously published examples. Subsequently, a new method is developed. By systematic subdivision of the above strips, new types (both non-compact and compact) will be constructed also yielding a positive answer to Hilbert's original question.

A positive answer of a special kind can be given immediately for the case of a fundamental region B of some discrete group $\mathfrak{G}$, if $B \neq \bar{B}$ (closure of B): For this case the polyhedron $\bar{B}$ can fill space under the "sharply transitive" action of $\mathfrak{G}$, and yet is not a fundamental region of $\mathfrak{G}$, since its boundaries contain equivalent points.

K. Reinhardt in 1928 was the first to answer the Hilbert question affirmatively by giving a 3-dimensional multiconnected polyhedron. H. Heesch gave a similar answer in 1935 for E^2 by constructing a special decagon (cf. [**3**]). Further results given by Heesch are cited at the end of §C, after the detailed analysis has been carried out. For a survey of the development in the Hilbert problem and its generalizations we refer to [**2**].

B. Definitions.

DEFINITION 1. A *tiling* Π of E^2 is a covering of E^2, without overlap, by any set of compact or non-compact topological disks (called tiles). In the special case that each tile of Π is congruent to a fixed region B, called the *prototile* of Π, Π is called *monohedral.*

In this paper the term "tiling" is used in the latter sense. The boundaries of the tiles are required to be piecewise differentiable. A peculiar case obtains if there is a tiling Π^h in which the infinite configurations round each tile are congruent. In this case Π^h is called *homogeneous* and an equivalent definition may be given:

DEFINITION 2. A tiling Π^h is called *homogeneous* if there exists at least one group $\mathfrak{G}$ acting transitively on Π^h.

Two homogeneous tilings Π_1^h and Π_2^h are said to belong to the same *topological class* if there exists a homomorphism Φ: $E^2 \to E^2$ with $\Phi(\Pi_1^h) = \Pi_2^h$.

From Definition 2 it follows that homogeneous tilings of the same topological class are topologically indistinguishable. For two homogeneous tilings with *compact* tiles the latter case obtains if they have the same cycle z defining the number k and order n_i of vertices of a tile within the tiling Π^h counterclockwise around its boundary. There is

DEFINITION 3. A *vertex* of Π is a boundary point of $n_i > 2$ tiles of Π; n_i is called the *order* of the vertex.

In the case of homogeneous tilings with compact tiles the only possible values for n_i are 3, 4, 6, 8 and 12, and for k 3, 4, 5 and 6. Furthermore there is the following theorem[1]: If a group $\mathfrak{G}$ acts transitively on a homogeneous tiling with *compact* tiles, then $\mathfrak{G}$ belongs to a type of the 17 *plane two-dimensional crystallographic* ones. It should be noted: If a group $\mathfrak{G}$ belonging to a type of this list acts transitively on a homogeneous tiling Π^h, the prototile of Π^h must not be compact (cf. Fig. 1a). Besides this for homogeneous tilings with non-compact tiles there are only the following possibilities (cf. [7]):

(i) $\mathfrak{G}$ has a subgroup of translations isomorphic to $\mathbf{R}$ ($\mathfrak{G}$ is then called *semi-discrete*) and the tiles have the form of parallel strips or semi-planes:

(ii) $\mathfrak{G}$ belongs to one of the seven types of discrete groups with only one direction for translations; or

(iii) $\mathfrak{G}$ is a cyclic group of rotations $\mathfrak{C}_n$ ($n = 2, 3, 4, \ldots$) or a dihedral group $\mathfrak{D}_m$ ($m = 1, 2, \ldots$) with a unique fixed-point.

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[1] The proof of the theorem is e.g. given by A. Speiser (*Die Theorie der Gruppen von endlicher Ordnung*, Berlin, 1937, p. 95).

It should be mentioned that most of the types of discrete groups involve free parameters (such as the ratio of the lengths of translation vectors, the angle between them, etc.) yielding one or several infinities of geometrical varieties of fundamental regions for each group of Euclidean motions (cf. [**1**]). We remark that the 17 discrete 2-dimensional types of groups are abstractly distinct, while the 7 discrete 1-dimensional types of groups only yield 4 distinct abstract ones.

In order to mark uniquely the transitive action of a group $\mathfrak{G}$ on a homogeneous tiling Π^h, it is appropriate to incorporate the diagrams of the symmetry system $S_{\mathfrak{G}}$ of $\mathfrak{G}$, defined by Niggli, into Π^h.

DEFINITION 4. The *symmetry system* $S_{\mathfrak{G}}$ of a discrete group $\mathfrak{G}$ of E^2 is the totality of its symmetry elements, where a *symmetry element* of $\mathfrak{G}$ is the set of points in E^2 left fixed by an element of $\mathfrak{G}$, that is:

1. the centre of a rotation,
2. the axis of a reflection,
3. the axis of a glide-reflection

(for the appropriate types of motions occurring in $\mathfrak{G}$).

We see that translational vectors do not belong to $S_{\mathfrak{G}}$. Rotations can be distinguished graphically by special symbols for their centres: lenses (digons), triangles, squares or hexagons mean that the corresponding rotations have periods 2, 3, 4 or 6. The axes of reflections are represented by lines, those of glide-reflections by dotted or broken lines.

If there are equal symmetry elements in $S_{\mathfrak{G}}$ which are inequivalent under the group $\mathfrak{G}$, they are distinguished graphically, e.g. two inequivalent rotations through 180° by a hollow and a striped lens (,), two inequivalent axes of glide-reflection by different dottings of the lines.

Another possibility of specifying the transitive action of a group $\mathfrak{G}$ upon a Π^h uniquely is given by the cycle Z of neighbor-transformations of some tile B of Π^h.

DEFINITION 5. The *cycle Z of neighbor-transformations* of a tile $B \in \Pi^h$ is the totality of transformations by which B is made congruent with its neighboring tiles (tiles possessing more than *one* boundary point with B in common) going counterclockwise around its boundary.

In the case that B has, say, stabilizer $\mathfrak{F}$ and neighbors B_i $(i = 1, \dots, m)$,

$$Z = \{N_1\mathfrak{F}, N_2\mathfrak{F}, \dots, N_m\mathfrak{F}\}^2 \quad \text{with } N_i\colon B \to B_i.$$

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2 Products of group elements are read from right to left in this paper.

If Z is given for a homogeneous tiling with *compact* tiles, explicit mention of $\mathfrak{G}$ is redundant, because for this case Z is a complete system of generators for $\mathfrak{G}$ (cf. [**9**]). In certain homogeneous tilings with *non-compact* tiles, however, Z is only a set of generators for a proper subgroup of the group $\mathfrak{G}$ of the tiling (cf. [**10**]).

Henceforth we shall distinguish between the type of a group, denoted by $\mathfrak{g}_i$ (e.g. p1, p2, etc.) and a special (geometrical) realization of $\mathfrak{g}_i$, denoted by $\mathfrak{G}_i$. For each homogeneous tiling Π^h there exists (by definition) at least one group $\mathfrak{G}$ of Euclidean motions acting transitively upon it. $\mathfrak{G}$ may belong to type $\mathfrak{g}$. If there is only one geometrical possibility for such a realization, the pair $(\Pi^h, \mathfrak{g})$ is single-valued. If further there exists only a single type $\mathfrak{g}$ with respect to Π^h, specific reference to $\mathfrak{g}$ can be dropped and Π^h itself can be interpreted as a special geometrical model $\mathfrak{G}$ for $\mathfrak{g}$: $\mathfrak{g}$ is realized as $\mathfrak{G}$ by Π^h (cf. Fig. 1b). If, however, there are different geometrical possibilities for transitive actions of groups $\mathfrak{G}_i$ $(i = 1, 2, \ldots, n)$ belonging to the same type $\mathfrak{g}$, the pair $(\Pi^h, \mathfrak{g})$ is many-valued. E.g. there are 3 different cases for groups of type p2 with respect to the homogeneous tiling $\tilde{\Pi}^h$ given in Fig. 2 (a, b, c), the unique determination of their actions being achieved by incorporating $S_{\mathfrak{G}_i}$ $(i = 1, 2, 3)$ into $\tilde{\Pi}^h$. Moreover there is still one possibility for a group of type p1 upon $\tilde{\Pi}^h$ (cf. Fig. 2d).

In order to classify homogeneous tilings, the concept of the corresponding symmetry group $\overline{\mathfrak{G}}$ of such a tiling Π^h is introduced: If there are in general different types $\mathfrak{g}_i$ $(i = 1, 2, \ldots, m)$, each of them moreover yielding different realizations $\mathfrak{G}_{ij}$ $(j = 1, 2, \ldots, n(i))$ for transitive action on Π^h, one group out of this list $\mathfrak{C}$ — this notation will still be used later — is distinguished, namely the largest group of Euclidean motions under which Π^h is self-congruent. This group shall be called the *corresponding symmetry of* Π^h and denoted $\overline{\mathfrak{G}}$. It is evident that $\overline{\mathfrak{G}}$ is uniquely determined by the geometrical properties of Π^h. In order to characterize $\overline{\mathfrak{G}}$ uniquely, it is necessary (1) to give its type (p1, p2, etc.), and (2) to mark its transitive action upon Π^h. (1) and (2) together are, e.g., given by incorporating the symmetry system $S_{\overline{\mathfrak{g}}}$ into Π^h. It can be shown easily (cf. [**10**]) that $\overline{\mathfrak{G}}$ contains all other possible groups of Π^h as subgroups and acts upon Π^h in such a way that the stabilizer of a tile of Π^h is maximal compared with all other possible actions of groups on Π^h.

For example, for the tiling $\Pi^h_\square$ by squares with $z = (4, 4, 4, 4)$, there is $\overline{\mathfrak{G}}$ of type p4m in a geometrical realization upon $\Pi^h_\square$ given in Fig. 3a with e.g. 4-fold centres of rotation in the centres of the tiles, the tiles having stabilizer $\mathfrak{D}_4$. Besides this case there are two other possibilities for geometrical realizations of a group of type p4m acting transitively on $\Pi^h_\square$, the tiles having stabilizers $\mathfrak{D}_1$ or $\mathfrak{D}_2$, respectively (cf. Fig. 3b and c). It should be noted that there are altogether 36 different possibilities for transitive actions on $\Pi^h_\square$ (cf. [**9**]).

For the tiling $\tilde{\Pi}^h$ of Fig. 2, there is $\overline{\mathfrak{G}}$ of type p2 geometrically realized on $\tilde{\Pi}^h$ in such a way that $\mathfrak{C}_2$ is the stabilizer of the tiles (cf. Fig. 2b).

In the special case that $(\Pi^h, \mathfrak{g})$ is single-valued and $\mathfrak{g}$ is the only possible type for Π^h, the above list $\mathfrak{C}$ contains only one element — the corresponding symmetry group $\overline{\mathfrak{G}}$ of Π^h (for example see Fig. 1a or 1b).

To distinguish homogeneous tilings, the following definition is made.

Definition 6. Two homogeneous tilings Π^h_i $(i = 1, 2)$ belong to the same *class* if:

(1) they belong to the same topological class,

(2) their corresponding symmetry groups $\overline{\mathfrak{G}}_i$ $(i = 1, 2)$ belong to the same type $\mathfrak{g}$;

(3) the homeomorphism Φ: $E^2 \to E^2$ with $\Phi(\Pi^h_1) = \Pi^h_2$ (existing by (1)) can be coupled with an automorphism f: $\overline{\mathfrak{G}}_1 \to \overline{\mathfrak{G}}_2$ in such a way that for each $h \in \overline{\mathfrak{G}}_1$ and each tile $B \in \Pi^h_1$,

$$\Phi(hB) = f(h) \cdot \Phi(B).$$

(3) means that the groups $\overline{\mathfrak{G}}_i$ $(i = 1$, resp. $2)$ act on the tilings Π^h_i $(i = 1$, resp. $2)$ in the same geometrical manner.

C. Detailed analysis. For any homogeneous tiling Π^h one can now consider the list $\mathfrak{C}$ of groups $\{\mathfrak{G}_{i,j(i)}\}$ defined above in §B, and with respect to each $\mathfrak{G}_{ij} \in \mathfrak{C}$ the corresponding stabilizer $\mathfrak{F}_{ij}$ of a tile B of Π^h. For B there is the alternative of having

(α) a trivial stabilizer $\mathfrak{F}_{ij}$, i.e. $\mathfrak{F}_{ij} = \mathfrak{E}$ (unit group) or

(β) a non-trivial stabilizer $\mathfrak{F}_{ij}$, i.e. $\mathfrak{F}_{ij} \neq \mathfrak{E}$.

Only in case (α) does $\mathfrak{G}_{ij}$ operate sharply transitively on Π^h (there is only one transformation $t \in \mathfrak{G}_{ij}$, t: $B \to B_i$ $(B, B_i \in \Pi^h)$). Then B can be chosen as a fundamental region of $\mathfrak{G}_{ij}$. In case (β) B cannot be a fundamental region of $\mathfrak{G}_{ij}$, but can be chosen as a union of $f > 1$ fundamental regions of $\mathfrak{G}_{ij}$, f being the cardinality of $\mathfrak{F}_{ij}$.

The Hilbert question cited in the introduction now leads to the following two subquestions:

(1) Do there exist tiles forming at least one homogeneous tiling Π^h but having for each possible Π^h and $\mathfrak{G}$ of Π^h only non-trivial stabilizers?

(2) Do there exist tiles which tile the plane but cannot form any homogeneous tiling?

The subquestion (1) can be discussed by restricting to the set of tiles forming at least one homogeneous tiling.

We consider a tile B with a non-trivial stabilizer $\mathfrak{F}$ within a homogeneous tiling Π^h with group $\mathfrak{G}$. $\mathfrak{G}$ is decomposed into cosets with respect to $\mathfrak{F}$ (say left cosets):

$$(+) \qquad \mathfrak{G} = h_1\mathfrak{F} + h_2\mathfrak{F} + \cdots \qquad (h_1 = e, \text{ the identity element}).$$

The order and index of $\mathfrak{F}$ may be finite or infinite. All elements of the same coset $h_\iota\mathfrak{F}$ carry B onto the same tile B_ι. These comprise exactly f different transformations, where f is the cardinality of $\mathfrak{F}$. If a representative is chosen out of each coset, we get a system of elements, say $\{h_1, h_2, \ldots\}$, called a *system of* (*left*) *representatives*. By elements of this set B may be carried onto each other tile of Π^h (and that by exactly one element), because $\mathfrak{G}$ acts transitively on Π^h.

A special situation obtains if the system of representatives $\{h_1, h_2, \ldots\}$ can be chosen so that they form a group $\mathfrak{H}$. Then the following theorem holds:

THEOREM 1. *If the system* $\mathfrak{H} = \{h_1, h_2, \ldots\}$ *of coset representatives is a group*, $\mathfrak{H}$ *acts sharply transitive upon* Π^h.

Proof. The transitive action of $\mathfrak{H}$ follows from the transitive action of $\mathfrak{G}$, as has been shown above. It has to be shown that every tile of Π^h has a trivial stabilizer with respect to $\mathfrak{H}$: If $\mathfrak{M}$ is any group acting transitively on any set X, the stabilizers of the elements of X are always conjugate in $\mathfrak{M}$. Therefore since, by construction, the tile B has a trivial stabilizer with respect to $\mathfrak{H}$, so does every tile in Π^h. □

In the case that $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$, the subgroups $\mathfrak{H} \subset \mathfrak{G}$ and $\mathfrak{F} \subset \mathfrak{G}$ are said to be complementary (or complements of each other) in $\mathfrak{G}$.

DEFINITION 7. Two subgroups $\mathfrak{U}$ and $\mathfrak{V}$ of a group $\mathfrak{G}$ are said to be *complementary in* $\mathfrak{G}$ if the elements of $\mathfrak{G}$ can be brought into matrix scheme such that the rows are left (or right) cosets of $\mathfrak{U}$ in $\mathfrak{G}$ and the columns are right (or left) cosets of $\mathfrak{V}$ in $\mathfrak{G}$.

For example, the dihedreal group $\mathfrak{D}_6$ can be decomposed as follows:

$$\mathfrak{D}_6 = \mathfrak{C}_6 + D_1\mathfrak{C}_6 = \mathfrak{D}_1 + \mathfrak{D}_1 C_6 + \mathfrak{D}_1 C_6^2 + \cdots + \mathfrak{D}_1 c_6^5$$

(C_6: rotation through $2\pi/6$ about O; D_1: reflection in a line through O; $\mathfrak{C}_6 := \langle C_6 \rangle$; $\mathfrak{D}_1 := \langle D_1 \rangle$). The groups $\mathfrak{C}_6 = \{E, C_6, C_6^2, \ldots, C_6^5\}$ and $\mathfrak{D}_1 = \{E, D_1\}$ therefore are complementary, and the matrix scheme is

E	C_6	C_6^2	C_6^3	C_6^4	C_6^5
D_1	D_1C_6	$D_1C_6^2$	$D_1C_6^3$	$D_1C_6^4$	$D_1C_6^5$

The columns are right cosets of $\mathfrak{D}_1$ in $\mathfrak{D}_6$ and the rows left cosets of $\mathfrak{C}_6$ in $\mathfrak{D}_6$. We can conclude from this: given a homogeneous tiling Π^h associated with $\mathfrak{D}_6$, the tiles being a union of fundamental regions of $\mathfrak{D}_6$ with stabilizer $\mathfrak{D}_1$, Π^h is also associated with $\mathfrak{C}_6$, the tiles being fundamental regions of $\mathfrak{C}_6$.

There is another definition equivalent to Definition 7:

DEFINITION 8. Two subgroups $\mathfrak{U}$ and $\mathfrak{V}$ of a group $\mathfrak{G}$ are said to be *complementary in* $\mathfrak{G}$ whenever $\mathfrak{U} \cap \mathfrak{V} = \mathfrak{E}$ and $\mathfrak{U} \cdot \mathfrak{V} = \mathfrak{G}$ ($\mathfrak{U} \cap \mathfrak{V}$ denotes the meet of $\mathfrak{U}$ and $\mathfrak{V}$ and $\mathfrak{U} \cdot \mathfrak{V}$ their product (the set of all elements of the form PQ with $P \in \mathfrak{U}$ and $Q \in \mathfrak{V}$)).

The special Hilbert question (1) can now be answered affirmatively if the following geometrical situation can be realized:

Let Π^h be a homogeneous tiling with prototile B, on which the group $\mathfrak{G}$ acts transitively, B having non-trivial stabilizer $\mathfrak{F}$. There is no system of coset representatives of $\mathfrak{F}$ forming a group, and this still holds for other possible groups acting upon Π^h and other possible homogeneous tilings with B.

