COMPACT QUOTIENTS BY C*-ACTIONS

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Let $X$ be a compact normal complex space on which $C^*$ acts ‘in a
nice manner’. We describe all invariant open subsets $U$ of $X$ such that
the holomorphic map $U \to U/C^*$ of $U$ onto the categorical quotient for
the category of compact complex spaces, $U/C^*$, is locally Stein. The
description depends on a partial ordering of the fixed point components
which arises from the Bialynicki-Birula decompositions of $X$.

Introduction. Let $\rho: T \times X \to X$ be a meromorphic action, (cf. $\S 1$),
of $T = C^*$ on an irreducible compact normal complex analytic space $X$.
Such an action is said to be locally linearizable if and only if given any
$x \in X$ there is a $T$-invariant neighborhood $V$ of $x$ and a proper $T$-equi-
variant holomorphic embedding of $V$ into $C^n$ with $T$ acting linearly on $C^n$.

In this paper we solve the following problem:

Describe all $T$-invariant Zariski open subsets $U$ of $X$, such that $U/T$
is a compact complex analytic space and $U \to U/T$ is a semi-geometric
quotient (i.e. a categorical quotient which is locally Stein cf. (1.8)).

This problem has been solved by A. Bialynicki-Birula and A. Sommese,
[B − B + S], under the above setting when $U$ contains no fixed points
and by A. Bialynicki-Birula and J. Święcieka, [B − B + Sw], when the
action is algebraic and $X$ is a compact algebraic variety.

As in [B − B + S], our description of semi-geometric quotients $U \to
U/T$ is intimately linked to a certain partial ordering of the fixed point
components $F_1, \ldots, F_r$. So that we can state our results precisely we shall
introduce the following notation. We assume that all analytic spaces are
Hausdorff, reduced and have countable topology.

Let $\{F_1, \ldots, F_r\}$ be the connected components of the fixed point set of
$T$, $X^T$. Define $\phi^+, \phi^-: X \to X^T$ by $\phi^+(x) = \lim_{t \to 0} tx$ and $\phi^-(x) =
\lim_{t \to \infty} tx$, respectively.

Let $X^+_i = \{x \in X| \phi^+(x) \in F_i\}, i = 1, \ldots, r$, and $X^-_i = \{x \in
X| \phi^-(x) \in F_i\}, i = 1, \ldots, r$.

An index $i$ is said to be directly less than an index $j$ if $C_{ij} = (X^+_i - F_i)
\cap (X^-_j - F_j) \neq \emptyset$. We say that $i$ is less than $j$, denoted $i < j$, if there
exists a sequence $i = i_0, \ldots, i_k = j$ such that $i_l$ is directly less than $i_{l+1}$ for
$l = 0, \ldots, k - 1$. This relation forms an ordering of the indices $\{1, \ldots, r\}$. 149
A cross section of \{1, \ldots, r\} is a division of \{1, \ldots, r\} into two non-empty disjoint subsets \(A^-\) and \(A^+\) satisfying the condition that \(i \in A^-\) and \(j < i\) implies that \(j \in A^-\).

A semi-cross section of \{1, \ldots, r\} is a division of \{1, \ldots, r\} into three disjoint subsets, \(A^-, A^0, A^+\), at least two of which are nonempty, which satisfy the following two conditions:

(a) if \(i < j\) and \(j \in A^0\) then \(i \notin A^0\)
(b) if \(A^+ \neq \emptyset\) then \((A^- \cup A^0, A^+)\) is a cross section and if \(A^- \neq \emptyset\) then \((A^-, A^0 \cup A^+)\) is a cross section.

A subset \(B\) of \(X\) is a semi-sectional set if \(B = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-\) for some semi-cross section \((A^-, A^0, A^+)\).

**Main Theorem.** Let \(\rho: T \times X \rightarrow X\) be as above and let \(U\) be a \(T\)-invariant Zariski open subset of \(X\). Then \(U/T\) is a compact complex analytic space and \(U \rightarrow U/T\) is a semi-geometric quotient if and only if \(U\) is a semi-sectional set with respect to some semi-cross section \((A^-, A^0, A^+)\). \(\square\)

Our proof uses the techniques of [B - B + S].

We conclude this paper with a simple illustration of the Theorem for the case of a diagonal action of \(C^*\) on \(\mathbb{P}^1 \times \mathbb{P}^1\).

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1. **Notation and background material.** In this section we establish the pertinent notation and background material needed for the proof of the Theorem. The principal reference for this material is [B - B + S].

Let \(T\) denote \(C^*\), the multiplicative group of non-zero complex numbers. A holomorphic action \(\rho: T \times X \rightarrow X\) of \(T\) on a normal compact analytic space \(X\) is said to be a *meromorphic action* if \(\rho\) extends to a meromorphic map \(\tilde{\rho}: \mathbb{P}^1 \times X \rightarrow X\), where \(\mathbb{P}^1\) in one-dimensional complex projective space. This condition is satisfied if \(X\) is a Kaehler manifold and \(X^T\) has non-empty intersection with every connected component of \(X\), [So].

The maps \(\phi^+, \phi^-: X \rightarrow X^T\) as defined in the introduction always exist for meromorphic actions, [Kor]. The collections of subsets \(\{X_i^+\}_{i = 1, \ldots, r}\) and \(\{X_i^-\}_{i = 1, \ldots, r}\) form two decompositions of the space \(X\), called respectively the plus and the minus Bialynicki-Birula decompositions. They satisfy the following properties:

\[(1.1)\] (a) \(X = \bigcup X_i^+ = \bigcup X_i^-\) is a disjoint union of \(T\)-invariant sets.
(b) There are two special components of $X^T$, $F_i$ called the source and $F_r$ called the sink (renumbering if necessary), such that $X_i^+$ and $X_r^-$ are Zariski open in $X$.

(c) Each $X_i^+$ and $X_j^-$ is a constructible set, i.e., the finite union of locally closed sets.

