3-MANIFOLDS WITH SUBGROUPS \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) IN THEIR FUNDAMENTAL GROUPS

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3-MANIFOLDS WITH SUBGROUPS $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
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In this paper we characterize those 3-manifolds $M^3$ satisfying $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M)$. All such manifolds $M$ arise in one of the following ways: (I) $M = M_0 \# R$, (II) $M = M_0 \# R^*$, (III) $M = M_0 \cup_3 R^*$. Here $M_0$ is any 3-manifold in (I), (II) and any 3-manifold having $P^2$ components in its boundary in (III). $R$ is a flat space form and $R^*$ is obtained from $R$ and some involution $\iota: R \to R$ with fixed points, but only finitely many, as follows: if $C_1, \ldots, C_n$ are disjoint 3-cells around the fixed points then $R^*$ is the 3-manifold obtained from $(R - \text{int}(C_1 \cup \cdots \cup C_n))/\iota$ by identifying some pairs of projective planes in the boundary.

1. Introduction. In [1] it was shown that the only possible finitely generated abelian subgroups of the fundamental groups of 3-manifolds are $\mathbb{Z}_n$, $\mathbb{Z} \oplus \mathbb{Z}_2$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The purpose of this paper is to characterize all $M^3$ satisfying $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M)$.

To explain this characterization recall that the Bieberbach theorem (see Chapter 3 of [8]) implies that if $M$ is a closed 3-dimensional flat space form then $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M)$. We let $M_1, \ldots, M_6$ denote the 6 compact connected orientable flat space forms in the order given on p. 117 of [8]. Similarly $N_1, \ldots, N_4$ will denote the non-orientable ones. For explicit descriptions see §2. One of the main theorems from [3] is

(1.1) Theorem. The only space forms from the orientable ones $M_1, \ldots, M_6$ which admit involutions having fixed points, but only finitely many, are $M_1$, $M_2$, $M_6$. Moreover these involutions are unique up to conjugacy and have 8, 4, 2 fixed points respectively.

If $\iota: M_i \to M_i$, $i = 1, 2$ or 6, is such an involution and $x_1, \ldots, x_n$ are the fixed points ($n = 8, 4, 2$) then there are disjoint 3-cells $C_1, \ldots, C_n$ so that

$$x_i \in \text{int } C_i \text{ and } \iota(C_i) = C_i, \quad 1 \leq i \leq n.$$ 

We let $M_i^*$ denote the orbit manifold $M_i - \text{int}(C_1 \cup \cdots \cup C_n)/\iota$. Thus $\partial M_i^*$ consists of 8 projective planes, $\partial M_2^*$ consists of 4 and $\partial M_6^*$ has 2. Canonical presentations of $M_i^*$, $i = 1, 2, 6$, are given in [3]. By making identifications of pairs of such projective planes in $\partial M_i^*$ we obtain new manifolds still containing $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ in their fundamental groups. We
refer to any such manifold obtained this way as a projectively flat space form. In this identification procedure it is not necessary to identify all boundary components.

(1.2) Example. $M_1$ is the torus $S^1 \times S^1 \times S^1$ and the involution can be taken to be $\iota(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure1}
\caption{8 $P^2$ boundary components, 2 pairs of which have been identified.}
\end{figure}

If $M_0$ is a manifold having $P^2$ components in its boundary let $M_0 \cup \partial R^*$ denote the manifold obtained from the disjoint union $M_0 \cup R^*$ of $M_0$ with a projectively flat space form $R^*$ by identifying some $P^2$ components of $\partial M_0$ with some from $\partial R^*$. Then $\pi_1(M_0 \cup \partial R^*)$ contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as a subgroup.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2}
\caption{Main Theorem. Suppose $M^3$ is a 3-manifold. Then $\pi_1(M)$ admits $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as a subgroup if and only if $M$ has one of the following forms:
(I) $M = M_0 \# R$ for some flat space form $R$,
(II) $M = M_0 \# R^*$ for some projectively flat space form $R^*$,
(III) $M = M_0 \cup \partial R^*$ for some projectively flat space form $R^*$, where, in case (III), $M_0$ is as above.}
\end{figure}
Throughout we work in the PL category and use [2] for a standard reference. Section 2 contains the descriptions of the space forms, section 3 contains a topological characterization of them, and §4 has the proof of the main theorem.