Concerning the Hilbert question the following answers are known. Let the following 4 classes of tiles of E^2 be defined:

K_{IA}: compact tiles } forming at least one homogeneous
K_{IB}: non-compact tiles } tiling but having only non-trivial stabilizers.
K_{IIA}: compact tiles } forming exclusively non-homogeneous
K_{IIB}: non-compact tiles } tilings of E^2.

Heesch (cf. [4]) showed K_{IA} to be empty. He constructed the complete list of compact tiles forming at least one homogeneous tiling by developing two different methods, a topological and a grouptheoretical one (cf. [4]). A detailed analysis of the homogeneous tilings with prototiles of this list led to the result cited above. The peculiarity that stabilizers of compact tiles within homogeneous tilings always have a complement in $\mathfrak{G}$ is caused by the special structure of the 17 crystallographic types of groups. Thus if a tiling with compact tiles has a transitive group, it also has a sharply transitive group.

Heesch, however, showed K_{IA} to be non-empty for the *sphere* S^2 by giving a homogeneous tiling Π^* of the S^2 consisting of 20 regular spherical triangles. The non-trivial stabilizer $\mathfrak{C}_3$ of these tiles does not have a complement in the icosahedral group, the group of Π^*, and the tiles do not form any other tiling of the S^2.

Besides this Heesch constructed tiles of the class K_{IIA}, finding the special heptagon (cf. [**4**]) and the decagon, cited above, by trial and error. Moreover he was probably the first to publish tiles belonging to K_{IB} and K_{IIB} (cf. [**4**]).]

D. Results. Next, new types of tiles belonging to the classes K_{IB}, K_{IIA}, K_{IIB} and completing those published in the literature will be constructed according to a certain scheme. A remarkable consequence is the existence of tiles of class K_{IIB} having a *symmetry group* of infinite or finite order.

We shall distinguish between the *stabilizer* $\mathfrak{F}$ and the *symmetry group* $\mathfrak{S}$ of a tile B:

The *stabilizer* $\mathfrak{F}$ of B within a homogeneous tiling Π^h is defined relative to the respective group $\mathfrak{G}$ of Π^h: $\mathfrak{F}$ is the largest subgroup of $\mathfrak{G}$ under the elements of which B is self-congruent. The *symmetry group* $\mathfrak{S}$ of B is the largest group of Euclidean motions under which B as a geometrical figure is self-congruent. Evidently we have $\mathfrak{F} \subseteq \mathfrak{S}$.

The tiles of K_{IB} in this article were obtained by the author by constructing the complete list of non-compact tiles belonging to at least one homogeneous tiling (cf. [**10**]). The examples of K_{IIB} are obtained from these by destroying the respective group $\mathfrak{G}$ of the tiling by incompatible geometrical realization of its subgroups, at the first stage the tiles still having a non-trivial symmetry group, and at the second stage the tiles being asymmetrical.

As a first example we consider a tiling Π whose tiles are in the shape of strips having a boundary consisting of two separate, incongruent curves k and k^+ (see Fig. 4). Both k and k^+ have a 1-dimensional group of glide-reflections as symmetry group, i.e. the largest group of Euclidean motions under which the curve is self-congruent. These groups of k and k^+ are geometrically realized in the form $\mathfrak{H} = \langle T \rangle + G\langle T \rangle$ and $\mathfrak{H}^+ = \langle T^+ \rangle + G^+ \langle T^+ \rangle$, respectively ($T$, T^+ : translations; G, G^+ : glide-reflections along the axes g_{-1}, g_1 with $G^2 = T$, $G^{+2} = T^+$, respectively).

The boundary curves k and k^+ are called *glide-reflection carriers*. A segment of k generating k by repeated application of $\mathfrak{H}$ is called a *glide-reflection segment*.

The neighbor-transformations of a tile $B_0 \in \Pi$ onto its two neighbors B_{-1} and B must be glide reflections along two parallel lines because of the special geometrical form of k and k^+. For each association of Π with a group $\mathfrak{G}$ leading to a homogeneous tiling, $\mathfrak{G}$ has to be a group possessing two inequivalent series of parallel axes of glide-reflection in its symmetry system, i.e. a 2-dimensional group. Whether such association is possible or not depends on the geometrical realization of k relative to k^+. The symmetries of k and k^+ have to fit together in a special way described thereafter if a group is to act transitively on a tiling Π with the prototile B. For the geometrical realization of k relative to k^+ there are three different possibilities leading to fundamentally different situations in the tilings built up by B.

Concerning the length of the translations T, T^+ within the glide-reflection group $\mathfrak{H}$, $\mathfrak{H}^+$, geometrically realized in k, k^+, respectively, the following cases arise:

1.1 The proportion $|T|:|T^+|$ is rational and the following special property holds: $|T|:|T^+| = m:n$ with m, n being odd coprime integers.
1.2 The proportion $|T|:|T^+|$ is rational, but the quotient in lowest terms does not have the property of 1.1, i.e. one of m, n is even.
1.3 The proportion of $|T|:|T^+|$ is irrational.

THEOREM 2a. *Case* 1.1 *leads to homogeneous tilings with non-compact tiles if their neighbor-transformations are suitably chosen, the tiles having a non-trivial stabilizer, namely a* 1*-dimensional translation group.*

Proof. Because $|T|:|T^+| = m:n$ with m and n being odd coprime integers, a 1-dimensional group $\tilde{\mathfrak{T}} = \langle \tilde{T} \rangle$ of translations with

$$(1) \qquad |\tilde{T}| \equiv n|T| = m|T^+|$$

can be chosen, the elements of which put a tile $B_0 \in \Pi$ into congruence with itself, this being true for k and k^+. (This is also true for any group $\tilde{\tilde{\mathfrak{T}}} = \langle \lambda T \rangle$ ($\lambda > 1$, $\lambda \in \mathbf{N}$)). $\tilde{\mathfrak{T}}$ is a subgroup of the symmetry group of k as well as of k^+. Furthermore groups of glide-reflection $\tilde{\mathfrak{H}}_{-1} \supset \tilde{\mathfrak{T}}$ and $\tilde{\mathfrak{H}}_{+1} \supset \tilde{\mathfrak{T}}$ with the same glide-modulus[3], namely $|\tilde{T}|/2$, can be chosen for k and k^+:

[3] For a glide-reflection σ: $(x, y) \to (-x, y + \mathfrak{g})$ ($g \neq 0$; x, y Cartesian coordinates) $\mathfrak{g}$ is known as the *glide-modulus* of σ.

For k: $\tilde{\mathfrak{H}}_{-1} = \langle \tilde{T} \rangle + \tilde{G}_{-1} \langle \tilde{T} \rangle$,
$\tilde{G}_{-1}$ is a glide-reflection along the axis g_{-1} with glide-modulus $|\tilde{G}_{-1}| = |\tilde{T}|/2$;

For k^+ : $\tilde{\mathfrak{H}}_{+1} = \langle \tilde{T} \rangle + \tilde{G}_{+1} \langle \tilde{T} \rangle$,
$\tilde{G}_{+1}$ is a glide-reflection along the axis g_{+1} with glide-modulus $|\tilde{G}_{+1}| = |\tilde{T}|/2$.

The groups $\tilde{\mathfrak{H}}_{-1}$ and $\tilde{\mathfrak{H}}_{+1}$ exist because there is an odd number of glide-reflection segments with regard to $\mathfrak{H}$, $\mathfrak{H}^+$, respectively between two points of k, k^+ , respectively with distance $|\tilde{T}|/2$, viz. according to (1),

$$|\tilde{G}_{-1}| = |\tilde{G}_{+1}| = \frac{|\tilde{T}|}{2} = n\frac{|T|}{2} = n|G| = m\frac{|T^+|}{2} = m|G^+|$$

(m, n odd coprime integers).

The glide-reflections $\tilde{G}_{-1} \in \tilde{\mathfrak{H}}_{-1}$, $\tilde{G}_{+1} \in \tilde{\mathfrak{H}}_{+1}$ are now chosen to be neighbor-transformations for a tile B_0 of a tiling Π_1^h constructed according to 1.1. $\tilde{G}_{+1}$ and $\tilde{G}_{-1}$ are generators of a group: Because of the validity of $\tilde{G}_{+1}^2 = \tilde{G}_{-1}^2$ $(= \tilde{T})$, the defining relation of a 2-dimensional group $\mathfrak{G}$ of type pg is fulfilled (see [**1**], p. 43). If the neighbor-transformations N_i $(i = 1, 2, \ldots, n)$ of a tile B_0 are generators of a discrete group $\mathfrak{G}$ of E^2, B_0 can be chosen as a fundamental region of $\mathfrak{G}$ or a union of these (see [**1**], 4.5). By equivalent reiteration of B_0 with respect to these elements N_i and their products, a homogeneous tiling belonging to $\mathfrak{G}$ arises. With $N_1 \equiv \tilde{G}_{+1}$ and $N_2 \equiv \tilde{G}_{-1}$ in 1.1, a homogeneous tiling Π_1^h with a group of type pg is obtained, the prototile B_0 of which has a nontrivial stabilizer of infinite order, namely $\tilde{\mathfrak{F}} = \langle \tilde{T} \rangle$, and therefore is non-compact (see Fig. 4). □

Because in 1.1 B_0 has the stabilizer $\tilde{\mathfrak{F}} = \langle \tilde{T} \rangle$, a group of infinite order, there exists an infinity of neighbor-transformations from B_0 to each of its neighbors B_{-1}, B_{+1}, respectively, namely the glide-reflections $\tilde{G}^{\mu}_{\pm 1} = \tilde{G}_{\pm 1}\tilde{T}^{(\mu-1)/2}$ $(\mu = 1, 3, 5, 7, \ldots)$ — $\mu = 1$ is the former case $\tilde{G}^1_{\pm 1} = \tilde{G}_{\pm 1}$. If pairs of neighbor-transformations of B_0 other than $\tilde{G}_{-1}$ and $\tilde{G}_{+1}$, $\tilde{G}^{\mu}_{-1}$ and $\tilde{G}^{\mu}_{+1}$, respectively $(\mu = 3, 5, \ldots)$, were chosen by constructing a tiling Π, e.g. G and G^+ for the case $m : n \neq 1$, Π would not be homogeneous due to the failure of the defining relation for a group of type pg. Besides a homogeneous tiling in 1.1, one may also construct an infinity of non-homogeneous tilings having prototile B_0. An example Π_1 is given in Figure 5.

THEOREM 2b. *The tile $B_0 \in \Pi_1$ belongs to the class K_{IB}.*

Proof. According to Theorem 2a B_0 can be the tile of a homogeneous tiling Π_1^h associated with a group of type pg and having non-trivial stabilizer $\langle \tilde{T} \rangle$. There is no other possibility for homogeneous tilings with the prototile B_0: Neighbor-transformations of B_0 can only be glide-reflections because of the boundary of B_0, consisting of k and k^+. Each group $\mathfrak{G}$ associable with a homogeneous tiling with prototile B_0 therefore must have a series of parallel axes of glide-reflections in its symmetry system S_g, with the specification of neighboring axes of S_g being *inequivalent* with respect to $\mathfrak{G}$ because of the incongruence of k and k^+. $\mathfrak{G}$ must be a 2-dimensional group, because there is no 1-dimensional one with such S_g. By analysing the complete list of the symmetry systems of the 17 types of groups, only two types with this property are found, namely pg and pmg. (The types cm, pgg, etc. only have parallel axes for glide-reflections all being *equivalent*.) The type pmg must be rejected because of the diads[4] lying upon its axes of glide-reflection and demanding rotational symmetries, which k and k^+ do not have. There only remains the type pg. For the transitive action of a group $\mathfrak{G}$ of type pg on a tiling with prototile B_0 there is only the possibility of the tiling Π_1^h above: The tiles have stabilizers $\langle \tilde{T} \rangle$. Therefore the glide-reflections of $\mathfrak{G}$ belong to the cosets of $\langle \tilde{T} \rangle$. But since the product of one of these glides with itself is a nontrivial element of $\langle \tilde{T} \rangle$, no system of coset representatives can form a group. □

THEOREM 3. *Case* 1.2 *always leads to inhomogeneous tilings with tiles having a* 1-*dimensional group of translations as their symmetry group.*

Proof. In case 1.2 a 1-dimensional group $\mathfrak{S} = \langle \tilde{T} \rangle$ of translations with $|\tilde{T}| = n|T| = m|T^+|$ may be found similarly to 1.1, under the elements of which k and k^+ (and also B_0) are self-congruent. It is evident that $\mathfrak{S}$ is the symmetry group of B_0. However it is impossible to find a group $\mathfrak{G}$ of glide-reflections for k and k^+ with $\mathfrak{G} \supset \mathfrak{S}$. This is due to condition 1.2 preventing an *odd* number of glide-reflection segments lying between two points of k as well as of k^+ with distance $|\tilde{T}|/2$. Because of this, association of a tiling according to 1.2 with a group of the unique possible type pg cannot be realized: the defining relation $\tilde{G}_{-1}^2 = \tilde{G}_{+1}^2$ for type pg is not valid because of the special choice of the boundary curves of a single tile. There is also no possibility of B_0 being a fundamental region of a discrete group, which can be shown in analogy to Theorem 2b (cf. Fig. 6). □

[4] 2-fold centres of rotations.

Because of the symmetry group $\mathfrak{S} = \langle \tilde{T} \rangle$ of B_0, there is an infinity of transformations from B_0 onto its neighbors. By congruent repetition of B_0 an infinity of different necessarily inhomogeneous tilings can be built up. The neighbor-transformations will be glide-reflections with glide-modulus an arbitrary odd multiple of the glide-modulus of the corresponding bounding glide-reflection carrier (cf. Fig. 7).

THEOREM 4. *Case* 1.3 *always leads to inhomogeneous tilings with asymmetrical tiles.*

Proof. The prototile B_0 is asymmetrical, since condition 1.3 leads to an incommensurability of the lengths of the translations of the symmetry group of k relative to those of k^+. Because of this there is exactly one transformation carrying B_0 onto each of its neighbors, e.g. B_1. This is a glide-reflection $\hat{G}$. The complete tiling cannot be self-congruent under $\hat{G}$, however, since $\hat{G}$ does not even take B_1 onto B_0. A tiling with B_0 therefore must always be inhomogeneous. (As in 1.2 B_0 leads to an infinity of different inhomogeneous tilings.) (Cf. Fig. 8.) □

We now consider a second tiling $\overline{\Pi}$ whose prototile $\bar{B}_0$ is a strip as in the first example. It differs from this in the substitution of one boundary curve, e.g. k^+, by a curve r having the 1-dimensional rotational group $\mathfrak{J}$ (containing translations and two inequivalent series of rotation centres through 180°) as symmetry group:

$$\mathfrak{J} = \langle T^+ \rangle + C_2 \langle T^+ \rangle$$

(C_2: rotation through 180° about a centre P_d), r being known as a *rotation carrier*. (The symmetry system $S_{\mathfrak{J}}$ is connected with r in Fig. 9.) The symmetry group of the second boundary curve k is as in the former case $\mathfrak{H} = \langle T \rangle + G \langle T \rangle$.

As in the first example certain conditions on the geometrical realization of k relative to r have to be fulfilled in order to make a tiling $\overline{\Pi}$, built up with congruent replicas of $\bar{B}_0$, be transitive with respect to some group $\mathfrak{G}$. The symmetry system of a potential tiling group must have a series of parallel axes of glide-reflections, and parallel to these a series of rows containing centres for rotations through 180°, called diads. This can only be a 2-dimensional group of type pgg, because this is the only type of discrete group possessing subgroups exactly with these symmetry systems in parallel arrangement.

For the length of the translations T, T^+ of the translational subgroups, geometrically realized by k (glide-reflection carrier) and r (rotation carrier) respectively, the following cases must be distinguished, leading to different types of tiles:

2.1 $|T|/|T^+| = m/n$ for coprime integers m, n, n being *odd*.
2.2 $|T|/|T^+| = m/n$ for coprime integers m, n, n being *even*.
2.3 The proportion $|T| : |T^+|$ is irrational.

Only in cases 2.1 and 2.2 does $\overline{B}_0$ have the symmetry group

$$\mathfrak{S} = \langle \tilde{T} \rangle = \langle nT \rangle = \langle mT^+ \rangle.$$

In order to generate a group $\mathfrak{G}$ acting transitively upon a $\overline{\Pi}^h$ with prototile $\overline{B}_0$, the 1-dimensional symmetry group $\mathfrak{S}$ of $\overline{B}_0$ or a subgroup of $\mathfrak{S}$ must be the stabilizer of $\overline{B}_0$ and be completed to a 2-dimensional group by adding suitable neighbor-transformations N_i of $\overline{B}_0$, or $\overline{B}_0$ must be chosen as the fundamental region of a 1-dimensional $\mathfrak{G}$. There are two neighbors of $\overline{B}_0$, $\overline{B}_{-1}$ and $\overline{B}_{+1}$. As N_1 there is a half-turn about any diad of r:

$$N_1 \equiv C_2 : \overline{B}_0 \to \overline{B}_{+1}.$$

Because of k N_2 has to be a glide-reflection $\tilde{G}$ along the axis g_0:

$$N_2 \equiv \tilde{G} : \overline{B}_0 \to \overline{B}_{-1}.$$

Case 2.1. Because in 2.1 $m|T^+| = n|T|$ with m, n coprime integers, n *being odd*, there is a glide-reflection $\tilde{G}$: $\overline{B}_0 \to \overline{B}_{-1}$ along g_0 with glide-modulus

$$(*) \qquad |\tilde{G}| = m\frac{|T^+|}{2} = n\frac{|T|}{2}$$

and $\tilde{G}^2 \in \mathfrak{S}$, $\mathfrak{S}$ therefore being the stabilizer of $\overline{B}_0$. As in 1.1 this case leads to a homogeneous tiling Π_4^h (cf. Fig. 9) with a group of type pgg in 2.1, the prototile $\overline{B}_0 \in \Pi_4^h$ being a union of infinite fundamental regions with a 1-dimensional group of translations as stabilizer. Since no other homogeneous tilings can be built with $\overline{B}_0$ (the proof is similar to 1.1), it follows as in case 1.1 that $\overline{B}_0 \in K_{\mathrm{IB}}$.