These properties were proven in the algebraic category by Bialynicki-Birula, [B — B], and in the Kaehler category by Carrell and Sommese, [C + S], and Fujiki [Fu_2].

We will now state a result found in [B — B + S] which is modeled on a result of Fujiki [Fu_2]. It provides the basis for the proof of the Main Theorem.

**Theorem (1.2).** Let $\rho: T \times X \to X$ be a meromorphic action of $T$ on an irreducible compact complex analytic space $X$. There is a diagram:

$$
\begin{array}{ccc}
Z & \xrightarrow{\mu} & X \\
\downarrow f & & \\
Q & &
\end{array}
$$

with the following properties:

(a) $f$ is a flat morphism of irreducible compact complex spaces $Z$ and $Q$.

(b) $\mu$ is a bimeromorphic holomorphic map of $Z$ onto $X$ such that the restriction of $\mu$ to each fiber $Z_q = f^{-1}(q)$ is an embedding.

(c) There is a natural holomorphic action of $T$ on $Z$ making $f$ and $\mu$ $T$-equivariant with respect to the trivial action on $Q$ and $\rho$ on $X$ respectively.

(d) There is a dense Zariski open subset $\mathcal{U}$ of $Q$ such that for every $q \in \mathcal{U}$, $Z_q$ is reduced and $\mu(Z_q)$ is the closure of a $T$-orbit from $X_i^+ \cap X_r^-$. 

(e) Every fiber $Z_q$ of $f$ is one-dimensional and for fibers $Z_q$, $Z_{q'}$ that are reduced, $\mu(Z_q) = \mu(Z_{q'})$ if and only if $q = q'$.

(f) $\mu(Z_q)$ is connected and meets $F_i$, the source, and $F_r$, the sink, for all $q \in Q$.

(g) For all $q \in Q$, $Z_q \cap Z^T$ is finite.

(h) Any continuous map $\tau: A \to Y$ of an open subset $A$ of $Q$ to a complex analytic space $Y$ which is holomorphic on a Zariski open dense subset of $A$ is holomorphic on all of $A$. □

Let $K$ be a compact complex space and let $\text{Comp}(K)$ be the set of all compact subsets of $K$. The *Hausdorff metric* on $\text{Comp}(K)$ is defined by:

$$
\text{dist}(A, B) = \max \left\{ \min_{a \in A} \text{dist}(a, b) \right\} + \max \left\{ \min_{b \in B} \text{dist}(b, a) \right\}
$$
where dist$(a, b)$ is the metric on $K$. Let $A, A_i, i \in I$, be elements of Comp$(K)$. When we say that the $A_i$'s converge to $A$ we mean they converge in the Hausdorff metric.

We have the following Corollary to (1.2).

**Corollary (1.3).** Let $\rho, X, Q, Z, F$ and $\mu$ be as in (1.2). Let $\{q_n\}$ be a sequence in $Q$. If $q_n$ converges to $q$ in $Q$ then $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in $X$.

**Proof.** We claim that $q_n$ converges to $q$ in $Q$ implies that $Z_{q_n}$ converges to $Z_q$ in the Hausdorff metric in $Z$, where $Z_{q_n} = f^{-1}(q_n), Z_q = f^{-1}(q)$. Let $z$ be an arbitrary point of $Z_q$, then any open neighborhood $V$ of $z$ must intersect $Z_{q_n}$ for $n \to 0$. Suppose not, since $f: Z \to Q$ is flat it is an open map and hence $f(V)$ is an open neighborhood of $q$. If $Z_{q_n}$ does not intersect $V$ then $q_n$ would not be an element of $f(V)$ and therefore $q_n$ would not converge to $q$. Thus we have that $Z_{q_n}$ converges to $Z_q$ and by the continuity of $\mu: Z \to X$ that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. \qed

**Definition (1.4).** Let $\rho: T \times X \to X$ be a meromorphic action of $T$ on a normal compact analytic space. We say that $\rho$ is a **locally linearizable action** if given any $x \in X$ there is a $T$-invariant neighborhood $V$ of $x$ and a proper $T$-equivariant holomorphic embedding of $V$ into $C^N$ with $T$ acting linearly on $C^N$.

**Proposition (1.5).** A holomorphic action $\rho: T \times X \to X$ on a normal irreducible compact complex space $X$ is locally linearizable if either of the following is true:

(a) $X$ is an algebraic variety and $\rho$ is an algebraic action or
(b) $X^T \neq \emptyset$ and $X$ can be equivariantly embedded in a compact Kaehler manifold $Y$ with a holomorphic action $\tilde{\rho}: T \times Y \to Y$.

**Proof.** (a) is due to Sumihiro [Su] and (b) is due to Koras [Kor2]. \qed

We shall also use extensively the following (cf. Corollary (0.2.4) of [B - B + S]).

**Proposition (1.6).** Let $\rho: T \times X \to X$ be a locally linearizable action of $T$ on a compact analytic space $X$. Given any $q \in Q$ we can choose $\{x_1, \ldots, x_k\}$ in $\mu(Z_q) - \mu(Z_q)^T$ with:

(a) $\phi^+(x_1) \in F_1$ and $\phi^-(x_k) \in F_r$
(b) $\phi^-(x_j) = \phi^+(x_{j+1})$ for $j = 1, \ldots, k - 1$
(c) if $\phi^-(x_j) = \phi^+(x_i)$, then $i = j + 1$
(d) $T\{x_1, \ldots, x_k\} = \mu(Z_q)$

Moreover, if $X$ is normal, then $\mu(Z_q) \cap F_1 = \{x_1\}$, $\mu(Z_q) \cap F_r = \{x_k\}$. □

We note that the last statement of Proposition (1.6) may not hold if $X$ is not normal, i.e., it is possible in such a case that $F_1 = F_r$, for example simply identify a point of $F_1$ with a point of $F_r$.