2. Flat 3-dimensional space forms. In this section we will briefly summarize some of the basic facts about flat space forms — see [8] for details. Recall that a complete connected riemannian manifold is a flat space form if its sectional curvature is constantly zero, and that the classical Bieberbach theorem states that a 3-manifold $M^3$ is a flat space form if and only if its universal covering space is $\mathbb{R}^3$ and the deck transformation group $\pi_1(M)$ is acting on $\mathbb{R}^3$ by rigid motions. This is also equivalent to the existence of a regular covering by a flat torus $S^1 \times S^1 \times S^1 \to M$ in case $M$ is closed.

For such a 3-manifold $M$ there is therefore an extension

$$1 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to G \to \Psi \to 1,$$

where $G = \pi_1(M)$ is torsion free and $\Psi$ is a finite group. Moreover the abelian subgroup $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ can be taken to be maximal abelian. Conversely we have the following result of Bieberbach: an abstract group $G$ is the fundamental group of a 3-dimensional flat closed space form if $G$ is torsion free and there exists an extension $1 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to G \to \Psi \to 1$ with $\Psi$ finite and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ maximal abelian in $G$. If $\rho: \Psi \to \text{Gl}_3(\mathbb{Z})$ is the representation associated to an extension then $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is maximal abelian if and only if $\rho$ is faithful. Thus the affine classification of flat space forms in dimension 3 proceeds by first classifying the finite subgroups of $\text{Gl}_3(\mathbb{Z})$ up to conjugacy and then by determining which ones correspond to torsion free extensions.

If $\rho: \Psi \to \text{Gl}_3(\mathbb{Z})$ is a representation then the congruence classes of extensions $1 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to G \to \Psi \to 1$ associated to $\rho$ are in 1-1 correspondence with $H^2(\Psi; \mathbb{R})$, where $\mathbb{R}$ is the $\Psi$ module $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Then it is easy to see that $G$ is torsion free if and only if the cohomology class $\chi \in H^2(G; \mathbb{R})$ restricts to a non-zero class in $H^2(\mathbb{Z}_p; \mathbb{R})$ for each subgroup $\mathbb{Z}_p \subseteq G$, $p$ a prime.

Proceeding in this way it is a routine matter to classify 3-dimensional flat space forms up to affine diffeomorphism. It turns out that the only possible holonomy groups $\Psi$ are $1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the orientable case and $\mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in the non-orientable case.

Let $a_1$, $a_2$, $a_3$ denote a fixed basis of $\mathbb{R}^3$ and let $t_1$, $t_2$, $t_3$ be the corresponding translations. If $A$ is a $3 \times 3$ matrix with respect to this
basis and \( v \in \mathbb{R}^3 \) then let \((A, t_v)\) denote the affine map
\[
(A, t_v) : u \rightarrow v + A(u)
\]
Then the classification of space forms up to affine equivalence is given by the following two theorems.

**Theorem (2.1).** Up to affine equivalence there are 6 orientable closed flat 3-dimensional space forms. They are represented by the manifolds \( \mathbb{R}^3 / G \), where \( G \) is one of the groups below:

1. \( \Psi = \{1\} \) and \( G \) is generated by \( t_1, t_2, t_3 \).
2. \( \Psi = \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_1/2}) \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & -1 & 0 \\
   0 & 0 & -1
   \end{bmatrix}.
   \]
3. \( \Psi = \mathbb{Z}_3 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_1/3}) \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 0 & -1 \\
   0 & 1 & -1
   \end{bmatrix}.
   \]
4. \( \Psi = \mathbb{Z}_4 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_1/4}) \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 0 & -1 \\
   0 & 1 & 0
   \end{bmatrix}.
   \]
5. \( \Psi = \mathbb{Z}_6 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_1/6}) \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 0 & -1 \\
   0 & 1 & 1
   \end{bmatrix}.
   \]
6. \( \Psi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_1/2}), \beta = (B, t_{(a_2+a_3)/2}) \), where
   \[
   A = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & -1 & 0 \\
   0 & 0 & -1
   \end{bmatrix}, \quad B = \begin{bmatrix}
   -1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & -1
   \end{bmatrix}.
   \]

**Remarks.**

(1) The normal subgroup \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) is generated by the translations \( t_1, t_2, t_3 \) and the corresponding representations \( \Psi \rightarrow GL_3(\mathbb{Z}) \) are given by the matrices \( A \) for cases 2, \ldots, 5 and by \( A, B \) for the last case.

(2) This theorem explicitly describes the way in which the groups act by affine motions on \( \mathbb{R}^3 \). In order to put a metric of constant curvature
zero on the space forms, that is, to make these motions rigid, we must impose certain metric conditions on the \( a_i \). But this does not concern us here.

**Theorem (2.2).** Up to affine equivalence there are 4 non-orientable closed flat 3-dimensional space forms. They are represented by the manifolds \( \mathbb{R}^3/G \), where \( G \) is one of the groups below:

1. \( \Psi = \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \varepsilon = (A, t_{a_i/2}) \), where
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

2. \( \Psi = \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \varepsilon = (A, t_{a_i/2}) \), where
\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.
\]

3. \( \Psi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_i/2}) \), \( \varepsilon = (B, t_{a_i/2}) \), where
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

4. \( \Psi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( G \) is generated by \( t_1, t_2, t_3 \) and \( \alpha = (A, t_{a_i/2}) \), \( \varepsilon = (B, t_{(a_2 + a_3)/2}) \), where
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\]

We let \( M_1, \ldots, M_6 \) denote the orientable space forms and \( N_1, \ldots, N_4 \) the non-orientable ones. Perusing the list of groups in the orientable case reveals that the subgroup generated by \( t_2, t_3 \) is normal in \( G \). In the first 5 cases \( M_1, \ldots, M_5 \) we have an extension
\[
1 \to \mathbb{Z} \oplus \mathbb{Z} \to G \to \mathbb{Z} \to 1
\]
and for \( M_6 \) we have an extension
\[
1 \to \mathbb{Z} \oplus \mathbb{Z} \to G \to \mathbb{Z}_2 \ast \mathbb{Z}_2 \to 1
\]
where \( \mathbb{Z} \oplus \mathbb{Z} \) is the subgroup generated by \( t_2, t_3 \). The matrices associated to the extension in the first 5 cases are given by conjugation by \( t_1 \) for \( M_1 \).
and by $\alpha$ for $M_2, \ldots, M_5$. They are, respectively,

$$(2.3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$ 

Notice that these matrices are in $\text{Sl}_2(Z)$ and have orders 1, 2, 3, 4 and 6.

Geometrically this means that $M_1, \ldots, M_5$ are orientable torus bundles over $S^1$ resulting from orientation preserving homeomorphisms $\phi: S^1 \times S^1 \to S^1 \times S^1$ having finite orders, and with $\phi_*: H_1(S^1 \times S^1) \to H_1(S^1 \times S^1)$ given by the matrices in (2.3) respectively. In terms of complex coordinates these homeomorphisms can be described as follows:

$$\phi_i(x, y) = \begin{cases} (x, y) & \text{if } i = 1, \\ (\bar{x}, \bar{y}) & \text{if } i = 2, \\ (y, \bar{x}\bar{y}) & \text{if } i = 3, \\ (y, \bar{x}) & \text{if } i = 4, \\ (y, \bar{x}y) & \text{if } i = 5. \end{cases}$$

Then $M_i$, $1 \leq i \leq 5$, is the mapping torus construction

$$M_i = S^1 \times S^1 \times [0, 1]/(x, y, 0) \sim (\phi_i(x, y), 1).$$

The torus bundle structure over $S^1$ is given by Figure 3.