Case 2.2. In case 2.2: $m|T^+| = n|T|$ with m, n coprime integers, n *being even*, there is no glide-reflection $\tilde{G}$: $\overline{B}_0 \to \overline{B}_{-1}$ with glide-modulus $|\tilde{G}|$ according to $(*)$ of 2.1 leading to $\tilde{G}^2 \in \mathfrak{S}$. The possible glide-reflections G_i: $\overline{B}_0 \to \overline{B}_{-1}$ along g_0 do not fit together with the symmetry group $\mathfrak{S}$ of

$\bar{B}_0$: Concerning the glide-modulus of G_i there is $|G_i| = i|T|/2$ ($i = 1, 3, 5, \ldots$) leading to $G_i^2 = iT$ (i odd) and, because of this, $G_i^2 \notin \mathfrak{S} = \langle nT \rangle$ (n even). The only remaining possibility of $\bar{B}_0$ being a fundamental region of a 1-dimensional group must also be rejected, the proof being similar to that of Theorem 2b.

In another interpretation, as a consequence of condition 2.2 there is always an *even* number of glide-reflection segments of k between two points of the glide-reflection carrier k with distance $m \cdot |\tilde{T}|/2$ (m odd or even) and, because of this, no tiling constructed by $\bar{B}_0$ could be transitive with respect to a group of the only possible type in question, pgg.

2.2 always yields inhomogeneous tilings, the tiles having translational symmetry as in 2.1, i.e. its group is $\mathfrak{S} = \langle nT \rangle = \langle mT^+ \rangle$ (cf. Fig. 10). That Π_5 is inhomogeneous can be seen directly too: if all axes of glide-reflection and diads (these are the only possible symmetry elements for a correspondence of Π_5 with a group) are incorporated into Π_5, no symmetry system of a group arises: the totality of the rotational centres of groups of type pgg forms a *rectangular* lattice, those incorporated into Π_5, however, form a rhombohedral lattice.

Case 2.3. 2.3 always leads to inhomogeneous tilings as in case 1.3, the tile having no non-trivial symmetries, because the lengths of the translations of $\mathfrak{J}$ and $\mathfrak{H}$ are incommensurable.

Since in 1.2 and 2.2 inhomogeneous tilings had been obtained with tiles B_0, $\bar{B}_0$, respectively having a symmetry group — namely $\mathfrak{S} = \langle \tilde{T} \rangle$ — other inhomogeneous tilings can be constructed by the method of subdividing these tiles.

Subdivision for a tile B with symmetry group $\mathfrak{S}$ means that B has to be divided into congruent regions equivalent with respect to the elements of $\mathfrak{S}$ or $\mathfrak{S}^* \subset \mathfrak{S}$. This is obtained by introducing a new boundary curve l in the region of B in a certain way, and its equivalent repetition with respect to $\mathfrak{S}$ or $\mathfrak{S}^*$.

For subdivision of a tile B_0 (case 1.2) into non-compact regions there are the following possibilities leading to classes of tilings with obviously distinct rules of construction:

(α) The initial point A of l lies upon k (or k^+) and l has an "asymptotic" course relative to k^+ (or k) without having a pair of points equivalent under $\mathfrak{S} = \langle \tilde{T} \rangle$ (see Fig. 11). l having an *asymptotic* course relative to k means that the Euclidean distance $D(l, k) := \inf d(x, y)$ ($x \in l$, $y \in k$, d: Euclidean distance) is zero and $\inf d(x, y)$ is not attained by any pair of points x, y of the finite plane.

(β, γ) l is without an initial point and has an asymptotic course relative to k as well as to k^+ without having a pair of equivalent points. The asymptotic course of l relative to k is "parallel" (case β) or "anti-parallel" (case γ) to that relative to k^+ (Figs. 12, 13).

The distinct cases (α), (β), (γ) of 1.2 also arise for a subdivision of a tile $\overline{B}_0$ (case 2.2) (Figs. 14–18). New cases must be added, however since the boundary curves of $\overline{B}_0$ possess geometrically different symmetry groups. Case (α) splits into three different subcases:

(α) A is a point of the glide-reflection carrier k (Fig. 14).

$(\alpha 2)$ A is a centre of rotation of the rotation carrier r (Fig. 15).

$(\alpha 3)$ A is a point of r without being a centre of rotation (Fig. 16).

A topological peculiarity of the last constructed inhomogeneous tilings with non-compact tiles is worthy of mention. For this purpose consider the tilings Π_i ($i = 6$ to 13). Each is constructed by subdivision of a tiling Π^+ with tiles having a 1-dimensional symmetry group $\mathfrak{S} = \langle \tilde{T} \rangle$. This was achieved by equivalent reiteration of a curve l having asymptotic course relative to a boundary curve k (glide-reflection carrier or rotation carrier) of Π^+. By this process the points of k (and all congruent reiterations of k) became accumulation points of boundary points of the first kind.

(*Definition* 9. A point $P^+ \in \Pi$ is called an *accumulation point of boundary points of the first kind* if every neighborhood U of P^+, chosen sufficiently small, has the following property: U is the union of two semi-neighborhoods U_1 and U_2, each having a countable infinity of separate segments of boundary curves of Π.) (Cf. Figs. 11 and 11a.)

By subdivision of the tile B_0 or $\overline{B}_0$ new asymmetrical *compact* tiles can also be constructed, e.g. there are the following two cases for $\overline{B}_0$:

(δ) l is bounded without having a pair of $\tilde{T}$-equivalent points. The initial point A of l lies on the glide-reflection carrier k and the final point E coincides with a point P of the rotation carrier r,

$(\delta 1)$ being a centre of rotation (Fig. 19), or

$(\delta 2)$ not being a centre of rotation (Fig. 20).

By modification of the inhomogeneous tilings constructed by $\overline{B}_0$ (case 2.2), further inhomogeneous ones can be obtained: Between two neighboring tiles having a rotation carrier r in common, this carrier is deleted. In this way a strip $\overline{B}$ has arisen, having two congruent glide-reflection carriers (turned by 180°) as its boundary, and possessing the 1-dimensional rotational group $\mathfrak{J} = \langle T^+ \rangle + C_2 \langle T^+ \rangle$ as its symmetry group. But this group is geometrically realized in such a manner that combination with the glide-reflection group of the boundary curves to generate the 2-dimensional tiling group of type pgg is impossible, because $\overline{B}_0$ had been a tile of an inhomogeneous tiling. (A homogeneous tiling Π^h with the prototile $\overline{B}$ can only be built up by finding a group $\mathfrak{G}$ acting transitively upon Π^h *without utilizing the symmetry group* $\mathfrak{J}$ *of* $\overline{B}$. $S_{\mathfrak{g}}$ must not contain the symmetry elements of $\mathfrak{J}$. Such a group $\mathfrak{G}$ for a homogeneous tiling Π^h with tile $\overline{B}$ is of type pg because of the boundary of $\overline{B}$.) A $\overline{B}$ having the symmetry group $\mathfrak{J}$ can be subdivided in several ways.

Case (a). Construction of non-compact tiles.

The symmetry system $S_{\mathfrak{J}}$ of $\mathfrak{J}$ is incorporated into $\overline{B}$ and l is chosen as follows: The initial point A of l coincides with a centre $P_d \in S_{\mathfrak{J}}$ of rotation and l has asymptotic course relative to a boundary curve k of $\overline{B}$ without having a pair of points equivalent with respect to $\mathfrak{J}$. By turning l through 180° about P_d, a line $\overline{l}$ arises having $\mathfrak{C}_2$ as symmetry group. By equivalent reiteration of $\overline{l}$ with respect to the elements of $\langle T^+ \rangle$, $\overline{B}$ is subdivided into tiles R_ι having the symmetry group $\mathfrak{C}_2$ and being equivalent to each other with respect to $\langle T^+ \rangle$ (Fig. 21).

By using a new unbounded line $\tilde{l}$ through the centre of symmetry Q_d (diad) of R_ι, a new asymmetrical tile R^+ can be constructed. Half of $\tilde{l}$, being disjoint to the two congruent boundary curves of R_ι, can be chosen arbitrarily and is equivalently reiterated beyond Q_d (Fig. 22).

Case (b). Construction of compact tiles.

l is chosen to be of finite length only, having points inequivalent under $\mathfrak{J}$. The initial point A coincides with a diad $P_d \in S_{\mathfrak{J}}$ and the final point E lies upon a boundary curve of $\overline{B}$. By equivalent reiteration of l with respect to $\mathfrak{J}$, $\overline{B}$ is subdivided into compact tiles U_ι equivalent with respect to $\langle T^+ \rangle$ and having the symmetry group $\mathfrak{C}_2$ (Fig. 23).

Further subdivisions of U_ι into asymmetrical tiles are possible. A new curve l^+ with inequivalent points is chosen in U_ι, the initial point A^+ of which coincides with the diad $Q_d \in S_{\mathfrak{C}_2}$. There are four different cases for the position of the final point E^+ of l^+:

b2.1) E^+ coincides with a point P_1 of a glide-reflection carrier k (boundary curve of $\overline{B}$) with $P_1 \notin l$ (Fig. 24);

b2.2) E^+ coincides with a point P_2 of a glide-reflection carrier k with $P_2 \in l$ (Fig. 25);

b2.3) E^+ coincides with a point $P_3 \in l$, which is not a digonal centre of rotation of $S_{\mathfrak{J}}$ and $P_3 \notin k$ (Fig. 26).

b2.4) E^+ coincides with a point $P_4 \in l$, which is a digonal centre $P_d \in S_{\mathfrak{G}}$ (Fig. 27).

By equivalent reiteration of l^+ with respect to $\mathfrak{C}_2$, U_ι is subdivided into two congruent tiles V_ι. Distinguishing the four cases b2.1–b2.4 is reasonable, since they lead to classes of tiles with obviously distinct constructional features and the tilings differ in number resp. order of their vertices. But only b2.1–b2.3 are new cases. The tiling Π^+ obtained according to b2.4 is the former case Π_{14} of Figure 19, as can be seen directly.

There remains the proof of

THEOREM 5. *The tiles of the tilings* Π_ι $(i = 6$ *to* $21)$ (*Fig.* 11 *to* 26) *can only form inhomogeneous tilings of* E^2.

Proof. The tiles of Π_ι $(i = 6 \text{ to } 16, 18)$ possess exactly one pair of congruent curves (of finite or infinite length) in translational positions on their boundaries. For the case of infinite length these curves run within a strip because of the way the curves are constructed. In the tilings Π_ι $(i = 17, 19, 20, 21)$ this situation obtains for regions built up by pairs of tiles. In each tiling built up with tiles of the respective tilings there must exist strips having the 1-dimensional translational group $\mathfrak{S} = \langle T \rangle$ as their symmetry group or subgroup (for Π_ι $(i = 16 \text{ to } 20)$) of symmetry. These strips are exactly those from which the tiles under consideration had been obtained by subdivision. In order to obtain a homogeneous tiling, one must construct a 2-dimensional group $\mathfrak{G}$ having $\mathfrak{S}$ as a subgroup and operating transitively on the totality of strips. This is impossible, however, because of the particular geometrical realization of the boundary curves of the strips. □

Theorem 5 also follows from the fact that the tiles under consideration do not belong to any class of the complete list of classes of compact ([**4**], [**5**]) or non-compact ([**10**]) tiles forming at least one homogeneous tiling.

Distributing the tiles obtained by construction of the homogeneous tilings Π_ι^h $(i = 1, 4)$ or inhomogeneous tilings Π_ι $(i = 2, 3, 5 \text{ to } 21)$ among the classes defined above, the following result can be stated:

K_{IB} : tiles of Π_1^h and Π_4^h (stabilizer $\mathfrak{F} = \langle \tilde{T} \rangle$):
K_{IIA}: tiles of Π_{14}, Π_{15}, Π_{19}, Π_{20}, Π_{21} (asymmetrical)
tiles of Π_{18} (symmetry group $\mathfrak{C}_2$);

K_{IIB}: tiles of tilings constructed according to cases 1.3 and 2.3 (asymmetrical);
tiles of Π_2, Π_5 (symmetry group $\mathfrak{S} = \langle \tilde{T} \rangle$);
tiles of Π_6, Π_7, Π_8, Π_9, Π_{10}, Π_{11}, Π_{12}, Π_{13}, Π_{17} (asymmetrical);
tiles of Π_{16} (symmetry group $\mathfrak{C}_2$).

Of course, by taking prisms based on the prototiles constructed above, it is easy to show that these are examples for an affirmative solution of the original (3-dimensional) version of the Hilbert question cited at the beginning.

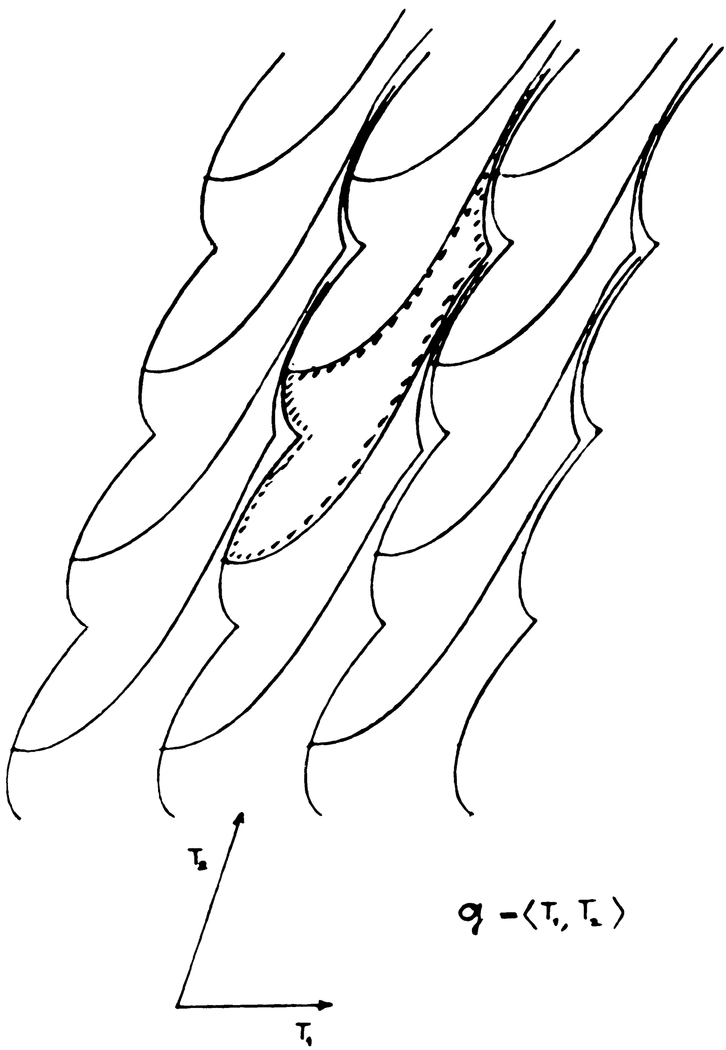


FIGURE 1a. *Homogeneous tiling of non-compact fundamental regions of a group of type* p1.

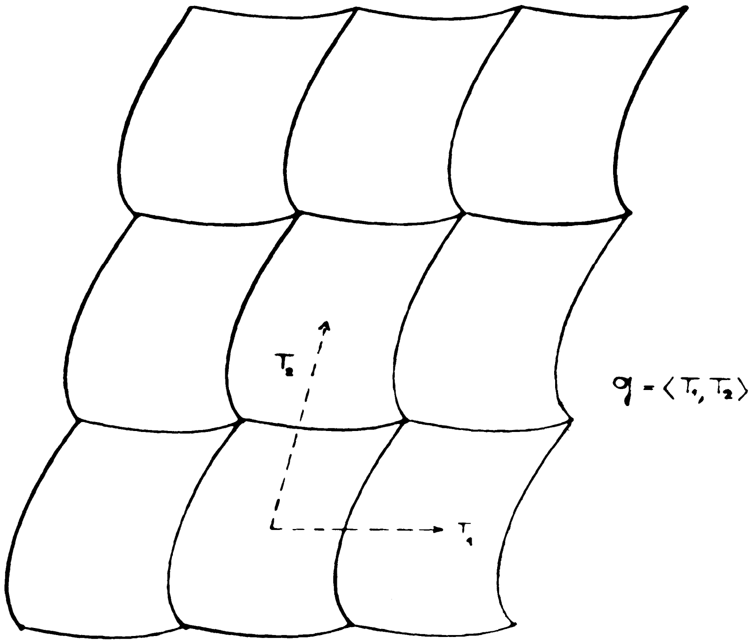
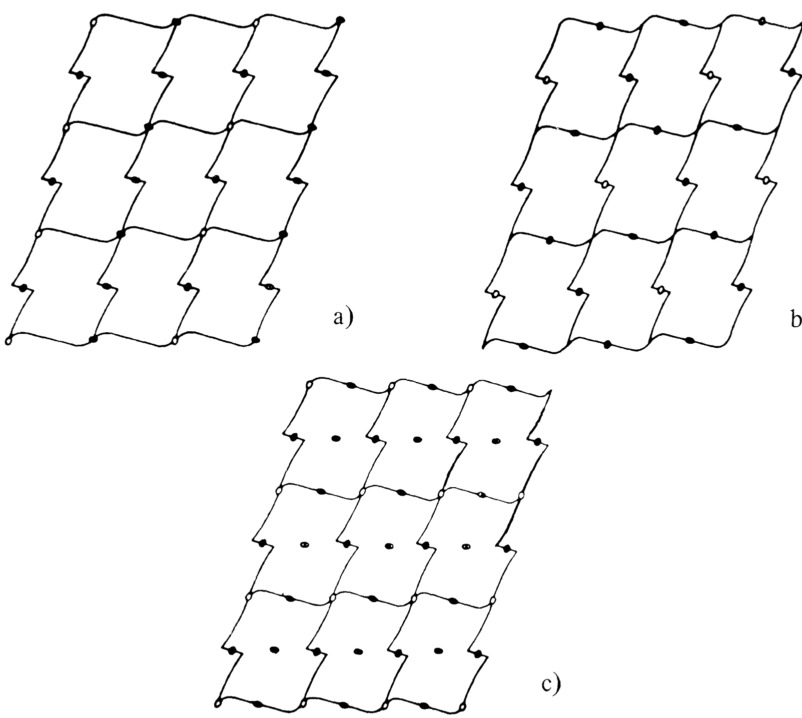
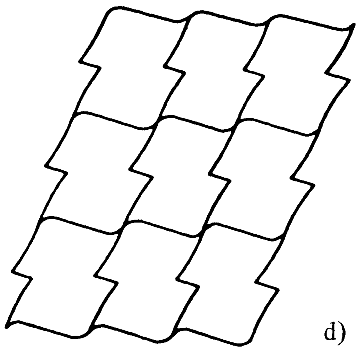


FIGURE 1b. *Homogeneous tiling* Π^h *as a geometrical model for a group* $\mathfrak{G}$ *of type* p1.