**Corollary** (1.7). Let $X$ and $\rho$ be as in (1.6). For any connected component $F_i$ of $X^T$, $F_1 < F_i < F_r$. □

Let $\overline{\rho} : G \times Z \to Z$ be an action of a reductive group $G$ on complex space $Z$. We can define an equivalence relation on the points of $Z$ by $x \sim y$ if and only if there is a sequence of points $x = x_0, x_1, \ldots, x_n = y$ in $Z$ such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$, $i = 0, \ldots, n - 1$. We define $Z/G$ to be the set of equivalence classes under the above relation and define a map $\pi : Z \to Z/G$ by $\pi(x) = [x]$, where $[x]$ denotes the equivalence class containing $x$. $Z/G$ is given the quotient topology, i.e. $V$ is an open subset of $Z/T$ if and only if $\pi^{-1}(V)$ is an open subset of $Z$. We call $\pi : Z \to Z/G$ the categorical quotient of $Z$ by $G$.

We note that in general our definition of a categorical quotient does not coincide with the usual definition, in which the equivalence relation is defined by the invariant holomorphic functions. Our definition implies that fibers of $\pi$ are connected and thus the quotient, $X/G$, need not be Hausdorff. When the quotient is assumed to be Hausdorff either definition will suffice.

**Definition** (1.8). A categorical quotient $\pi : Z \to Z/G$ is a semi-geometric quotient if it is locally Stein, i.e. given any point $y \in Z/G$ there is a neighborhood $W$ of $y$ such that $\pi^{-1}(W)$ is Stein.

**Lemma** (1.9). Let $\rho : T \times X \to X$ be a meromorphic action of $T$ on $X$ a compact complex analytic space. Let $U$ be a $T$-invariant open subset of $X$. If $\pi : U \to U/T$ is a semi-geometric quotient then each fiber contains at most one fixed point.

**Proof.** Let $x, y \in U^T$ and suppose $x \sim y$. Then we can find a sequence of fixed points in $U$, $x = z_0, z_1, \ldots, z_n = y$ such that $z_i \in [x]$ and $z_i$ is directly related to $z_{i+1}$, i.e. there is a point $z \in U$ with $\phi^+(z) = z_i$ and $\phi^-(z) = z_{i+1}$ or $\phi^+(z) = z_{i+1}$ and $\phi^-(z) = z_i$. Thus, if $x \neq y$ then
\( \pi^{-1}(\{x\}) \) contains \( \phi^+(z) \cup Tz \cup \phi^-(z) \) which is homeomorphic to \( \mathbb{P}^1 \) contradicting the assumption that \( \pi \) is locally Stein.

\[ \boxed{\text{COROLLARY (1.10). Let } \rho, U, \text{ and } \pi \text{ be as in (1.9). Then } \pi \text{ restricted to } U^T \text{ is one to one onto } \pi(U^T).} \]

The above allows us to identify \( U^T \) with a subset of \( U/T \), namely \( \pi(U^T) \).

\[ \boxed{\text{LEMMA (1.11). Let } \rho, U, \text{ and } \pi \text{ be as in (1.9). Then fibers of } \pi \text{ are either orbits or } x^+ \cup x^- \text{ for some } x \in U^T, \text{ where } x^+ = \{ z \in X | \phi^+(z) = x \} \text{ and } x^- = \{ z \in X | \phi^-(z) = x \}.} \]

\[ \textbf{Proof.} \] By (1.9) each fiber contains at most one fixed point. If the fiber does not contain a fixed point then it must consist of a single orbit since the intersection of the closures of two distinct orbits is either empty or contained in the set of fixed points. If the fiber contains a fixed point then since \( U \) is open and \( T \)-invariant, it follows that the fiber contains \( x^+ \cup x^- \). For the fiber to contain anything else it must contain a second fixed point which is impossible. Thus the fibers are as stated.

\[ \boxed{\text{LEMMA (1.12). Let } \rho: T \times X \rightarrow X \text{ be a meromorphic action of } T \text{ on } X \text{ a normal compact complex analytic space } X. \text{ Let } U \text{ be a } T \text{-invariant open subset of } X \text{ such that } \rho: U \rightarrow U/T \text{ is a semi-geometric quotient. If } U/T \text{ is Hausdorff it possess the structure of a complex analytic space and } \pi \text{ is a holomorphic map.}} \]

\[ \textbf{Proof.} \] The definition of a semi-geometric quotient implies that we may cover \( U \) with \( \pi \)-saturated Stein sets, \( A_i \). Each \( A_i/T \) is a complex Stein space such that \( \pi: A_i \rightarrow A_i/T \) is holomorphic, [Sn]. Since the structure on \( A_i/T \) is induced by the invariant holomorphic functions on \( A_i \), it follows easily that the structure on the \( A_i/T \)'s are compatible. Thus, if \( U/T \) is Hausdorff it is a complex analytic space and \( \pi \) is holomorphic.

\[ \boxed{2. \text{ Semi-geometric quotients. Throughout this section we shall assume that } \rho: T \times X \rightarrow X \text{ is a locally linearizable action of } T = \mathbb{C}^* \text{ on an irreducible normal compact complex analytic space } X \text{ with fixed point components } F_1, \ldots, F_r. \text{ We want to describe all } T \text{-invariant Zariski open subsets } U \text{ of } X \text{ whose quotient } U/T \text{ is semi-geometric and a compact complex space. The} } \]
following propositions enable us to partition all such \( U \) into these three disjoint classes:

**Class I.** \( U \) contains no fixed point components, i.e. \( U \subset X - X^T \).

**Class II.** The only fixed point component \( U \) contains is either the source \( F_i \) or the sink \( F_r \).

**Class III.** \( U \) contains fixed point components \( F^i \varphi \), and if \( U \) contains \( F^t \) and \( F^j \) they are not directly related to each other.