The last orientable flat space form $M_6$, the so-called Hantzsche-Wendt manifold, is not a torus bundle over the 1-sphere. However, $M_6$ is the union of 2 copies of the orientable twisted $I$-bundle over the Klein bottle. A particular model $W$ for this twisted $I$-bundle is $W = S^1 \times S^1 \times [0, 1]/(x, y, 0) \sim (\tau(x, y), 0)$ where $\tau: S^1 \times S^1 \to S^1 \times S^1$ is any fixed point free orientation reversing homeomorphism, e.g., $\tau(x, y) = (-x, -\bar{y})$.
or \( \tau(x, y) = (-x, -y) \). The \( I \)-bundle structure is given by sliding down the \( t \)-axis. Notice that \( S^1 \times S^1 / (x, y) \sim \tau(x, y) \) is the Klein bottle \( K \) and \( \partial W \) is a torus. (See Figure 4.)

![Figure 4](image)

**Figure 4**

To see how \( M_6 \) is the union of 2 copies of \( W \) we first analyze the structure of \( G = \pi_1(M_6) \). From the above description of \( G \) it follows that there is a decomposition of \( G \) as an amalgamated free product \( G_1 \ast_{Z \oplus Z} G_2 \), where

\[
G_1 = \text{the subgroup generated by } t_1, t_2, \alpha, \\
G_2 = \text{the subgroup generated by } t_1, t_2, \beta.
\]

In fact \( G_1, G_2 \) are Klein bottle groups and \( M_6 \) decomposes as follows:

\[
M_6 = S^1 \times S^1 \times [0, 1] / (x, y, 0) \sim (-x, -y, 0) \text{ and } (x, y, 1) \sim (-\bar{x}, -y, 1)
\]

\( W_0, W_1 \) are copies of \( W \) and \( W_0 \cap W_1 \) is an incompressible torus.

![Figure 5](image)

**Figure 5**

We can summarize this construction as follows. Let \( W \) be an orientable twisted \( I \)-bundle over a Klein bottle. Then \( H_1(W) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \), and \( H_1(\partial W) \cong \mathbb{Z} \oplus \mathbb{Z} \) has a natural basis \( b_0, b_1 \) such that, with respect to the inclusion \( \iota : \partial W \to W \), \( \iota_*(b_0) \) is a generator of \( 2\mathbb{Z} \) and \( \iota_*(b_1) \) is the
non-zero element of $\mathbb{Z}_2$. Now let $W_0, W_1$ be two copies of $W$ and let $\phi$:
$\partial W_0 \to \partial W_1$ be an orientation preserving homeomorphism with $\phi_*:
H_i(\partial W_0) \to H_i(\partial W_1)$ given by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with respect to the natural basis. Then $M_6 = W_0 \cup W_1/ x \sim \phi(x)$.

Up to conjugacy a complete list of matrices of finite order in $\text{Sl}_2(\mathbb{Z})$ is given by (2.3). The corresponding list in $\text{Gl}_2(\mathbb{Z})$ has 2 more representatives, namely
\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}
\]
Note that although these matrices are similar over the ring $\mathbb{Z}[\frac{1}{2}]$, they are not similar over $\mathbb{Z}$. The corresponding torus bundles over $S^1$ turn out to be $N_1, N_2$ respectively. To see this for $N_1$ we note that $G = \pi_1(N_1)$ satisfies an extension
\[1 \to \mathbb{Z} \oplus \mathbb{Z} \to G^\theta \to \mathbb{Z} \to 1, \text{ where } \mathbb{Z} \oplus \mathbb{Z} \]
is generated by $t_2, t_3$ and $\theta(\epsilon)$ is a generator of $\mathbb{Z}$. Since $\epsilon t_2 \epsilon^{-1} = t_2$ and $\epsilon t_3 \epsilon^{-1} = t_3^{-1}$ it follows that the matrix of this extension is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Accordingly $N_1$ is the mapping torus construction
$N_1 = S^1 \times S^1 \times [0,1]/ (x, y, 0) \sim (x, \bar{y}, 1)$.