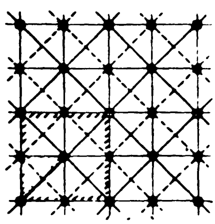


FIGURES 2a, b, c. *Homogeneous tiling* $\tilde{\Pi}^h$ *with* 3 *different realizations of tiling groups of type* p2.



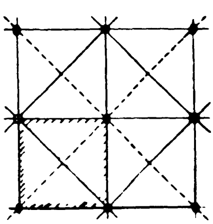
d)

FIGURE 2d. *Homogeneous tiling* $\tilde{\Pi}^h$ *with tiling group of type* p1.

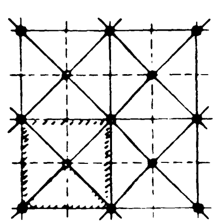


a)

FIGURE 3a. *Homogeneous tiling* $\Pi^h_\square$ *by squares* ($z = 4, 4, 4, 4$) *with corresponding symmetry group* $\overline{\mathfrak{G}}$ *of type* p4m.



b)



c)

FIGURE 3b, c. *Homogeneous tiling* $\Pi^h_\square$ *by squares* $z = (4, 4, 4, 4)$ *with two other groups of type* p4m *not being the symmetry group*.

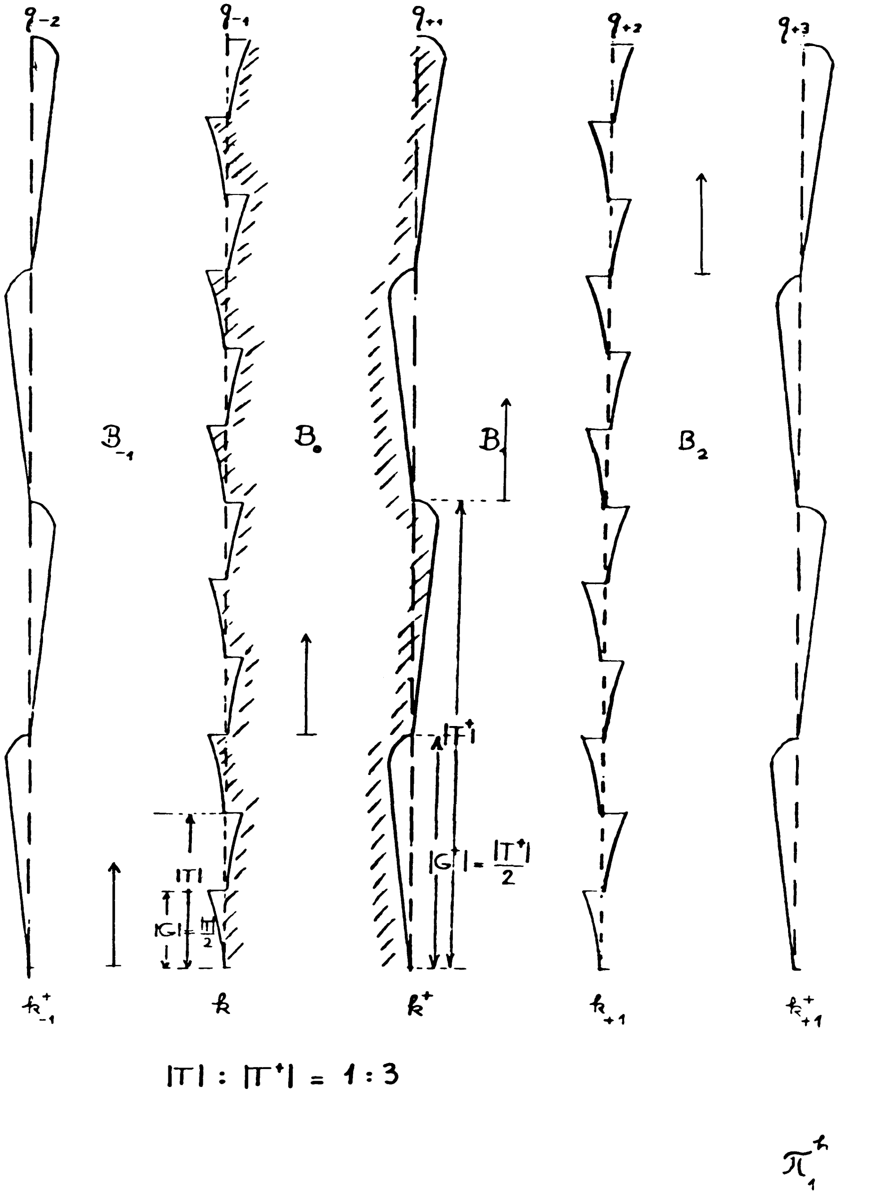


FIGURE 4. *Homogeneous tiling π_1^h with prototile B_0 according to* 1.1. The boundary curves of B_0, k (resp. k^+), have symmetry groups of glide-reflections: $\mathfrak{H} = \langle T \rangle + G\langle T \rangle$ and $\mathfrak{H}^+ = \langle T^+ \rangle + G^+ \langle T^+ \rangle$, respectively, with the chosen proportion $|T| : |T^+| = 1 : 3$ (case 1.1). As neighbor-transformations of B_0, glide-reflections along g_{+1} (resp. g_{-1}) are chosen: $\tilde{G}_{+1}$: $B_0 \to B_1$ (resp. $\tilde{G}_{-1}$: $B_0 \to B_{-1}$) with the same glide-modulus $|\tilde{G}_{+1}| = |\tilde{G}_{-1}| = 3|T|/2 = 3|G| = 1|T^+|/2 = |G^+|$. In general there is $\tilde{G}_{+i+1}$: $B_i \to B_{i+1}$ or $\tilde{G}_{-i-1}$: $B_{-i} \to B_{-i-1}$ as glide-reflection along the axis g_{+i+1} or g_{-i-1}, respectively, with $|\tilde{G}_{+i+1}| = |\tilde{G}_{-i-1}| = |G^+|$. The neighbor-transformations are made plain by the aid of corresponding arrows.

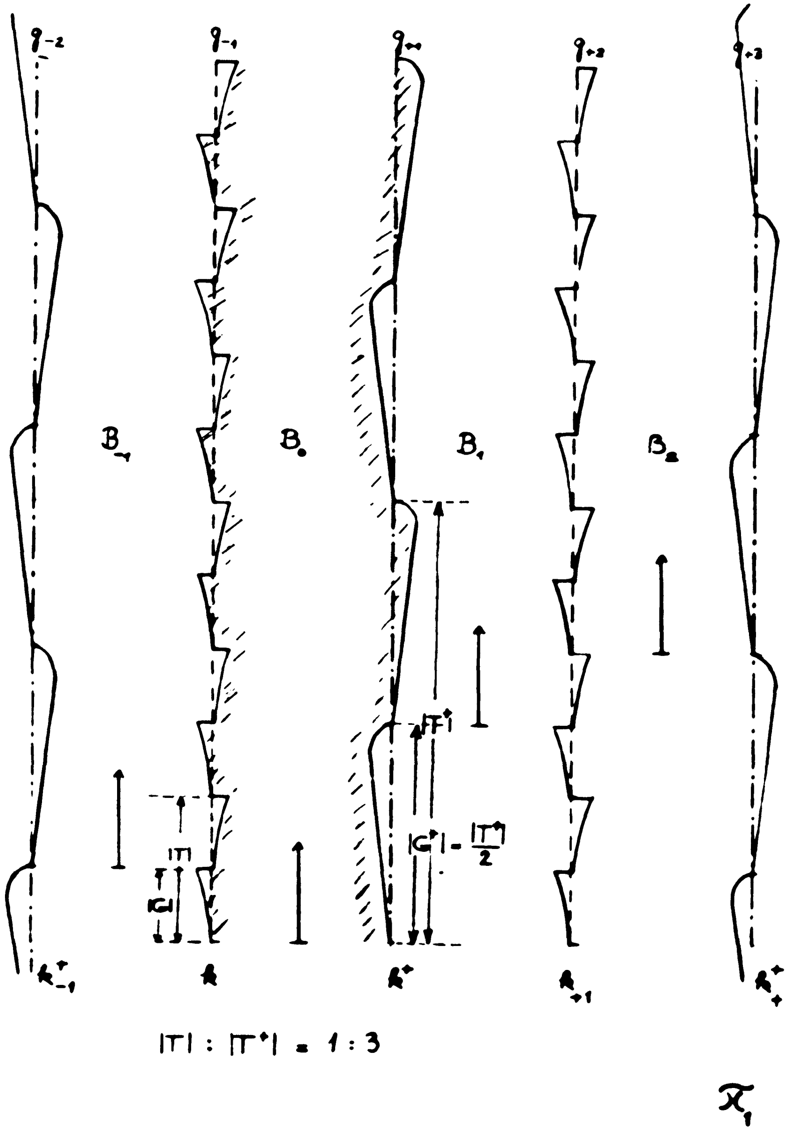
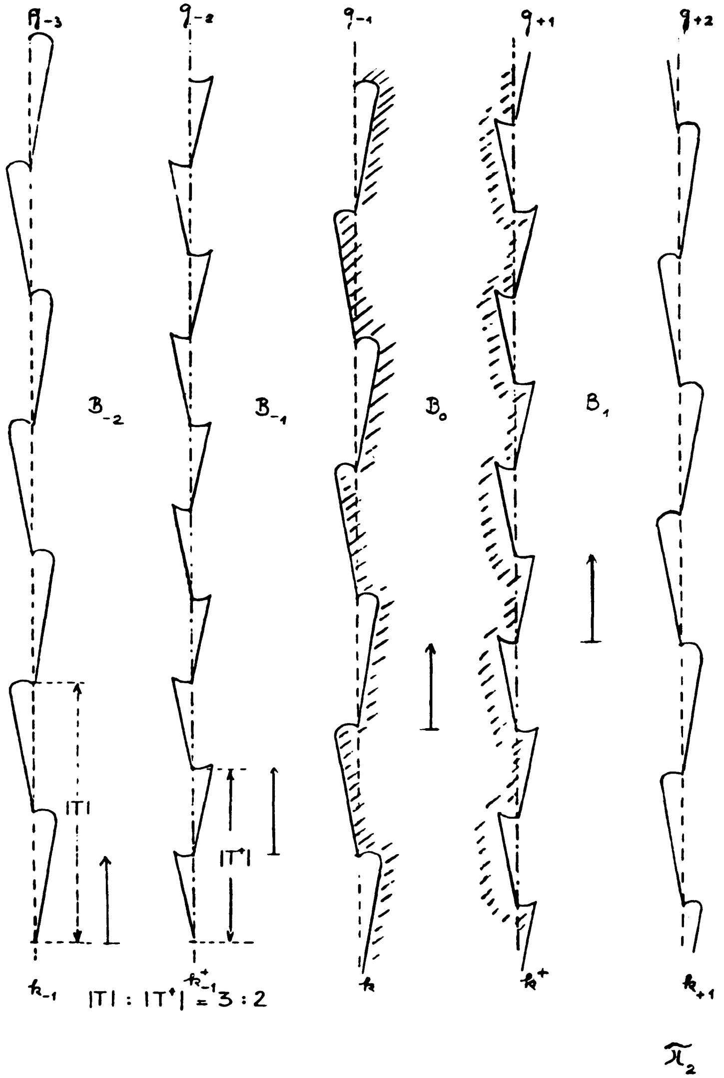


FIGURE 5. *Inhomogeneous tiling* Π_1 *with prototile* B_0 *of* Π_1^h. By constructing Π_1, the neighbor-transformations of $B_0 \in \Pi_1$ are glide-reflections chosen with *different* glide-moduli: $\tilde{G}_{+1}$: $B_0 \to B_1$ as in Figure 4 with $|\tilde{G}_{+1}| = |G^+| = 3|G|$, $\bar{G}_{-1}$: $B_0 \to B_{-1}$ with $|\bar{G}_{-1}| = |G|$. Because of this, $\tilde{G}_{\pm 1}^2 \neq \bar{G}^2$ and, therefore, $\tilde{G}_{+1}$ and $\bar{G}_{-1}$ are not generators of a group of type pg. Further, $\bar{G}_{+2}$: $B_1 \to B_2$ with $|\bar{G}_2| = |G|$, etc.



FIGURES 6, 7. *Inhomogeneous tilings* Π_2 *and* Π'_2 *with prototile* B_0 *according to* 1.2. The boundary curves of B_0, k (resp. k^+), have symmetry groups of glide-reflections as in Figure 4, but the chosen proportion is now $|T|:|T^+| = 3:2$ (case 1.2) which prevents the construction of a homogeneous tiling with prototile B_0. B_0 has symmetry group $\mathfrak{S} = \langle 2T \rangle = \langle 3T^+ \rangle$. Chosen neighbor-transformations for Π_2 (Figure 6):

$G^+ : B_0 \to B_1$, axis g_{+1}, direction $\uparrow$, $|G^+| = \frac{2}{3}|G|$.

G_0: $B_0 \to B_{-1}$, axis g_{-1}, direction $\downarrow$, $|G_0| = |G| = \frac{3}{2}|G^+|$.

G_{-1}: $B_{-1} \to B_{-2}$, axis g_{-2}, direction $\downarrow$, $|G_{-1}| = |G^+|$, etc.

for Π'_2 (Figure 7):

$G^+ : B_0 \to B_1$, axis g_{+1}, direction $\downarrow$, $|G^+| = \frac{2}{3}|G|$.

G_0: $B_0 \to B_{-1}$, axis g_{-1}, direction $\uparrow$, $|G_0| = |G| = \frac{3}{2}|G^+|$.

G_{-1}: $B_{-1} \to B_{-2}$, axis g_{-2}, direction $\downarrow$, $|G_{-1}| = 3|G^+|$.