**Proposition (2.1).** Let \( U \) be a \( T \)-invariant Zariski open subset of \( X \) whose quotient \( U/T \) is semi-geometric and a compact complex space. If \( F_i \cap U \neq \emptyset \), then \( X_i^+ \cup X_i^- \subset U \).

**Proof.** Let \( x \in F_i \cap U \). Since \( U \) is open and \( T \)-invariant the sets \( x^+ \) and \( x^- \) must both be contained in \( U \). Since \( X_i^+ = \bigcup_{x \in F_i} x^+ \) and \( X_i^- = \bigcup_{x \in F_i} x^- \) we must only show that if \( F_i \cap U \neq \emptyset \), then \( F_i \subset U \). Furthermore, since \( F_i \cap U \) is open in \( F_i \), this reduces to showing that \( F_i \cap U \) is closed.

Let \( x \in F_i \cap U \subset F_i \) and let \( \{x_n\} \) be a sequence of distinct points contained in \( F_i \) converging to \( x \). Since \( U \to U/T \) is a semi-geometric quotient each distinct \( x_n \) must have a distinct image in \( U/T \), and so we may consider \( \{x_n\} \) as a sequence in \( U/T \). Now \( U/T \) is assumed to be compact and so, passing to a subsequence and renumbering if necessary, we have that \( x_n \) converges to some \( y \in U/T \). The locally Stein condition of semi-geometric quotients implies that we can find a neighborhood \( W_T \subset U/T \) of \( y \) and a Stein set \( W = \pi^{-1}(W_T) \subset U \).

We can assume that \( \{x_n\} \) is contained in \( F_i \cap W \), which is a closed \( T \)-invariant subset of \( W \). By Corollary 3.6 of [Sn] since \( W \) is Stein we have that \( \pi(F_i \cap W) = F_i \cap W_T \) is closed in \( W_T \) (we note that in this case our definition of categorical quotient coincides with that of [Sn]). This implies that \( y \in \pi(F_i \cap W) \) and thus by identification, cf (1.10), \( y \in F_i \cap W \subset U \). The convergence of the \( \{x_n\} \) yields \( y = x \).

**Proposition (2.2).** Let \( U \) be as in (2.1). Let \( F_i \) and \( F_j \) be two fixed point components and suppose that \( F_i \subset U \). If \( F_j \) is directly related to \( F_i \) then \( F_j \not\subset U \).

**Proof.** Assume \( F_i < F_j \). Suppose \( F_j \subset U \) then we can find an \( x \in U \) such that \( \phi^+(x) \in F_i \) and \( \phi^-(x) \in F_j \). (2.1) implies that \( U \cup \phi^+(x) \cup Tx \cup \phi^-(x) \) which is biholomorphic to \( \mathbb{P}^1 \). This contradicts the local Steinness of the quotient since there can be no neighborhood of \( \pi(x) \) in \( U/T \) whose inverse image in \( U \) is Stein.
**Proposition (2.3).** Let $U$ be as in (2.1). If $U$ contains the source, $F_s$, or the sink, $F_r$, then $U$ does not contain any other fixed point component.

**Proof.** Assume $U \supset F_s$. Since $F_r$ is directly related to $F_s$, $F_r \not\subset U$. Let $F_i \subset U$, $i \neq 1, r$. Let $x \in F_i$ and choose $q \in Q$ such that $x \in \mu(Z_q)$, where $Q, Z_q$ and $\mu$ are as in (1.2). Let $Q$ also be as in (1.2) and choose a sequence $\{q_n\} \subset Q$ converging to $q$. $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ by (1.3). Thus we can find a sequence of points $\{x_n\} \subset U$ such that $x_n \in \mu(Z_{q_n})$ and $x_n$ converges to $x$. Let $\{y_n\}$, $y$ be the image of $\{x_n\}$, $x$ respectively in $U/T$. $\{y_n\} \subset$ image of $F_s$, which is identified with $F_s$, but since $x \not\in F_s$, $y \not\in F_s$. But every open neighborhood of $y$ meets $\{y_n\}$, so $y$ is in the closure of $F_s$ in $U/T$. Since $F_s$ is closed this contradiction proves the proposition. □

We now show our Main Theorem holds for each of the three Classes separately.

Assume $U$ is of Class I, i.e. $U \subset X - X_T$. This case was done in [B − B + S]. Their description is given in terms of cross sections. However, by considering a cross section $(A^-, A^0, A^+)$ as a semi-cross section $(A^-, A^0, A^+)$ with $A^0 = \emptyset$, their result coincides with our Main Theorem.

We next assume $U$ is of Class II, i.e. the only fixed point component $U$ contains is either $F_s$ or $F_r$. To simplify things we assume $F_s \subset U$. 

**Proposition (2.4).** Let $U$ be as in the preceding paragraph. If the quotient $U/T$ is semi-geometric and a compact complex space then there exists a semi-cross section $A = (A^-, A^0, A^+)$ such that $U$ is a semi-sectional set with respect to $A$.

**Proof.** Since $F_s \subset U$ we have by (2.1) that $X^+_s \subset U$. The proof of Proposition (2.3) can in fact be used to show that $U$ contains only $X^+_s$, i.e. $U = X^+_s$. Hence:

$$U = X^+_s = X - \bigcup_{i=2}^{r} X^+_i = X - \bigcup_{i \in A^+} X^+_i - \bigcup_{j \in A^-} X^-_j$$

where $A^+ = \{2, \ldots, r\}$, $A^- = \emptyset$. Taking $A^0 = \{1\}$ we have the desired semi-cross section. □

**Proposition (2.5).** Suppose $U$ is the semi-sectional set associated to the semi-cross section $(A^-, A^0, A^+)$ where $A^- = \emptyset$, $A^0 = \{1\}$, $A^+ = \{2, \ldots, r\}$. Then $U$ is a $T$-invariant Zariski open subset of $X$ whose quotient $U/T$ is semi-geometric and a compact complex space.
Proof. By definition we have:

\[ U = X - \bigcup_{i \in A^+} X^+_i - \bigcup_{j \in A^-} X^-_j. \]

Using the facts that \( A^- = \emptyset \) and that \( X = \bigcup_{i=1}^r X^+_i \) which is a disjoint union we have that \( U = X^+_1 \). Thus we have that \( U \) is a \( T \)-invariant Zariski open subset of \( X \).