In the case of $N_2$ we also have an extension
\[1 \to \mathbb{Z} \oplus \mathbb{Z} \to G^\theta \to \mathbb{Z} \to 1, \text{ where } G = \pi_1(N_2), \]
$\mathbb{Z} \oplus \mathbb{Z}$ is generated by $t_1 t_2, t_3$, and $\theta(\epsilon) = \text{ a generator of } \mathbb{Z}$. Now $\epsilon t_1 t_2 \epsilon^{-1} = t_1 t_2 t_3^{-1} = t_1 t_2 t_3^{-1}$ and so the matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Therefore
$N_2 = S^1 \times S^1 \times [0,1]/ (x, y, 0) \sim (xy, \bar{y}, 1)$.

We now have a complete list of conjugacy classes of matrices of finite order in $\text{Gl}_2(\mathbb{Z})$. There are five orientable matrices leading to the space forms $M_1, \ldots, M_5$ and two non-orientable matrices corresponding to $N_1, N_2$. The other space forms are not torus bundles over a circle. As noted above $M_6$ is the union of two twisted $I$-bundles over the Klein bottle.

The non-orientable space forms $N_1, \ldots, N_4$ admit Klein bottle bundle structures (for a different approach see [4]). In the following we derive explicit descriptions. First of all $N_1$ is homeomorphic to $K \times S^1$, where $K$ is the Klein bottle. To see this note that
$N_1 = S^1 \times S^1 \times [0, 1]/ (x, y, 0) \sim (x, \bar{y}, 1)$
$= S^1 \times \{ S^1 \times [0, 1]/ (y, 0) \sim (\bar{y}, 1) \} = S^1 \times K.$
Now let \( \sigma: S^1 \times S^1 \to S^1 \times S^1 \) be the homeomorphism \( \sigma(x, y) = (xy, y) \). Thus \( N_2 \) is the torus bundle over \( S^1 \) associated to \( \sigma \). Then consider the circle \( S_\theta = \{(x, y) \in S^1 \times S^1 | x^2 y = e^{2\pi i \theta}, 0 \leq \theta \leq 1 \} \). It is easy to verify that \( S_\theta \) is invariant under \( \sigma \) and that the following diagram commutes:

\[
\begin{array}{ccc}
S_\theta & \xrightarrow{\approx} & S^1 \\
\downarrow \sigma & & \downarrow \rho_\theta \\
S_\theta & \xrightarrow{\approx} & S^1
\end{array}
\]

where \( S_\theta \to S^1 \) is the homeomorphism \( x, y \to x \) and \( \rho_\theta: S^1 \to S^1 \) is \( x \to e^{2\pi i \theta \bar{x}} \).

Put \( K_\theta = S_\theta \times [0, 1]/(x, y, 0) \sim (\sigma(x, y), 1) \subseteq N_2 \). Then \( K_\theta \) is a Klein bottle and \( N_2 = \bigcup_{0 \leq \theta \leq 1} K_\theta \). In other words, \( N_2 \) is a twisted product of \( K = S^1 \times [0, 1]/(x, 0) \sim (\bar{x}, 1) \) and \( S^1 \). To see how this twisting works consider the map

\[
f: S^1 \times [0, 1] \times [0, 1] \to S^1 \times S^1 \times [0, 1],
\]

\[
f(x, t, \theta) = (xe^{2\pi i \theta t}, x^{-2}e^{2\pi i \theta(1-2t)}, t).
\]

The following are easy to verify:

(i) \( f(S^1 \times [0, 1] \times \theta) = S_\theta \times [0, 1] \), in fact \( f \) induces a homeomorphism \( S^1 \times [0, 1] \times \theta \to S_\theta \times [0, 1] \).