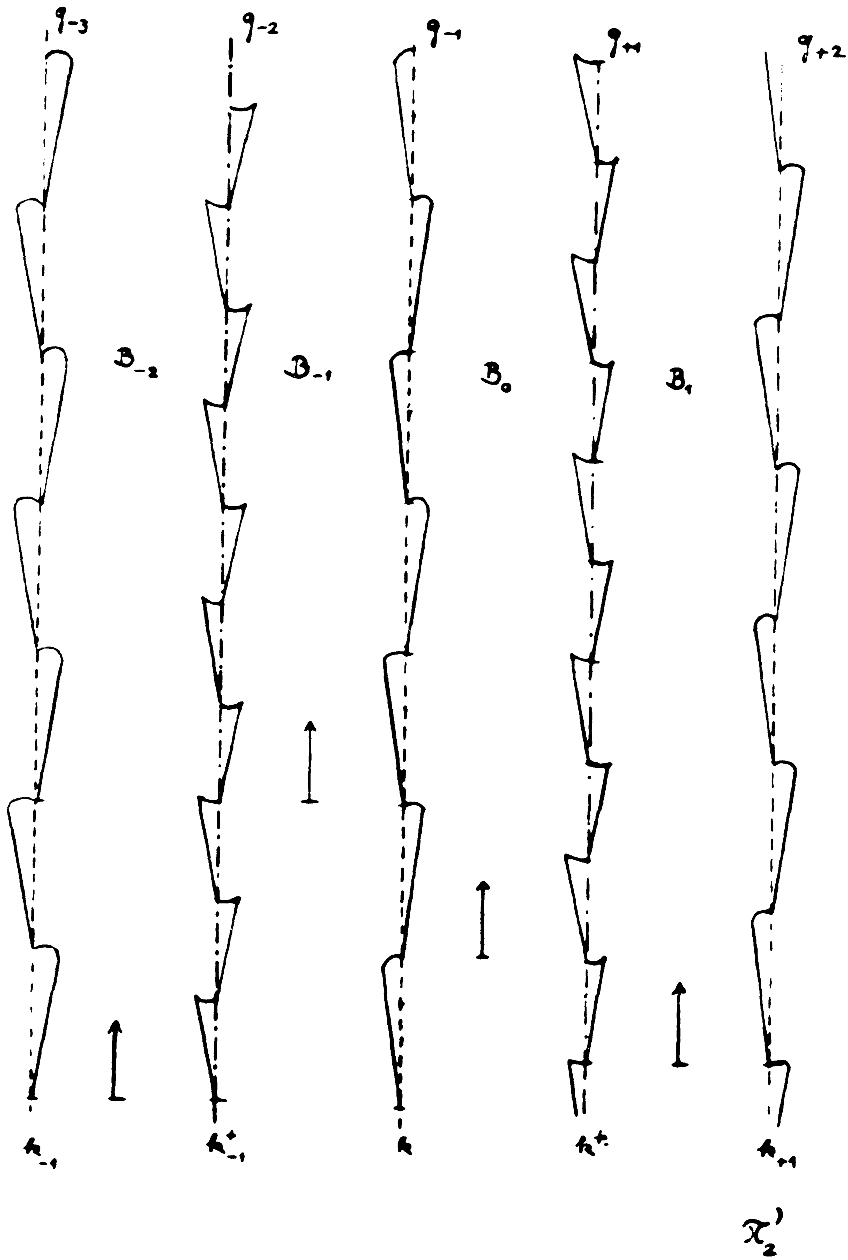


FIGURE 7

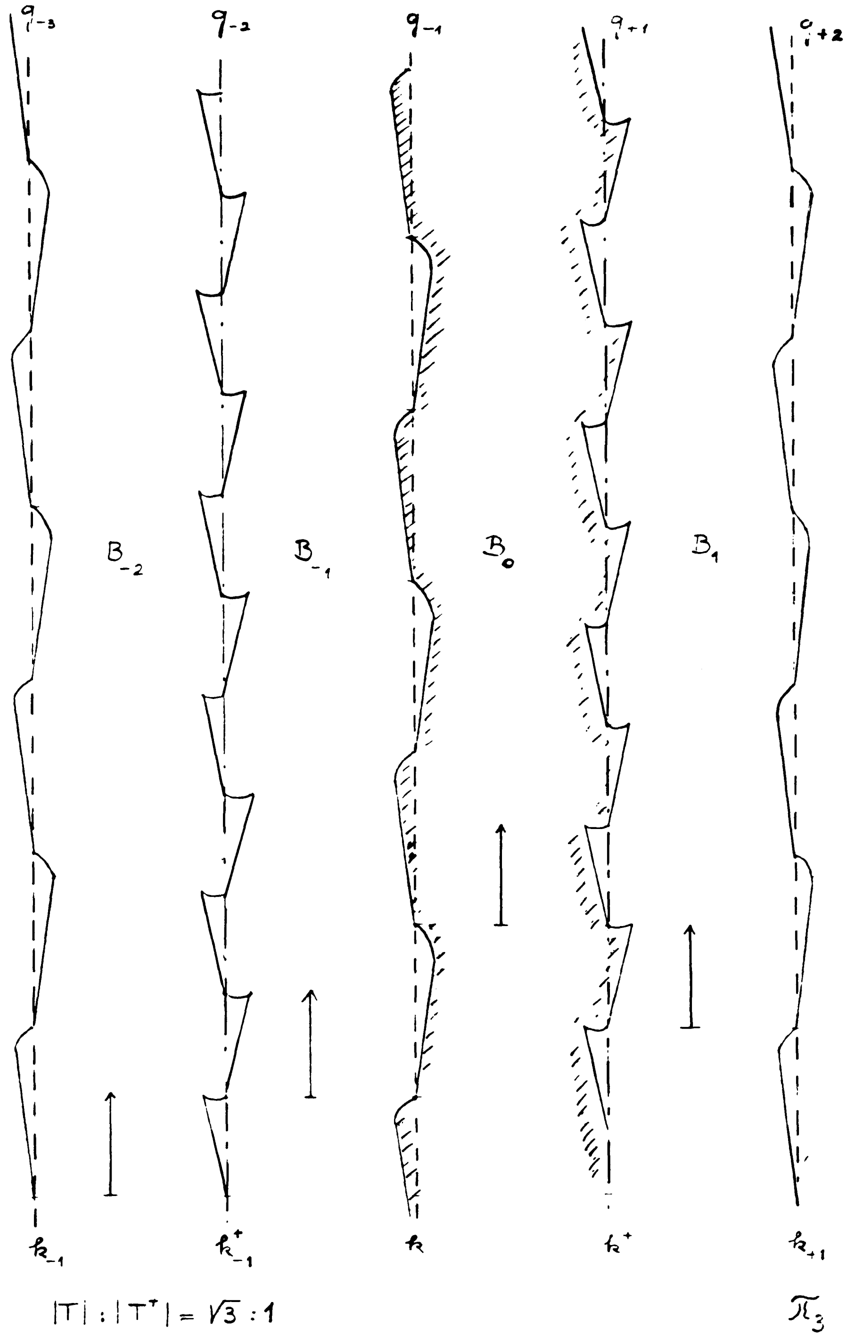


FIGURE 8. *Inhomogeneous tiling* Π_3 *with prototile* B_0 *according to* 1.3. k and k^+ have symmetry groups of glide-reflection as in Figure 4; the chosen proportion is, however, irrational, $|T| : |T^+| = \sqrt{3} : 1$ (case 1.3). The prototile $B_0 \in \Pi_3$ is asymmetrical and the construction of Π_3 is similar to Π'_2, varying arbitrarily the glide-modulus for neighbor-transformations of the tiles.

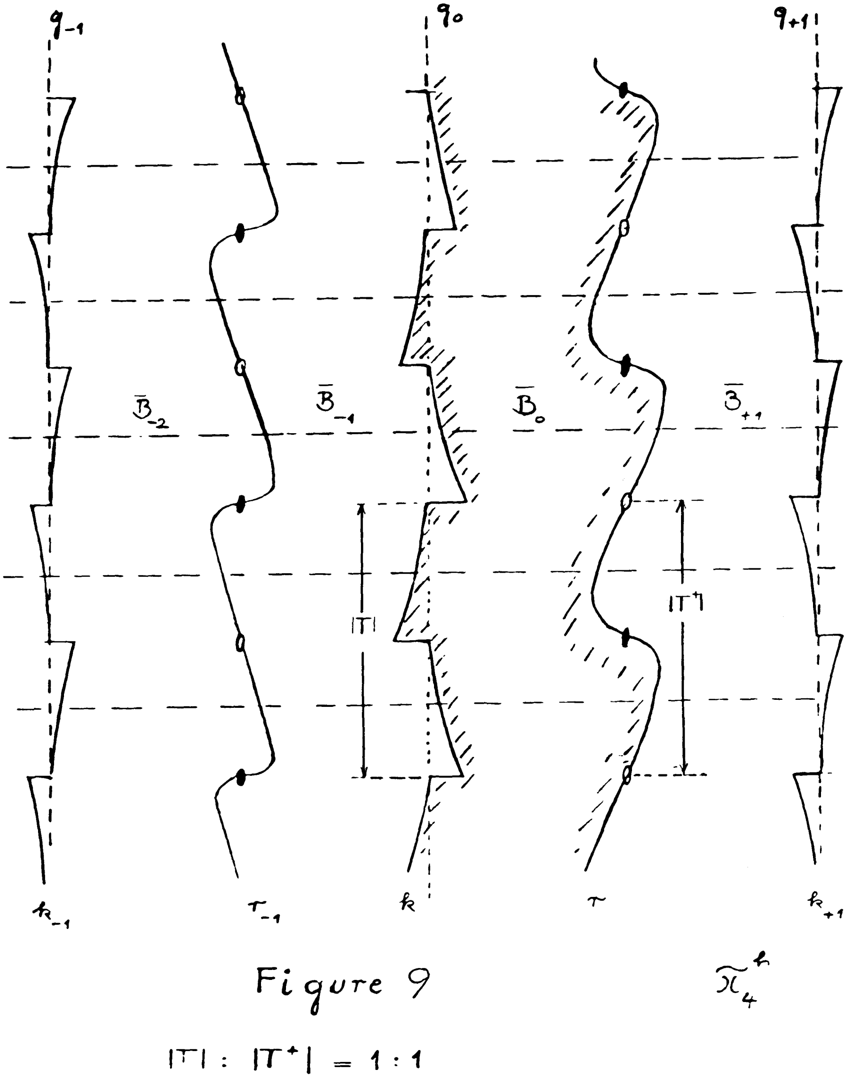


FIGURE 9. *Homogeneous tiling* Π_4^h *with prototile* B_0 *according to* 2.1. The boundary curves of $\overline{B}_0$, k and r, respectively, have symmetry groups $\mathfrak{H} = \langle T \rangle + G\langle T \rangle$ and $\mathfrak{F} = \langle T^+ \rangle + C_2\langle T^+ \rangle$ with the chosen proportion $|T| : |T^+| = 1 : 1$ (case 2.1). As neighbor-transformations of $\overline{B}_0$ there are chosen (1) glide-reflection G along axis $g_0 : G : \overline{B}_0 \to \overline{B}_{-1}$ with $|G| = |T|/2 = |T^+|/2$, and (2) half-turn C_2 about any diad of $S_{\mathfrak{F}} : C_2 : \overline{B}_0 \to \overline{B}_{+1}$. There is further C_2': $B_{-1} \to B_{-2}$ (half-turn about a diad of r_{-1}), G_{+1}: $\overline{B}_{+1} \to \overline{B}_{+2}$ (glide-reflection with axis g_{+1} and $|G_{+1}| = |G|$), etc. B_0 has stabilizer $\mathfrak{F} = \langle T \rangle = \langle T^+ \rangle$ and the group of Π_4^h is of type pgg.

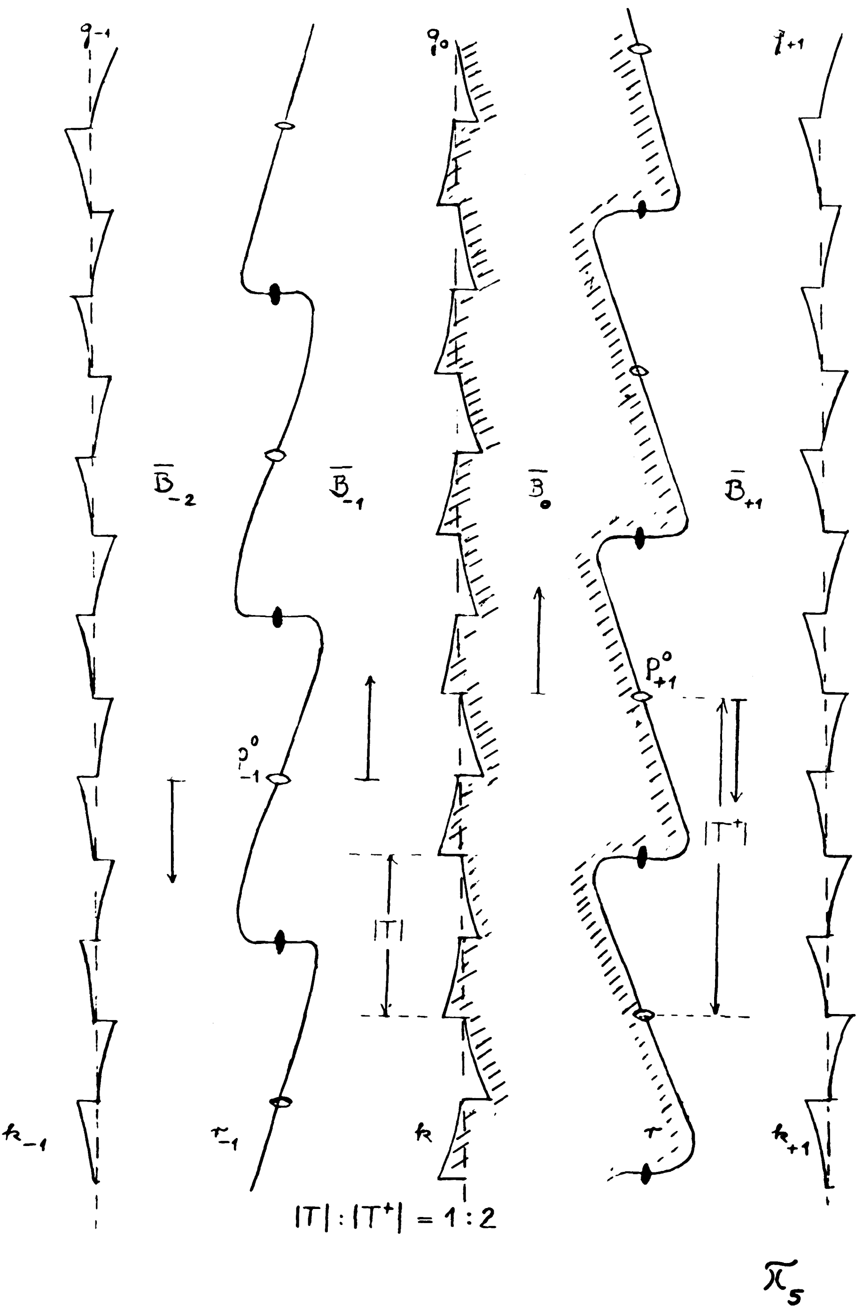
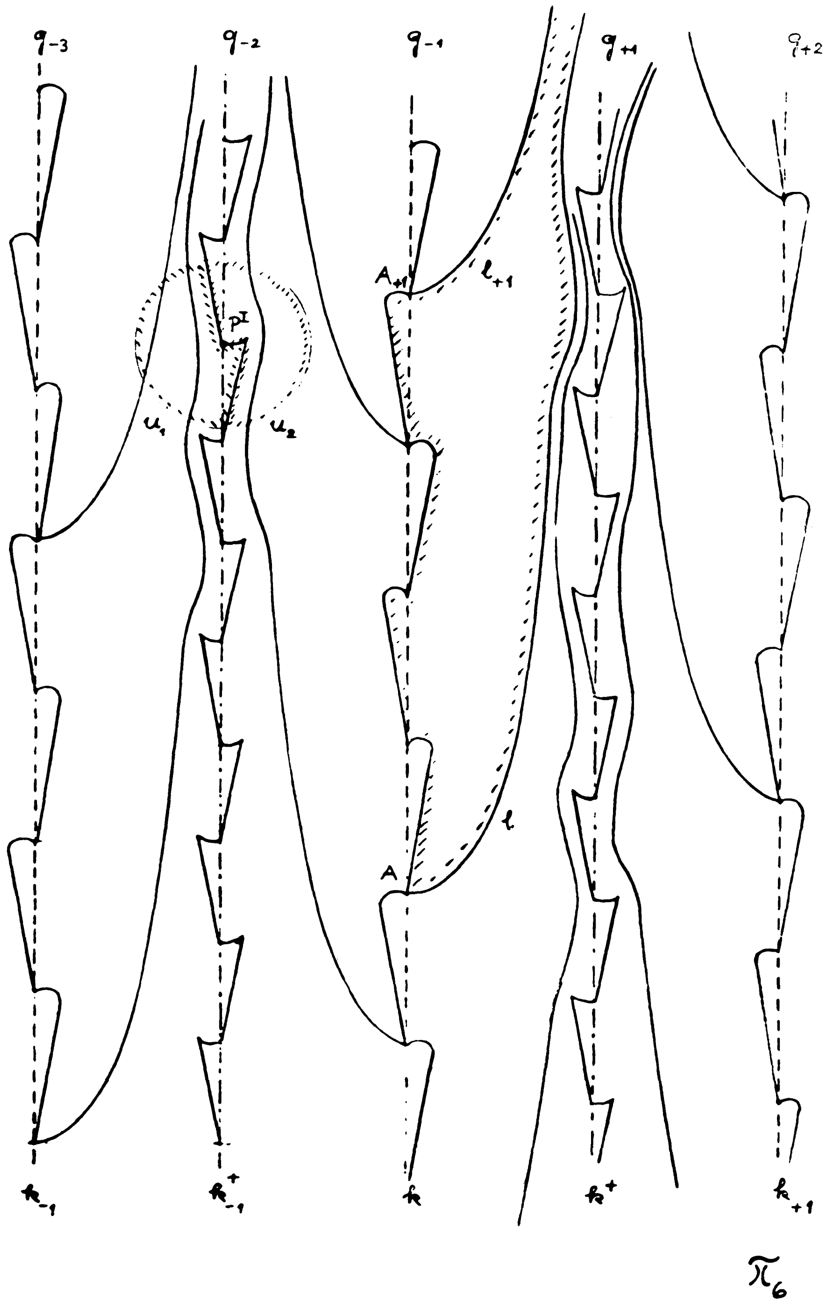


FIGURE 10. *Inhomogeneous tiling* Π_5 *with prototile* B_0 *according to* 2.2. The boundary curves of $\bar{B}_0$, k and r, respectively, have symmetries of the same type as in Figure 9, but the chosen proportion is now $|T|:|T^+|= 1:2$ (case 2.2). The symmetry group of $\bar{B}_0$ is $\mathfrak{S} = \langle 2T\rangle = \langle T^+\rangle$. As neighbor-transformations of $\bar{B}_0$ there are chosen 1) glide-reflection G along axis g_0: $G: B_0 \to \bar{B}_{-1}$ (direction $\downarrow$, $|G|=|T|/2=|T^+|/4$), and 2) half-turn C_2 about the diad $P^0_{+1} \in r$: $C_2: \bar{B}_0 \to \bar{B}_{+1}$. Further there is $C_2': \bar{B}_{-1} \to \bar{B}_{-2}$ (half-turn about P^0_{-1}), $G_{+1}: \bar{B}_{+1} \to \bar{B}_{+2}$ (axis g_{+1}, $|G_{+1}|= m|G|$, m arbitrarily odd), etc.



FIGURES 11, 12, 13. *Inhomogeneous tilings* Π_6, Π_7, Π_8. Π_6 is generated by subdivision of Π_2 by introducing a new boundary curve l and its equivalent reiteration by the elements of the symmetry groups $\mathfrak{S}_l$ of the tiles of Π_2. For l one has *condition* (α) in Π_6 (Figure 11), (β) in Π_7 (Figure 12), (γ) in Π_8 (Figure 13).