Since \( U/T = X^+_1/T = F_1 \) we have that \( U/T \) is a compact complex analytic space. For each \( x \in F_1 \) let \( V_x \) be a \( T \)-invariant Stein neighborhood of \( x \) in \( U \) given by definition of the action being locally linearizable. Then we have covered \( U/T \) by sets whose inverse images in \( U \) are Stein. \( U/T \) is obviously a categorical quotient and so it is a semi-geometrical quotient. The proposition is proven. \( \square \)

Combining (2.4) and (2.5) gives the Main Theorem for Class II sets.

From now on unless stated otherwise, we assume that \( U \) is of Class III, i.e. \( U \) contains fixed point components \( F_i, i \neq 1, r \) and any two are not directly related.

**Lemma** (2.6). Let \( U \) be a \( T \)-invariant Zariski open subset of \( X \) whose quotient \( U/T \) is semi-geometric and a compact complex space. Then \( X^+_1 \cap X^-_r \subset U \).

**Proof.** Let \( C = X^+_1 \cap X^-_r \), then \( C \) is a Zariski open subset of \( X \). Since \( X \) is assumed to be irreducible \( C \) must also be irreducible. By Zariski openness and denseness \( C \) must intersect \( U \). The same proof as that of Lemma (1.1.1) in \([B - B + S]\) yields that \( C \) is contained in \( U \). \( \square \)

Let \( Q \) be the subset of \( Q \) from Theorem (1.2). The above Lemma allows us to identify \( Q \) with a dense open subset of \( U/T \).

We have need of the following fact. Let \( A \) be a complex space and let \( B \) be a dense subset of \( A \). Let \( \{x_n\} \) be a sequence of points of \( A \), then we can find a sequence of points contained in \( B \), \( \{y_n\} \), such that \( \text{dist}(x_n, y_n) < 1/n \) where \( \text{dist} \) is the metric on \( A \). If \( \{x_n\} \) diverges then \( \{y_n\} \) diverges and if \( \{x_n\} \) converges then \( \{y_n\} \) converges to the same point. Thus if we have a sequence in \( U/T \) we can assume it is contained in \( Q \).

We shall make the following convention. Let \( y \in U/T \), when we choose a point \( x \in \pi^{-1}(y) \) we assume \( x \) is the unique fixed point if \( \pi^{-1}(y) \) contains one, otherwise \( x \) may be any point of \( \pi^{-1}(y) \).
Lemma (2.7). Let $U$ be a $T$-invariant Zariski open subset of $X$. Then $U/T$ is a semi-geometric quotient and a compact complex space if and only if given $q \in 0$, either:

(a) There exists a $y \in X - X^T$ such that $\mu(Z_q) \cap U = Ty$ or
(b) There exist $y_1, y_2 \in X - X^T$ with $\phi^-(y_1) = \phi^+(y_2)$ such that $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$, where $Q, Z_q$ and $\mu$ are as given in (1.2).

Proof. To prove the necessity of (a) or (b) we first shall show that $\mu(Z_q) \cap U \neq \emptyset$. Suppose not, we can find a sequence $\{q_n\} \subset {\mathcal Q} \subset Q$, such that $q_n$ converges to $q$ and thus $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in the Hausdorff metric. We note this implies that any open neighborhood of $\mu(Z_q)$ contains $\mu(Z_{q_n}), n \gg 0$. By (2.6) we can consider $\{q_n\} \subset U/T$. By assumption $U/T$ is compact and therefore, after passing to a subsequence and renumbering if necessary, $q_n$ converges to an element $y$ of $U/T$. Let $x \in \pi^{-1}(y)$. Let $V_1$ and $V_2$ be disjoint open subsets of $X$ which contain $\mu(Z_q)$ and $x$ respectively. We can assume that $V_2 \subset U$. $\pi(V)$ contains a dense open subset consisting of elements of $Q$. Since $q_n$ converges to $y$ and $y \in \pi(V)$ we can replace elements of $\{q_n\}$ for $n \gg 0$, with elements of $\pi(V) \cap Q$ without affecting convergence, so we may consider $q_n \in \pi(V)$ for $n \gg 0$. This implies $\pi^{-1}(q_n) \cap V_2 \neq \emptyset$. But $\pi^{-1}(q_n) = \mu(Z_{q_n}) \cap U \subset V_1$. This contradiction implies that $\mu(Z_q) \cap U \neq \emptyset$.

We now claim that $\mu(Z_q) \cap U$ is connected. Obviously this is true if $q \in Q$. Suppose $q$ is not an element of $Q$ and that $\mu(Z_q) \cap U$ is not connected. Then we can find two disjoint closed invariant sets, $S_1$ and $S_2$, with $S_1 \cup S_2 = \mu(Z_q) \cap U$. Note $\pi(S_1) \neq \pi(S_2)$. As before we can find a sequence $\{q_n\}$ contained in $Q$, such that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. Let $x_n = \pi(\mu(Z_{q_n}))$, then by continuity we have that $x_n$ converges to both $\pi(S_1)$ and $\pi(S_2)$ in $U/T$. This contradicts $U/T$ being Hausdorff.

$\mu(Z_q) \cap U$ can contain at most one fixed point since if it contained two, connectivity would imply that it contains $\mathbb{P}^1$ and then $U/T$ could not be a semi-geometric quotient. If $\mu(Z_q) \cap U$ contains no fixed point it has the form of (a), if it has a fixed point, $x$, since $x^+ \cup x^- \subset U$ by (2.1) it has the form of (b).