(ii) \( f(x, 0, \theta) \sim f(\bar{x}, 1, \theta) \), as points in \( N_2 \), for \( 0 \leq \theta \leq 1 \).

Therefore \( f \) induces: \( F: S^1 \times [0, 1] \times [0, 1]/(x, 0, \theta) \sim (\bar{x}, 1, \theta) \to N_2 \),

\[
F[x, t, \theta] = [xe^{2\pi i \theta t}, x^{-2}e^{2\pi i \theta(1-2t)}, t].
\]

In fact \( F \) induces homeomorphisms \( K \times \theta \to K_\theta, 0 \leq \theta \leq 1 \). For \( \theta = 0, 1 \) we have

\[
F[x, t, 0] = [x, x^{-2}, t], F[x, t, 1] = [xe^{2\pi it}, x^{-2}e^{-4\pi it}, t]
\]

and hence

\[
N_2 = K \times [0, 1]/[x, t, 0] \sim [xe^{-2\pi it}, t, 1].
\]

To determine the Klein bottle bundle structure on \( N_3 \) we first show that there exists an extension \( 1 \to Q \to G \to \mathbb{Z} \to 1 \), where \( G = \pi_1(N_3) \) and \( Q \) is the fundamental group of the Klein bottle. From the description of \( G \) as a group of rigid motions on \( \mathbb{R}^3 \) we can derive the following relations:

\[
\alpha^2 = t_1, \quad \varepsilon^2 = t_2, \quad \varepsilon\alpha\varepsilon^{-1} = t_2\alpha, \quad \alpha t_2\alpha^{-1} = t_2^{-1},
\]

\[
\alpha t_3\alpha^{-1} = t_3^{-1}, \quad \varepsilon t_1\varepsilon^{-1} = t_1, \quad \varepsilon t_3\varepsilon^{-1} = t_3^{-1}.
\]

Now it follows that the subgroup generated by \( \varepsilon, t_3 \) is a choice for \( Q \) and \( \theta(\alpha) \) is a generator for \( \mathbb{Z} \). Thus \( N_3 = K \times [0, 1]/(p, 0) \sim (\sigma(p), 1) \), where \( \sigma: K \to K \) is the homeomorphism inducing conjugation by \( \alpha \) on \( \pi_1(K) = Q \).
But

$$\alpha \varepsilon \alpha^{-1} = \varepsilon^{-1} \quad \text{and} \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

and therefore we may choose $\sigma$ to be $\sigma[x, t] = [x, 1 - t]$. To see this consider the corresponding homeomorphism on $S^1 \times [0, 1]$. Therefore $N_3 = K \times [0, 1]/[x, t, 0] \sim [x, 1 - t, 1]$.

![Figure 6](image1)

Finally it remains to describe the $K$-bundle structure on $N_4$. The fundamental group $G = \pi_1(N_4)$ has the relations

$$\alpha^2 = t_1, \quad \varepsilon^2 = t_2, \quad \varepsilon \alpha \varepsilon^{-1} = t_2 t_3 \alpha, \quad \alpha t_2 \alpha^{-1} = t_2^{-1},$$

$$\alpha t_3 \alpha^{-1} = t_3^{-1}, \quad \varepsilon t_1 \varepsilon^{-1} = t_1, \quad \varepsilon t_3 \varepsilon^{-1} = t_3^{-1}.$$  

Again we choose $Q = \langle \varepsilon, t_3 \rangle$. Then $Q$ is normal and is isomorphic to $\pi_1(K)$. There is an extension $1 \rightarrow Q \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$, where $G = \pi_1(N_4)$ and $\theta(\alpha)$ is a generator for $\mathbb{Z}$. One can easily check that

$$\alpha \varepsilon \alpha^{-1} = t^{-1} \varepsilon^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}.$$  

The homeomorphism $\sigma: K \rightarrow K$ inducing conjugation by $\alpha$ is $\alpha[x, t] = [xe^{-2\pi it}, 1 - t]$. Thus

$$N_4 = K \times [0, 1]/[x, t, 0] \sim [xe^{-2\pi it}, 1 - t, 1].$$

![Figure 7](image2)
We can summarize the preceding results as follows. Let $\psi_i: K \to K$ be homeomorphisms, $i = 1, 2, 3, 4$, so that the induced isomorphisms $\psi_i\ast$ on $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are given by the following “matrices”:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 \\
1 & 1
\end{bmatrix}.
\]

Then $N_i$ is the mapping torus construction

$N_i = K \times [0, 1]/ (x, y, 0) \sim (\psi_i(x, y), 1)$.