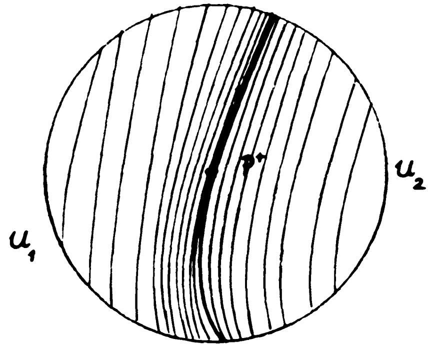


FIGURE 11a. *Accumulation point of boundary points of the first kind.*

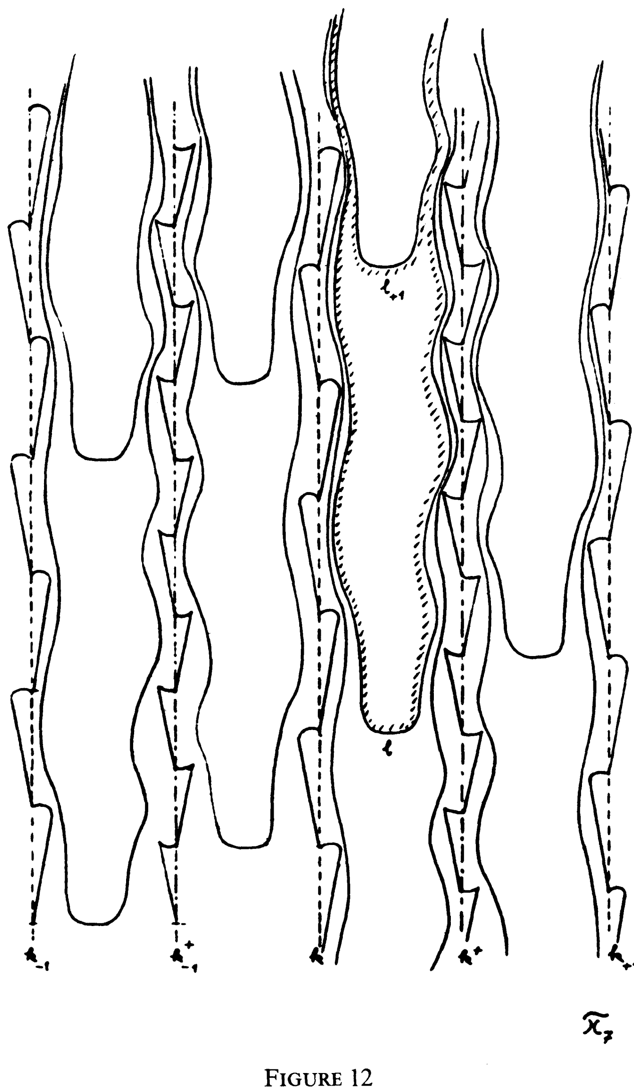


FIGURE 12

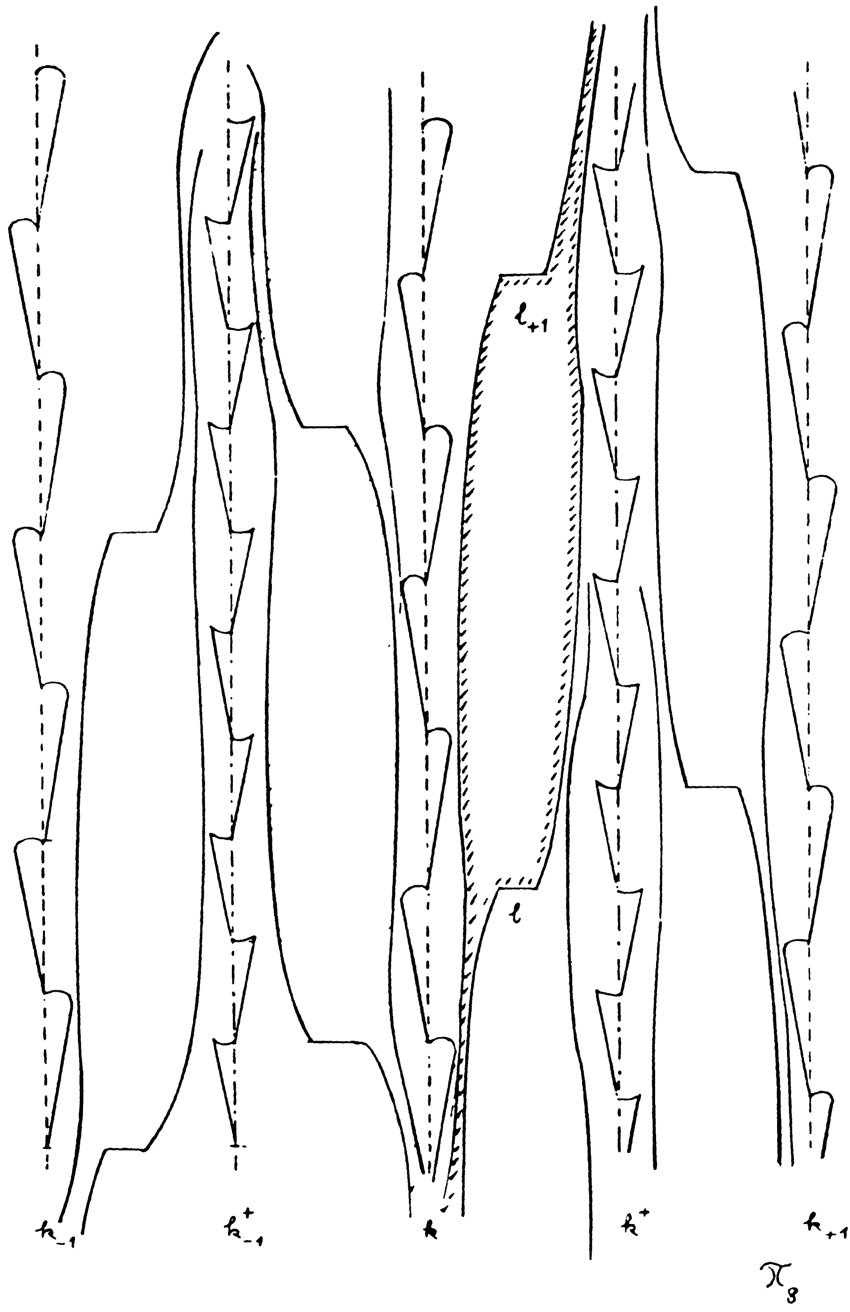
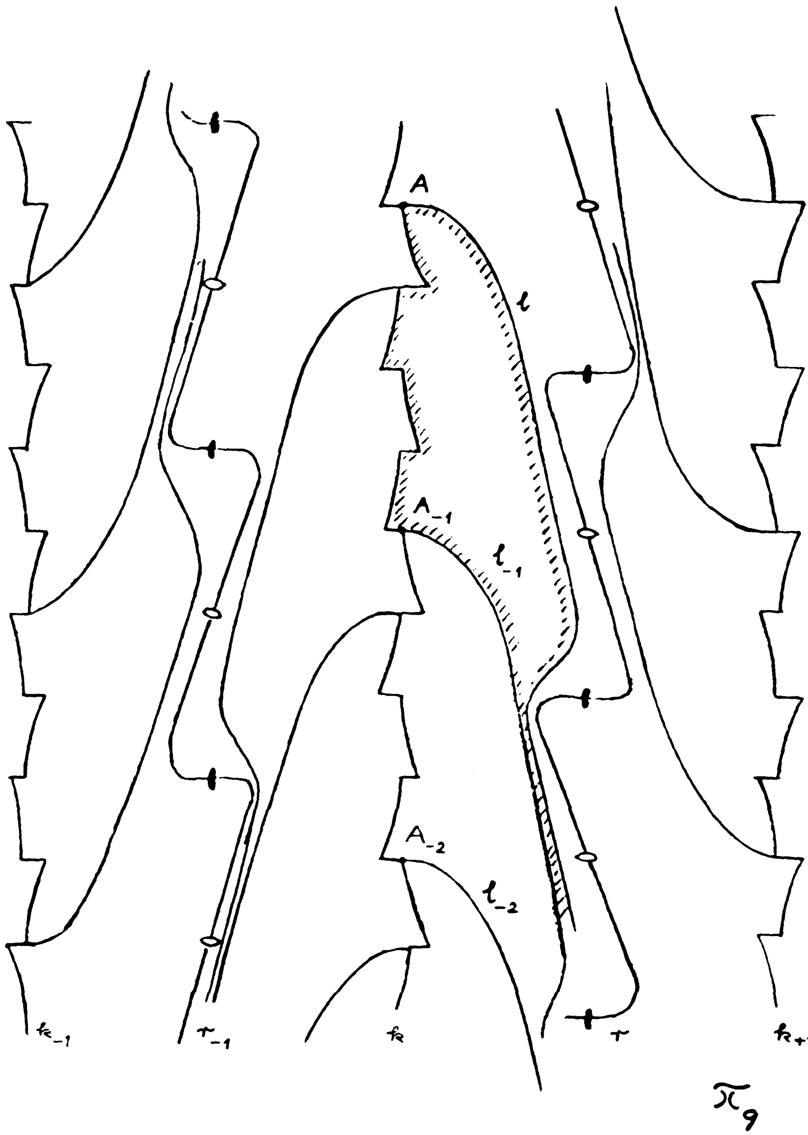


FIGURE 13



FIGURES 14, 15, 16. *Inhomogeneous tilings* Π_9, Π_{10}, Π_{11}. The tilings Π_i ($i = 9, 10, 11$) are generated by subdivision of Π_5. For l one has condition ($\alpha 1$) in Π_9 (Figure 14), ($\alpha 2$) in Π_{10} (Figure 15) and ($\alpha 3$) in Π_{11} (Figure 16).

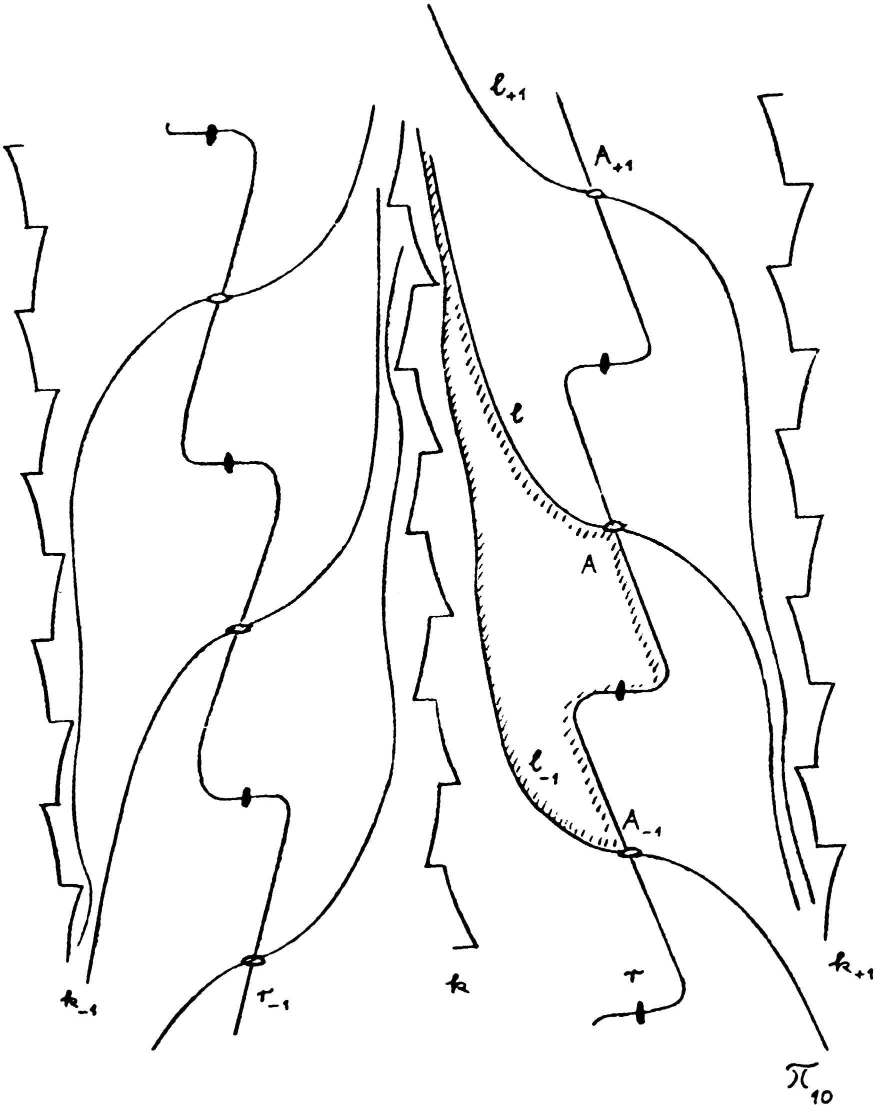


FIGURE 15

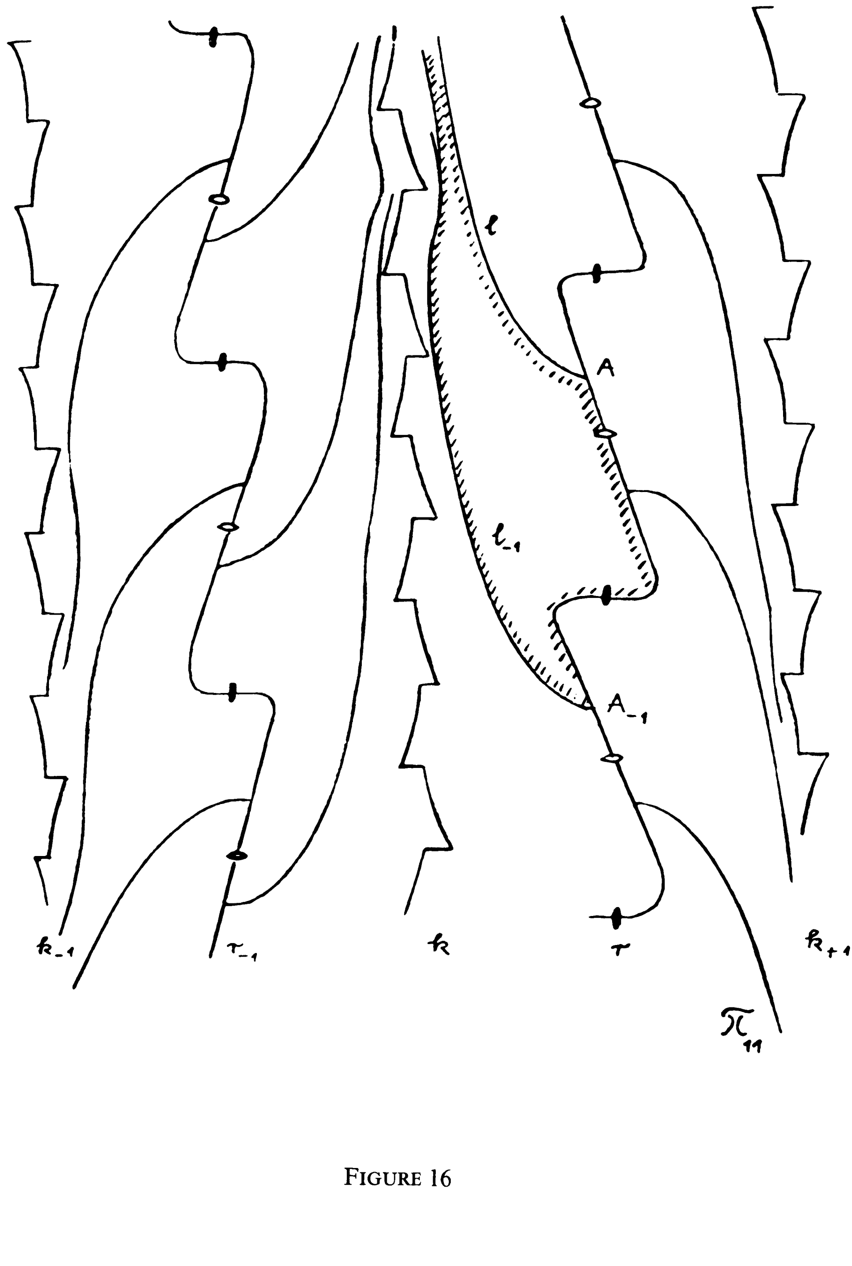
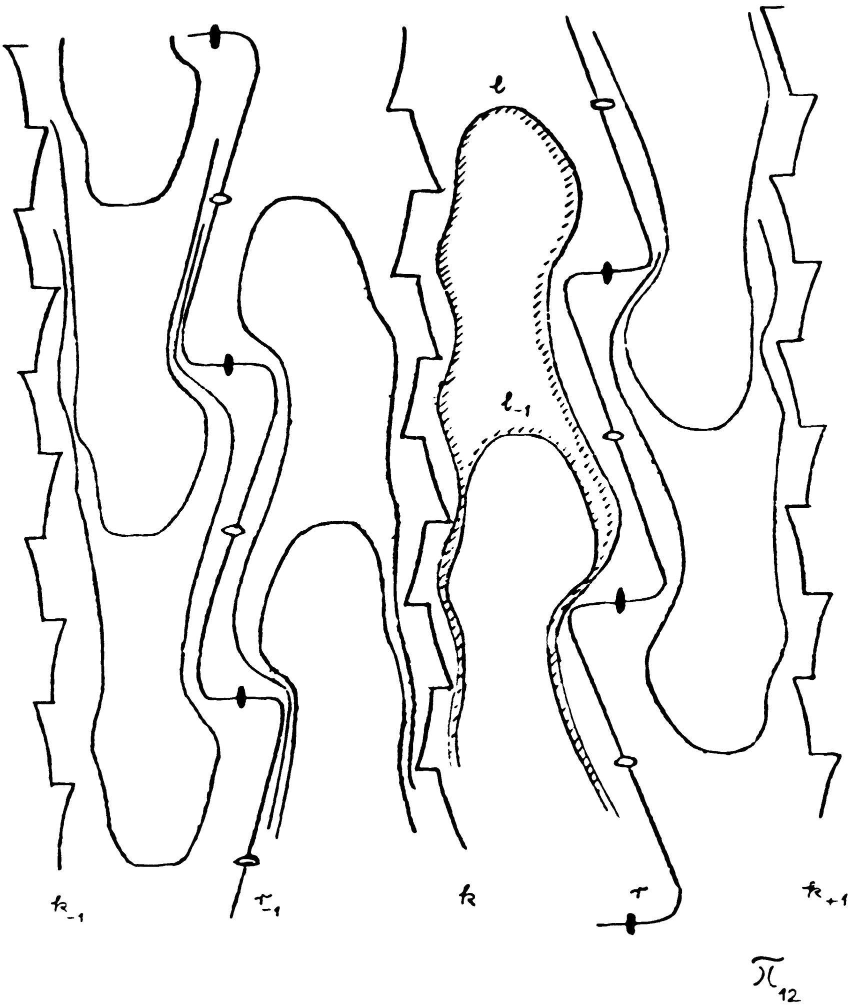


FIGURE 16



FIGURES 17, 18, 19, 20. *Inhomogeneous tilings* Π_{12}, Π_{13}, Π_{14}, Π_{15}. The tilings Π_i ($i = 12$ to 15) are generated by subdivision of Π_5. For l one has condition (β) in Π_{12} (Figure 17), respectively (γ) in Π_{13} (Figure 18), respectively ($\delta 1$) in Π_{14} (Figure 19), respectively ($\delta 2$) in Π_{15} (Figure 20).