Suppose $\mu(Z_q) \cap U$ is of the form (a) or (b) for any $q \in Q$. We will first show that $\pi: U \to U/T$ is a semi-geometric quotient. The fiber over any point in $U/T$ must either be a single orbit or $x^+ \cup x^-$ for some fixed point $x$. This can be seen by considering $\mu(Z_q) \cap U$. If it is just an orbit then it goes to a point in $U/T$ and is the fiber over that point. If it contains a fixed point then every $\mu(Z_q) \cap U$ which contains the $x$ will go to the same point in $U/T$. Thus the fibers are as stated above and it is
easily seen that this implies that \( U/T \) is a categorical quotient. For each \( y \in U/T \) choose \( x \in \pi^{-1}(y) \). For each \( x \) let \( V_x \) be the \( T \)-invariant Stein neighborhood of \( x \) in \( X \) given by the action being locally linearizable. Since \( U \) is \( T \)-invariant we can consider that \( V_x \subseteq U \) and by the description of the fibers that the \( V_x \) are \( \pi \)-saturated. Thus we can cover \( U/T \) with sets whose inverse images are Stein and, so the quotient \( U/T \) is semi-geometric.

Assume \( U/T \) is not Hausdorff, then we can find \( \{y_n\} \subseteq U/T \) with \( y_n \) converging to two distinct points \( z_1 \) and \( z_2 \). We may assume \( \{y_n\} \subseteq \mathcal{U} \) and so \( \{y_n\} \subseteq Q \) which is compact. We may assume, after passing to a subsequence and renumbering if necessary that \( y_n \) converges to \( q \in Q \). Let \( x_i \in \pi^{-1}(z_i) \) and \( V_i \) be an open neighborhood of \( z_i \) in \( U/T \). \( V_i \supset y_n \) for \( n \gg 0 \) and so we can find a sequence of points \( \{x_{i,n}\} \subseteq U \) with \( x_{i,n} \in \mu(Z_q) \cap \pi^{-1}(V_i) \) such that \( x_{i,n} \) converges to \( x_i \). Since \( \mu: Z \to X \) is continuous the above implies that \( \mu(Z_q) \) contains both \( x_1 \) and \( x_2 \). But since \( x_1 \) and \( x_2 \) are both contained in \( U \) this implies that their image in \( U/T \) must be the same, i.e. \( z_1 = z_2 \). This contradiction implies \( U/T \) is Hausdorff. Applying Proposition (1.10) gives us that \( U/T \) is a complex analytic space.

It remains to show that \( U/T \) is compact. Let \( \{x_n\} \) be a sequence in \( U/T \). We can assume it is contained in \( \mathcal{U} \) and therefore in \( Q \) which is compact and so we can find a convergent subsequence \( \{x'_m\} \) with \( x'_m \) converging to \( q \in Q \). Let \( x \in \mu(Z_q) \cap U \). Since \( x'_m \) converges to \( q \) we have that \( \mu(Z_{x'_m}) \) contained in \( U \) with \( z_m \in \mu(Z_{x'_m}) \cap U \) and such that the \( z_m \) converges to \( x \). By the continuity of \( U \to U/T \) we have that \( x'_m \) converges to \( y \) where \( y \) is the image of \( x \) in \( U/T \). Thus we have shown that every sequence in \( U/T \) has a convergent subsequence which converges to a point in \( U/T \). Therefore, \( U/T \) is compact.

This completes the proof of Lemma (2.7).

\[ \square \]

**THEOREM (2.8).** Let \( A = (A^-, A^0, A^+) \) be a semi-cross section. If \( U \) is the semi-sectional set which corresponds to the semi-cross section then \( U \) is a \( T \)-invariant Zariski open subset of \( X \) whose quotient \( U/T \) is semi-geometric and a compact complex space.

**Proof.** Recall

\[ U = X - \bigcup_{i \in A^+} X^+_i - \bigcup_{j \in A^-} X^-_j. \]


It is obvious that \( U \) is \( T \)-invariant. The proof that \( U \) is Zariski open is the same as that given in Theorem (1.3) of \([B - B + S]\).

Let \( q \in Q \). We claim that \( \mu(Z_q) \cap U \neq \emptyset \). Suppose not, then \( \{ h \in \{1, \ldots, r\} \mid F_h \cap \mu(Z_q) \neq \emptyset \} \) would all lie in \( A^- \) or all in \( A^+ \). This follows from Proposition (1.6) since otherwise we could find an \( x \in \mu(Z_q) \) and an \( i \) and \( j \) with \( j \in A^- \) and \( i \in A^+ \), such that \( \phi^+(x) \in F_j \) and \( \phi^-(x) \in F_i \). By the above description of \( U \) we see that this implies that \( x \in U \) and thus that \( \mu(Z_q) \cap U \neq \emptyset \). Therefore the set of \( h \) with \( F_h \cap \mu(Z_q) \neq \emptyset \) lies totally in either \( A^- \) or \( A^+ \). By (1.6) we would have that either \( r \in A^- \) or \( 1 \in A^+ \). The former implies that \( A^+ = \{1, \ldots, r\} \) and the latter that \( A^- = \{1, \ldots, r\} \). In either case this would imply that \( U = \emptyset \). Thus for all \( q \in Q \) we must have that \( \mu(Z_q) \cap U \neq \emptyset \).

Let \( q \in Q \) and let \( y_1 \) and \( y_2 \) be two points in \( X - X^T \) such that \( Ty_1 \cup Ty_2 \) is contained in \( \mu(Z_q) \cap U \). We claim that either \( Ty_1 = Ty_2 \) or either \( \phi^+(y_1) = \phi^-(y_2) \) or \( \phi^-(y_1) = \phi^+(y_2) \). Suppose \( Ty_1 \neq Ty_2 \). Under this condition assume also that \( \phi^+(y_1) \in F_b \) and \( \phi^-(y_1) \in F_a \) and \( a < b \) or \( \phi^+(y_1) \in F_b \) and \( \phi^-(y_2) \in F_a \) and \( a < b \). Either way we get a contradiction. In the former case if \( a \in A^- \) then \( y_1 \in F_a^- \) and is not in \( U \), if \( a \in A^0 \cup A^+ \) then \( b \in A^+ \), (since \( A^-, A^0, A^+ \) is a semi-cross section), and therefore \( y_2 \in X_b^+ \) and thus not in \( U \). Likewise the latter case also implies that either \( y_1 \) or \( y_2 \) is not an element of \( U \).

Assume that in fact \( \phi^-(y_1) = \phi^+(y_2) = x \). Let \( x \in F_k \), then \( k \in A^0 \), since otherwise if \( k \in A^- \) this would mean \( y_1 \) is not an element of \( U \) and if \( k \in A^+ \) this would mean \( y_2 \) is not an element of \( U \). Hence \( x \in U \).

Therefore, we have shown that for every \( q \in Q \) either \( \mu(Z_q) \cap U = Ty \) for some \( y \in X - X^T \) or \( \mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2 \) for some \( y_1 \) and \( y_2 \in X - X^T \). Applying Lemma (2.7) finishes the proof. \( \square \)

**Lemma (2.9).** Let \( U \) be a \( T \)-invariant Zariski open subset of \( X \) whose quotient \( U/T \) is semi-geometric and a compact complex space. Let \( \{ F_k \} \) be the set of fixed point components contained in \( U \). Let \( U' = U - \bigcup X_k^- \). Then \( U' \) is a Class I \( T \)-invariant Zariski open subset of \( X \) whose quotient \( U'/T \) is semi-geometric and a compact complex space.

**Proof.** \( U' \) is obviously \( T \)-invariant and contained in \( X - X^T \). Lemma (1.3.1) of \([B - B + S]\) shows that \( \bigcup X_k^- \) is a closed set and thus \( U' \) is an open constructible set and therefore is Zariski open.

For all \( q \in Q \) we can consider \( \mu(Z_q) \cap U' \) which is contained in \( \mu(Z_q) \cap U \). If \( \mu(Z_q) \cap U = Ty \) for some \( y \in X - X^T \) then \( y \) is not an
element of $X_k$ for any $F_k$ contained in $U$ and so $\mu(Z_q) \cap U' = Ty$. If $\mu(Z_q) \cap U = Ty_1 \cup \phi(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$, we have that $y_1 \in X_k^-$ for some $F_k$ contained in $U$ but that $y_2$ is not an element of $X_k^-$ for any $F_k$ contained in $U$ and thus that $\mu(Z_q) \cap U' = Ty_2$. Hence we have that for every $q \in Q$ there is an $y \in X - X^T$ such that $\mu(Z_q) \cap U' = Ty$. Applying Lemma (1.2) of [B - B + X] gives the desired result. □

**Remark (2.10).** In [B - B + S] it is shown that a Class I $T$-invariant Zariski open subset $U$ of $X$ has a compact complex space as quotient if and only if $(X - U)$ has two connected components, one which contains the source, $F_1$, and the other which contains the sink, $F_r$. We will use this fact in the next theorem.

**Theorem (2.11).** Let $U$ be a $T$-invariant Zariski open subset of $X$ whose quotient $U/T$ is semi-geometric and a compact complex space. Then $U$ is a semi-cross sectional set with respect to some semi-cross section $(A^-, A^0, A^+)$. 

**Proof.** Given $U$ let $U'$ be the corresponding Class I set given by Lemma (2.9), i.e. $U' = U - \cup (X^-_k)$ where $\{F_{k_1}, \ldots, F_{k_n}\}$ is the set of fixed point components contained in $U$. As noted in Remark (2.10), $(X - U')$ has two connected components, one containing $F_1$ and the other $F_r$. Since we assume that $U$ does not contain either $F_1$ or $F_r$ we must have that $\cup (X^-_k)$ does not contain them either. Therefore $(X - U') - \cup (X^-_k)$ must be disconnected, since $F_1$ and $F_r$ are still in different components. But $(X - U') - \cup (X^-_k) = (X - U)$. Thus we have that $(X - U)$ is disconnected and that $F_1$ and $F_r$ are in different components.

Let $A_1$ be the connected component of $X - U$ which contains $F_1$ and let $A_2$ be the connected component of $X - U$ which contains $F_r$. Assume there was another connected component of $X - U$ besides $A_1$ and $A_2$, call it $A_3$. Let $x \in A_3$ and choose a $q \in Q$ such that $x \in \mu(Z_q)$. By Lemma (2.7) we know that $\mu(Z_q) \cap U$ is either $Ty$ for some $y \in X - X^T$ or $Ty_1 \cup \phi(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$. In either case (1.6) implies that $\mu(Z_q) \cap (X - U)$ has two connected components, one which intersects $F_1$ and another which intersects $F_r$. Thus $x$ must be in the same connected component of $X - U$ as $F_1$ or $F_r$, i.e. $A_3 = A_1$ or $A_3 = A_2$. Therefore, $X - U$ has exactly two connected components.