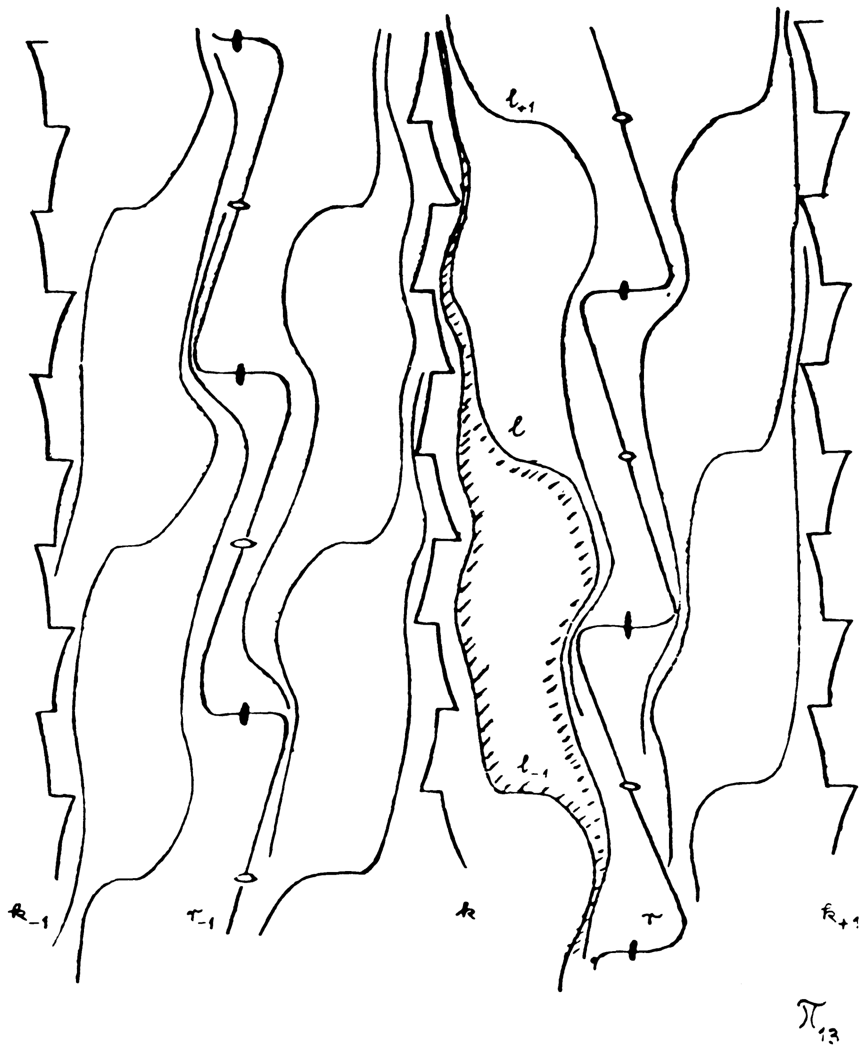


FIGURE 18

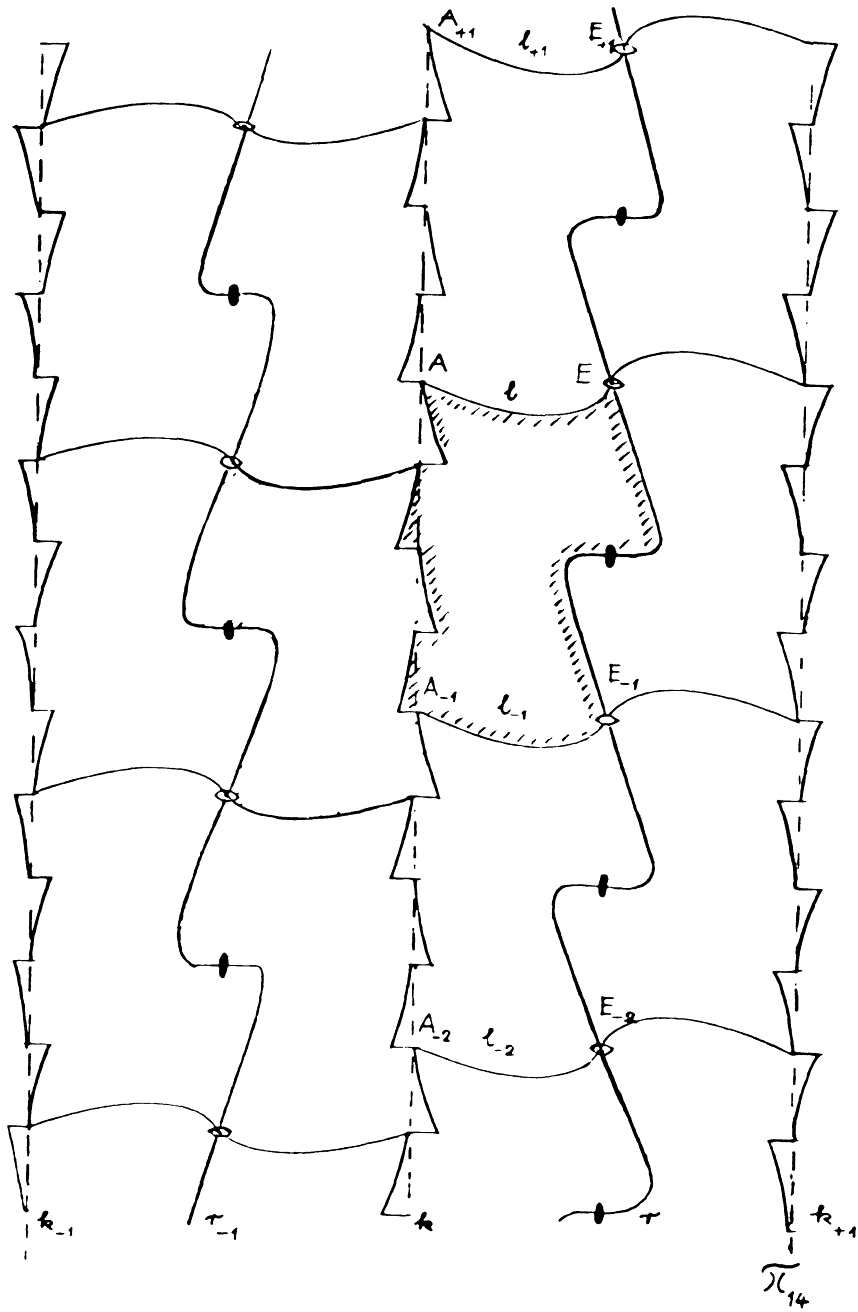


FIGURE 19

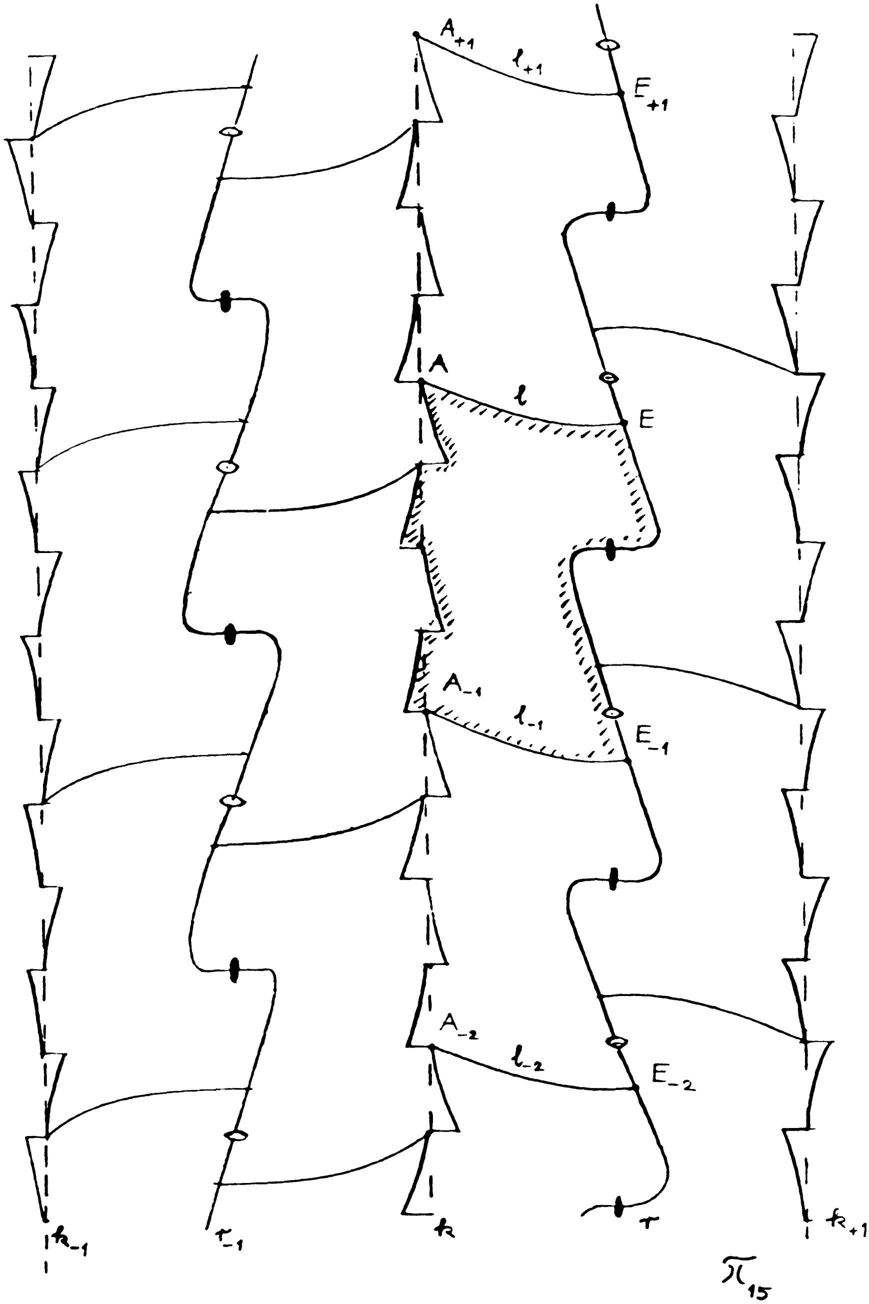


FIGURE 20

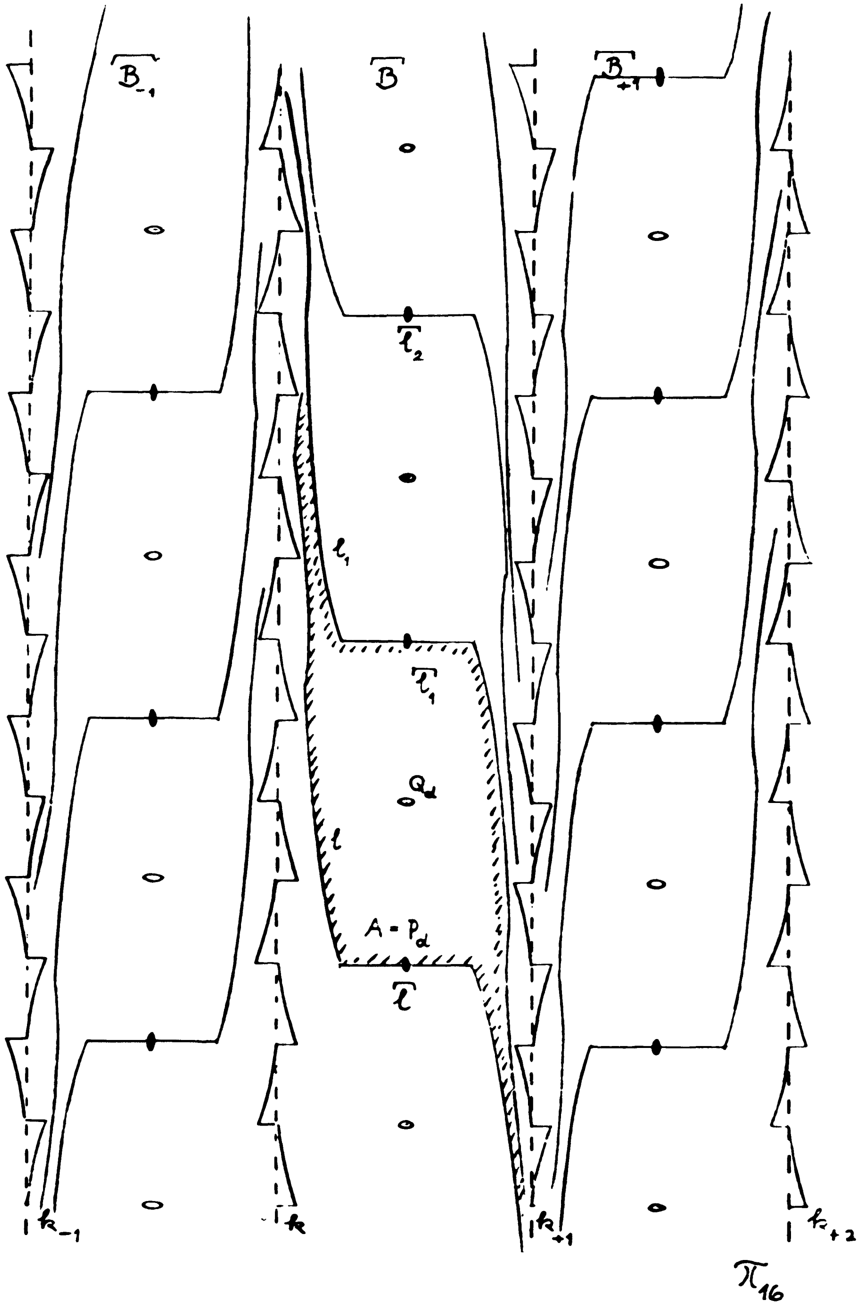


FIGURE 21. *Inhomogeneous tiling* Π_{16}. The tiling Π_{16} is generated by subdivison of a special tiling $\overline{\Pi}$. $\overline{\Pi}$ is generated from Π_5 by deleting all boundary curves which are rotation carriers, the tiles $\overline{B}_l \in \overline{\Pi}$ having symmetry group $\mathfrak{S} = \langle T^+ \rangle + C_2 \langle T^+ \rangle$. The newly introduced boundary curve l for subdividing $\overline{\Pi}$ satisfies condition (a). The tiles of Π_{16} have rotational symmetry $\langle C_2 \rangle$.

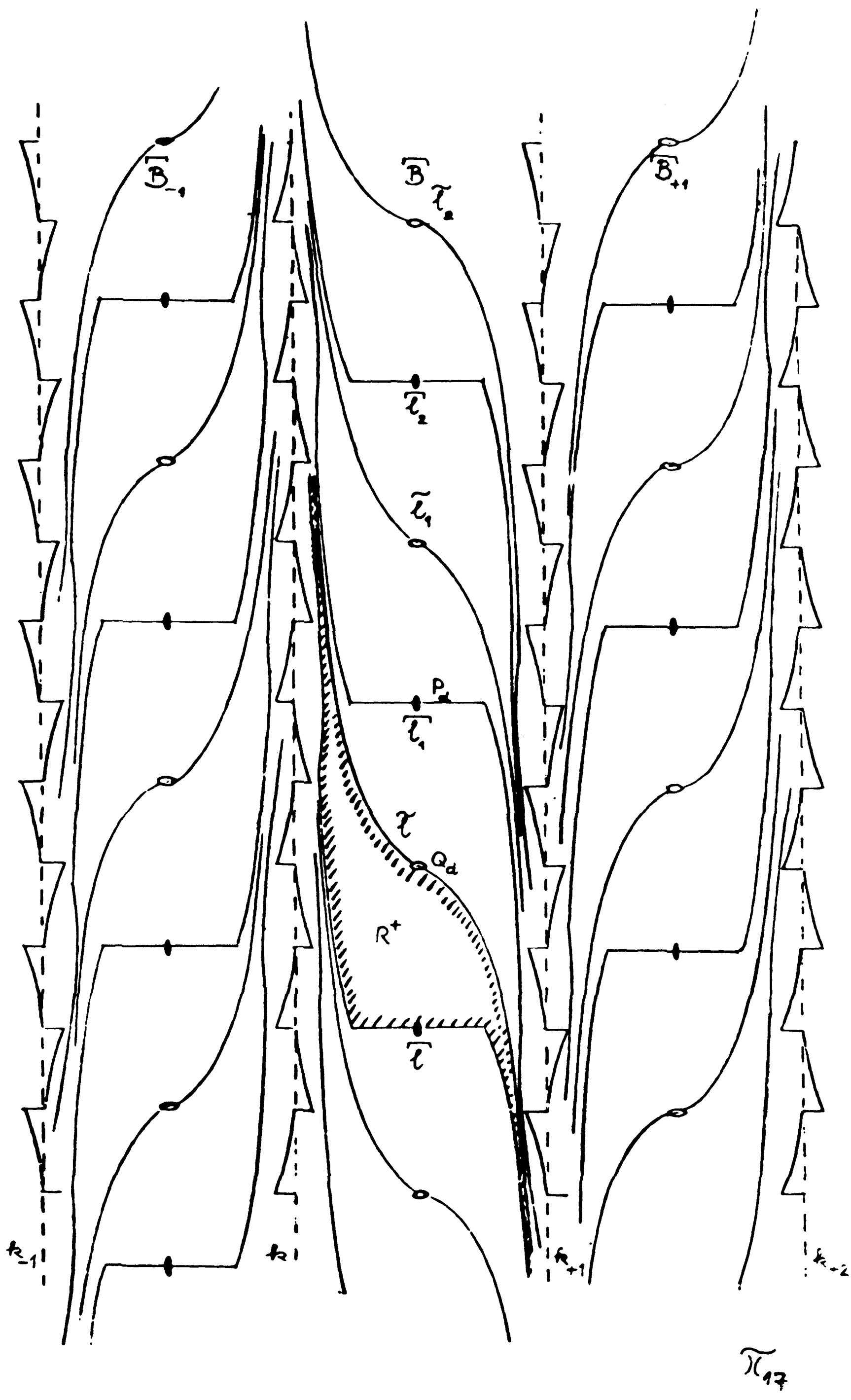


FIGURE 22. *Inhomogeneous tiling* Π_{17}. The tiling Π_{17} is generated by subdivision of tiling Π_{16} by introducing a new boundary curve $\tilde{l}$ similar to condition (a). The tiles of Π_{17} are asymmetrical.

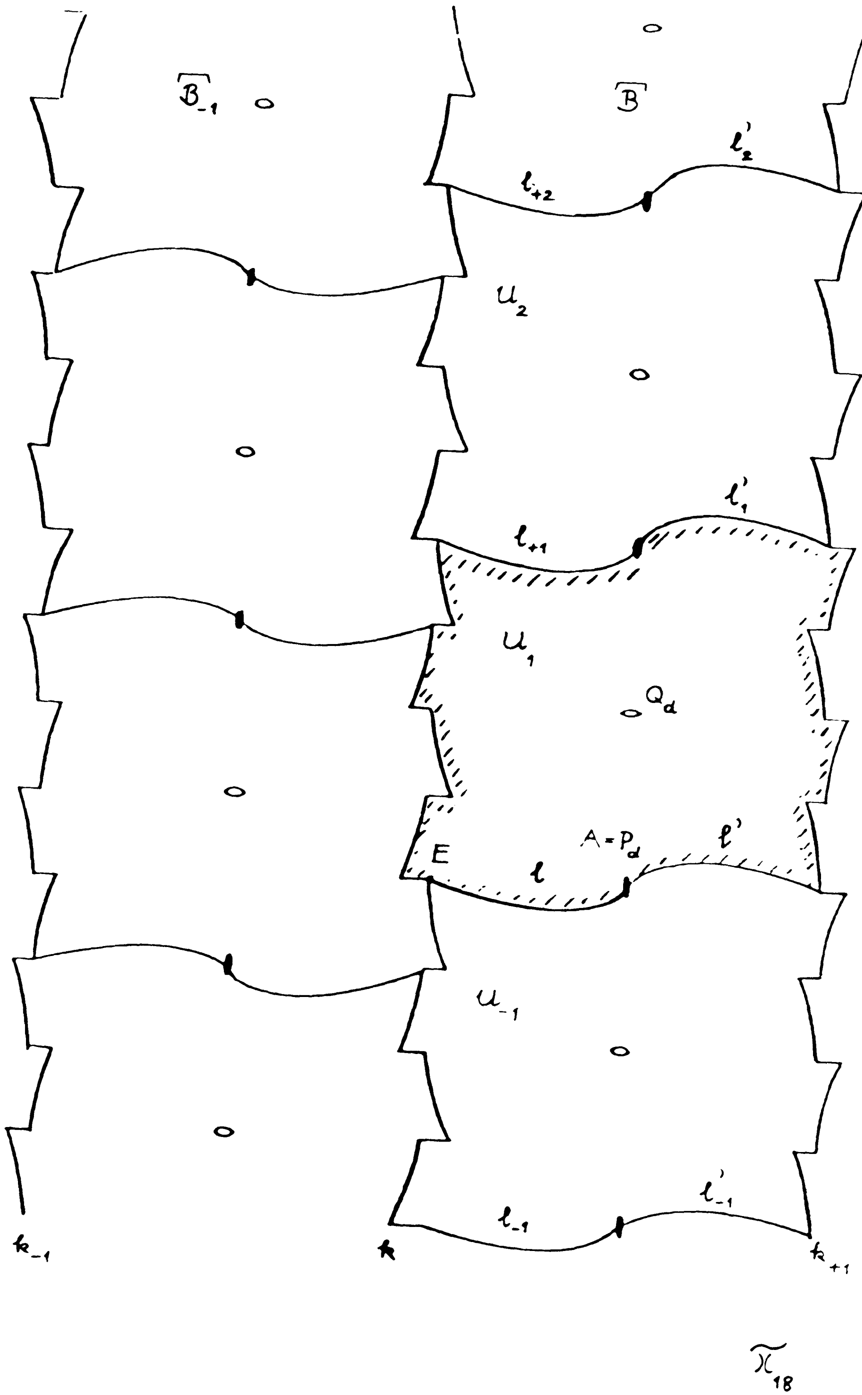
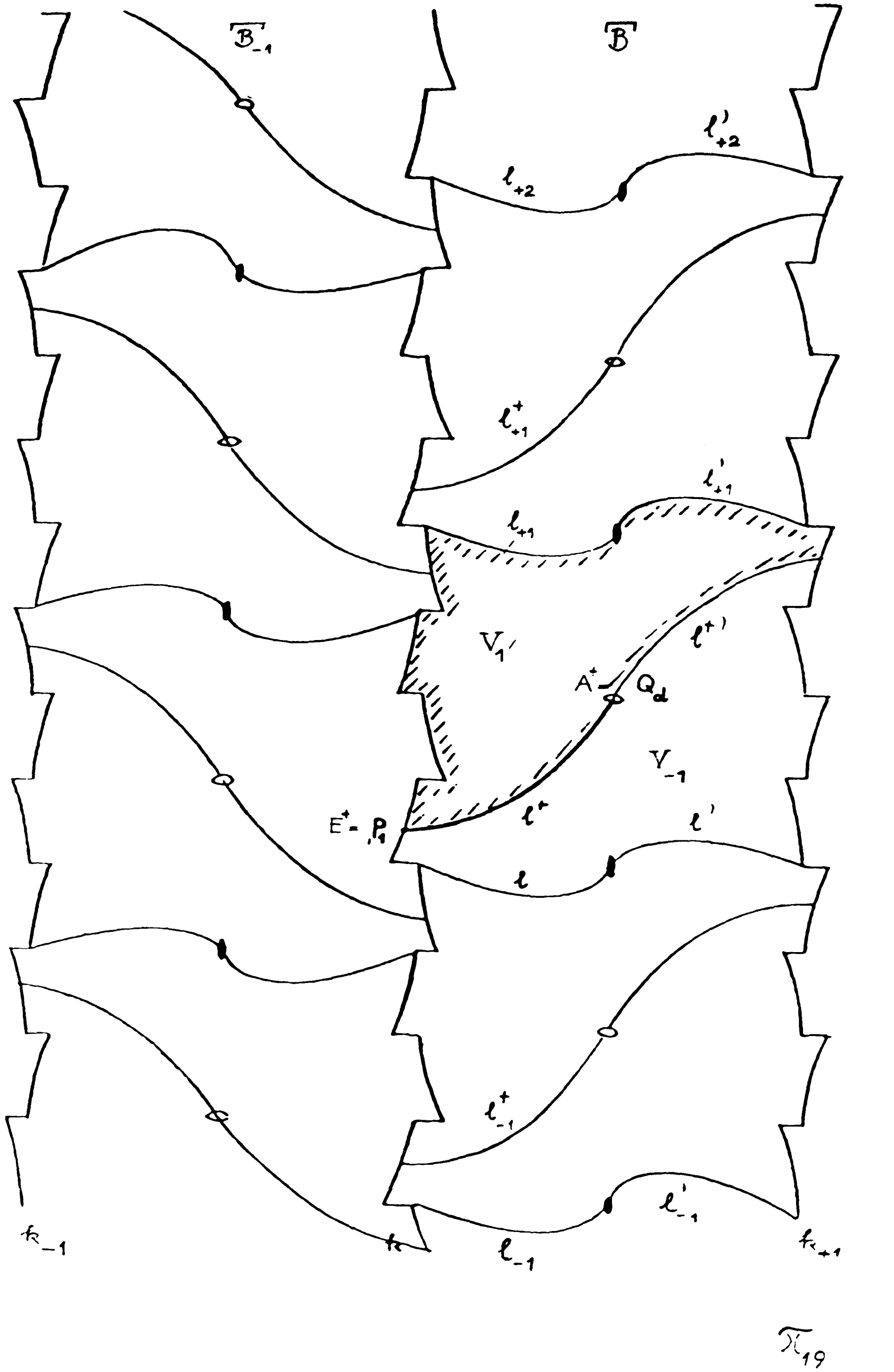
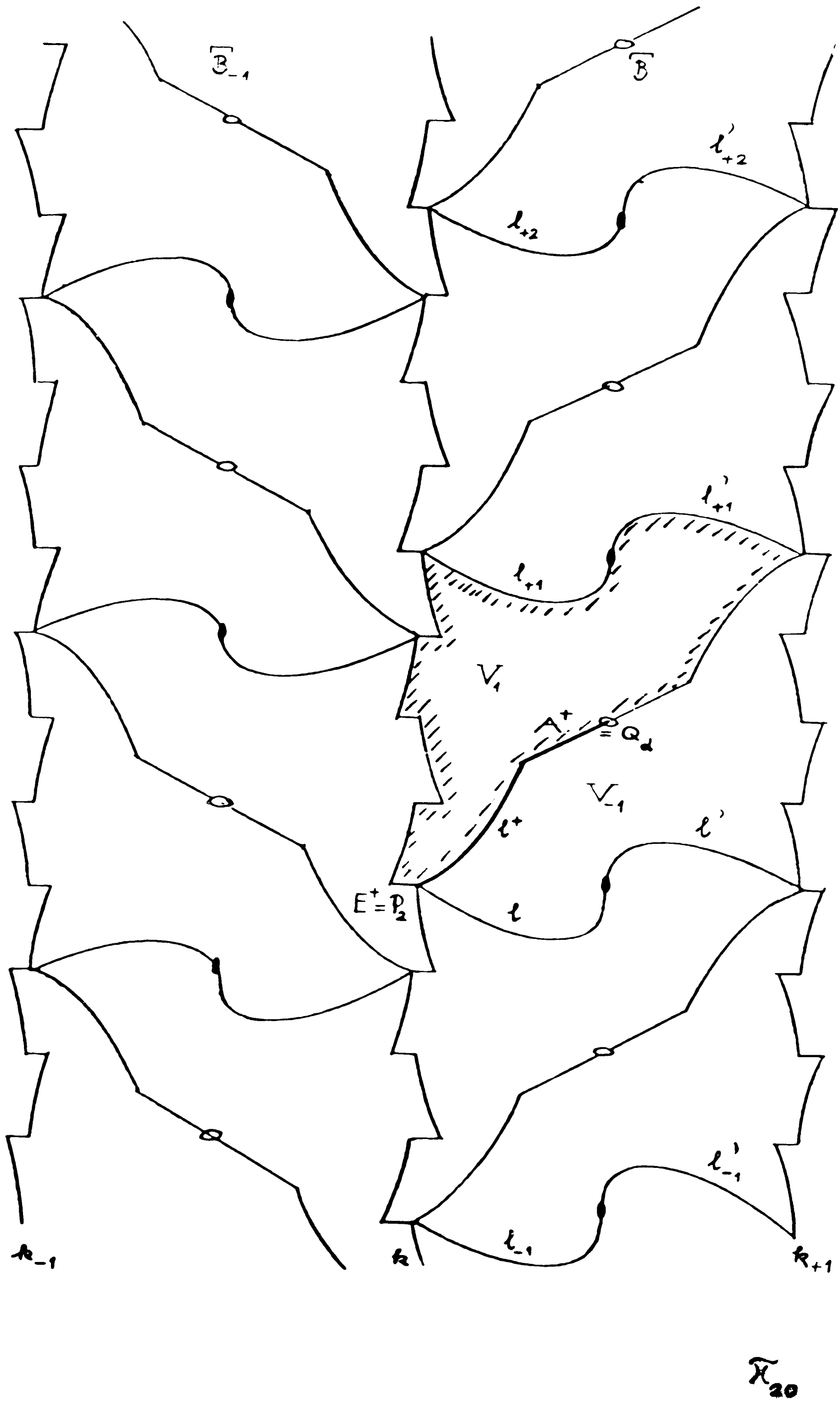


FIGURE 23. *Inhomogeneous tiling* Π_{18}. The tiling Π_{18} is generated by subdivision of the special tiling $\overline{\Pi}$ described under Figure 21. The newly introduced boundry curve l satisfies condition (b).



FIGURES 24, 25, 26. *Inhomogeneous tilings* Π_{19}, Π_{20}, Π_{21}. The tilings Π_i ($i = 19, 20, 21$) are generated by subdivision of Π_{18}. The newly introduced boundary curve l^+ satisfies condition (b2.1) in Π_{19} (Figure 24), (b2.2) in Π_{20} (Figure 25), (b2.3) in Π_{21} (Figure 26).



$\overline{B}_{-1}$
$\overline{B}$
l'_{+2}
l_{+2}
l'_{+1}
l_{+1}
V_1
A^+
$=Q_2$
V_{-1}
l^+
l'
$E^+=P_2$
l
l'_{-1}
l_{-1}
k_{-1}
k
k_{+1}
$\overline{K}_{20}$

FIGURE 25

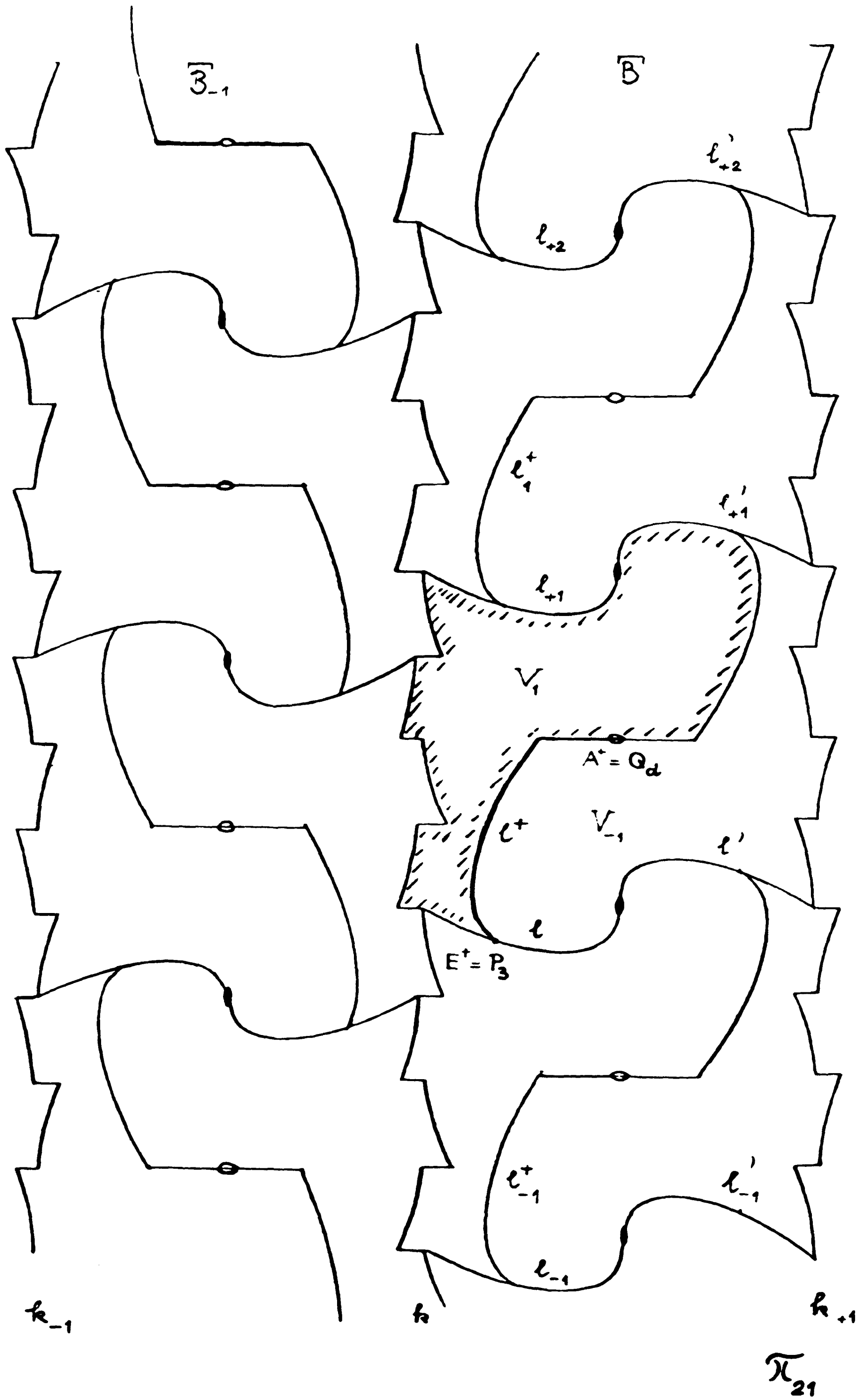


FIGURE 26

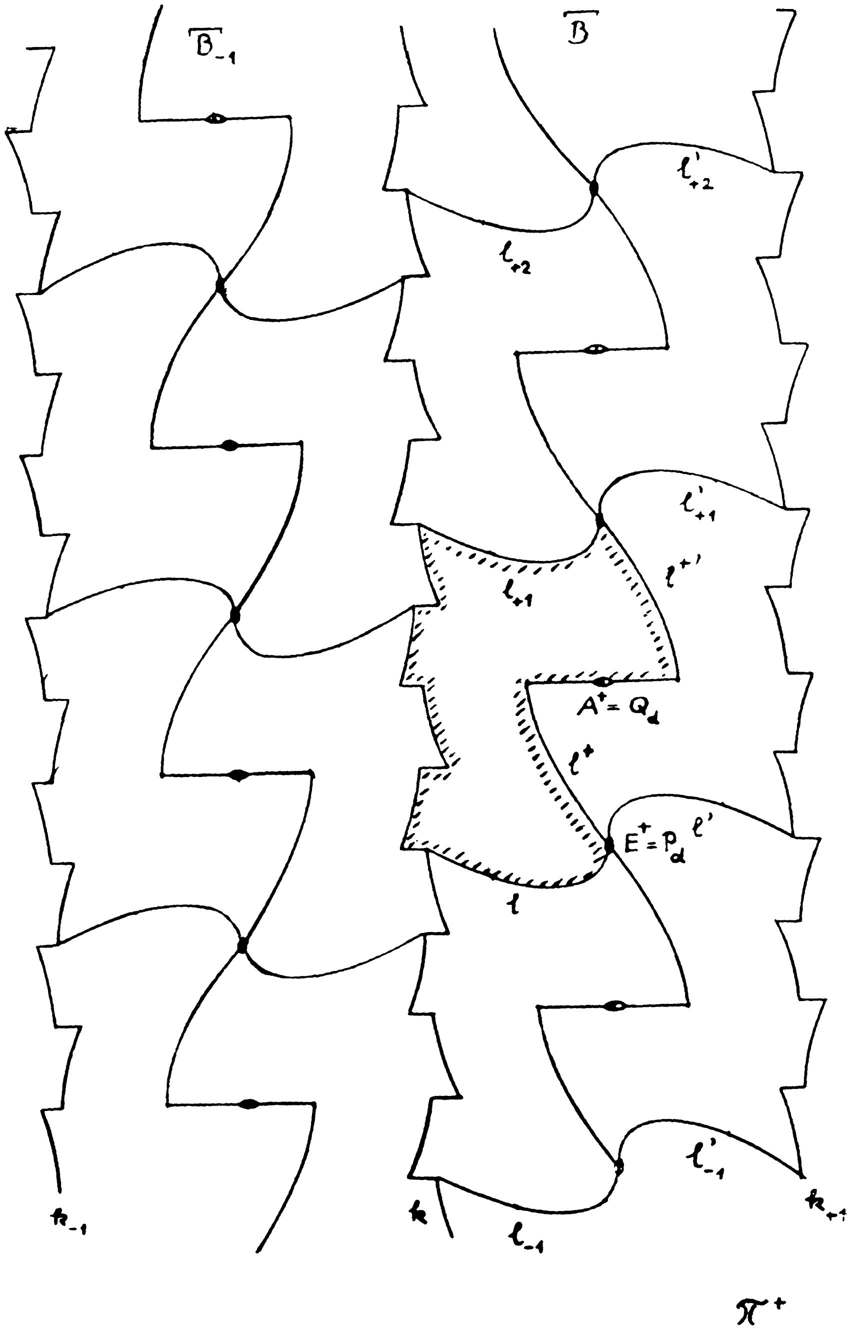


FIGURE 27. *Inhomogeneous tiling* Π^+. The tiling Π^+ constructed in Figure 27 is generated by subdivision of Π_{18}. The newly introduced boundary curve l^+ satisfies condition (b2.4). Π^+, however, belongs to the same class of inhomogeneous tilings as Π_{14} (Figure 19), which can be seen directly. There is no topological difference between Π^+ and Π_{14}: $z = (3, 3, 3, 4, 4)$, and the construction rules for their tiles are the same.

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