Let $\{F_1, \ldots, F_r\}$ be the set of connected components of $X^T$. Set $A^- = \{j : F_j$ is contained in $A_1\}$, $A^0 = \{k : F_k$ is contained in $U\}$ and $A^+ = \{i : F_i$ is contained in $A_2\}$. We claim that $(A^-, A^0, A^+)$ forms a semi-cross section of $\{1, \ldots, r\}$. Let $j \in A^-$ and suppose $j'$ is directly less than $j$. We can find $x_j \in F_j$, $x_j' \in F_{j'}$, and $x \in X$ such that $\phi^+(x) = x_j'$,
and \( \phi^{-}(x) = x_j \). Thus we can find a \( q \in Q \) with \( \mu(Z_q) \) containing \( \{x_j, x_j', x\} \). Looking at \( \mu(Z_q) \cap (X - U) \) we see that \( F_j \) is contained in \( A_1 \), i.e. \( j' \in A^{-} \). A finite application of the above step shows that \( j' \in A^{-} \) for any \( j' < j \). Now let \( k \in A^0 \) and suppose \( j \) is directly less than \( k \). Proposition (2.2) shows that \( F_j \) is not contained in \( U \) and (1.6) implies that it must be contained in \( A_1 \). Thus \( j \in A^{-} \) and therefore so is any \( j' < k \). Now suppose there was a \( k' \in A^0 \) with \( k < k' \), the above implies that \( k \in A \). Therefore we see that if \( k \in A^0 \) and if \( k' \) is related to \( k \) then \( k' \) is not an element of \( A^0 \). It is also obvious by what we have shown that if \( k \in A^0 \) and if \( k < i \) then \( i \in A^+ \). Hence \((A^{-}, A^0, A^+)\) forms a semi-cross section of \( \{1, \ldots, r\} \).

Let \( x \in U \). Then either \( x \in (X_k^+ \cup X_k^-) \), for some \( k \in A^0 \), or \( x \in X_i^- \cap X_i^+ \), for some \( i \in A^+ \) and \( j \in A^- \). Therefore since \( U, A_1 \) and \( A_2 \) are \( T \)-invariant and the points of \( U \) satisfy the conditions stated above we have that \( U \) is given by:

\[
U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.
\]

Hence \( U \) is a semi-sectional set with respect to the semi-cross section \((A^{-}, A^0, A^+)\).

Combining (2.8) and (2.11) yields the Main Theorem for Class III sets. \(\square\)

3. An Example. Let \( T \) act on \( \mathbb{P}^1 \times \mathbb{P}^1 \) by \( t([z_0 : z_1], [w_0 : w_1]) = ([z_0 : tz_1], [w_0 : tw_1]) \). There are four fixed points of this section, \( F_1 = ([1 : 0], [1 : 0]), F_2 = ([1 : 0], [0 : 1]), F_3 = ([0 : 1], [1 : 0]) \) and \( F_4 = ([0 : 1], [0 : 1]) \). The plus decomposition is given by

\[
X_1^+ = \mathbb{P}^1 \times \mathbb{P}^1 - \{z_0 = 0 \text{ or } w_0 = 0\},
\]

\[
X_2^+ = \{([z_0 : z_1], [0 : 1]) : z_0 \neq 0\},
\]

\[
X_3^+ = \{([0 : 1], [w_0 : w_1]) : w_0 \neq 0\}
\]

and

\[
X_4^+ = ([0 : 1], [0 : 1]).
\]

The minus decomposition is given by

\[
X_1^- = ([1 : 0], [1 : 0]),
\]

\[
X_2^- = \{([1 : 0], [w_0 : w_1]) : w_1 \neq 0\},
\]

\[
X_3^- = \{([z_0 : z_1], [0 : 1]) : z_1 \neq 0\}
\]

and

\[
X_4^- = \mathbb{P}^1 \times \mathbb{P}^1 + \{z_1 = 0 \text{ or } w_1 = 0\}.
\]

Hence \(([1 : 0], [1 : 0])\) is the source and \(([0 : 1], [0 : 1])\) is the sink.
The following chart describes the possible $T$-invariant open subsets $U$ of $X$ whose quotient $U/T$ is semi-geometric and a compact complex space:

<table>
<thead>
<tr>
<th>Class</th>
<th>$A^-$</th>
<th>$A^0$</th>
<th>$A^+$</th>
<th>$U$</th>
<th>$U/T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>${1,2,3}$</td>
<td>$\emptyset$</td>
<td>${4}$</td>
<td>$C^2 - 0$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>I</td>
<td>${1,2}$</td>
<td>$\emptyset$</td>
<td>${3,4}$</td>
<td>$C^* \times \mathbb{P}^1$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>I</td>
<td>${1,3}$</td>
<td>$\emptyset$</td>
<td>${2,4}$</td>
<td>$C^* \times \mathbb{P}^1$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>I</td>
<td>${1}$</td>
<td>$\emptyset$</td>
<td>${2,3,4}$</td>
<td>$C^2 - 0$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>II</td>
<td>$\emptyset$</td>
<td>${1}$</td>
<td>${2,3,4}$</td>
<td>$C^2$</td>
<td>point</td>
</tr>
<tr>
<td>II</td>
<td>${1,2,3}$</td>
<td>${4}$</td>
<td>$\emptyset$</td>
<td>$C^2$</td>
<td>point</td>
</tr>
<tr>
<td>III</td>
<td>${1,2}$</td>
<td>${3}$</td>
<td>${4}$</td>
<td>$\mathbb{P}^1 \times \mathbb{P}^1 - {(0:1),(0:1)}$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>III</td>
<td>${1,3}$</td>
<td>${2}$</td>
<td>${4}$</td>
<td>$\mathbb{P}^1 \times \mathbb{P}^1 - {(0:1),{1:0}}$</td>
<td>$\mathbb{P}_1$</td>
</tr>
<tr>
<td>III</td>
<td>${1}$</td>
<td>${2,3}$</td>
<td>${4}$</td>
<td>$\mathbb{P}^1 \times \mathbb{P}^1 - {(1:0),(0:1)}$</td>
<td>$\mathbb{P}_1$</td>
</tr>
<tr>
<td>III</td>
<td>${1}$</td>
<td>${3}$</td>
<td>${2,4}$</td>
<td>$\mathbb{P}^1 \times \mathbb{P}^2 - {(1:0),(1:0)}$</td>
<td>$\mathbb{P}^1$</td>
</tr>
<tr>
<td>III</td>
<td>${1}$</td>
<td>${2}$</td>
<td>${3,4}$</td>
<td>$\mathbb{P}^1 \times \mathbb{P}^1 - {(1:0),(0:1)}$</td>
<td>$\mathbb{P}^1$</td>
</tr>
</tbody>
</table>

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