Note $N_1, \ldots, N_4$ comprise all possible $K$-bundles over $S^1$.

Abelianizing the fundamental groups, the first homology groups of the space forms are easily computed to be as follows:

\[
\begin{align*}
H_1(M_1) &= H_1(S^1 \times S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\
H_1(M_2) &= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
H_1(M_3) &= \mathbb{Z} \oplus \mathbb{Z}_3 \\
H_1(M_4) &= \mathbb{Z} \oplus \mathbb{Z}_2 \\
H_1(M_5) &= \mathbb{Z}_4 \\
H_1(M_6) &= \mathbb{Z}_4 \oplus \mathbb{Z}_4
\end{align*}
\]

Since the identification maps in the bundle structures of all ten space forms $M_1, \ldots, M_6$, $N_1, \ldots, N_4$ have finite order it follows that all ten space forms admit Seifert fibrations. The exceptional fibers correspond to fixed points of the group actions generated by the identification maps. See also [4].

The canonical involutions on $M_1, M_2, M_6$ can now be easily described in terms of bundle coordinates:

1. $M_1$, $\iota_8[x, y, t] = [\bar{x}, \bar{y}, 1 - t]$ has 8 fixed points $[\pm 1, \pm 1, 0], [\pm 1, \pm 1, \frac{1}{2}]$.
2. $M_2$, $\iota_4[x, y, t] = \begin{cases} 
[\bar{-x}, \bar{-y}, \frac{1}{2} - t] & \text{if } 0 \leq t \leq 1/2, \\
[\bar{-x}, \bar{-y}, \frac{3}{2} - t] & \text{if } 1/2 \leq t \leq 1.
\end{cases}$

The 4 fixed points are $[\pm i, \pm i, \frac{3}{4}]$.

$M_2 = S^1 \times S^1 \times [0, 1]/ (x, y, 0) \sim (\bar{x}, \bar{y}, 1)$. 

Finally, there are 2-fold coverings $M_1 \rightarrow M_2$ and $M_2 \rightarrow M_6$. By varying $\iota_2$, $\iota_4$, $\iota_8$ within their conjugacy class we can make these involutions compatible with these coverings. To do this consider the three involutions of $M = S^1 \times S^1 \times S^1$ defined in terms of complex coordinates as follows:

$$\sigma(x, y, z) = (-x, \bar{y}, \bar{z}), \quad \rho(x, y, z) = (\bar{x}, -\bar{y}, z),$$

$$\iota(x, y, z) = (-x, -\bar{y}, \bar{z}).$$

Then it is easily checked that $\sigma$, $\rho$, $\iota$ pairwise commute, $\sigma: M_1 \rightarrow M_1$ is the covering translation for $M_1 \rightarrow M_2$, and the induced involution $\bar{\rho}: M_2 \rightarrow M_2$ is the covering translation for $M_2 \rightarrow M_6$. Since $\iota$ commutes with $\sigma$, $\rho$ it gives involutions on $M_2$ and $M_6$. These induced involutions are the canonical ones.

For a more explicit description of $M_1^*$, $M_2^*$ and $M_6^*$ see [3]. In conclusion we have the following hierarchy of coverings:

3. The topology of flat space forms. In this section we characterize the compact 3-dimensional flat space forms as those connected $P^2$-irreducible $M^3$ satisfying $\pi_1(M)$. (A 3-manifold $M$ is irreducible if each 2-sphere in $M$ bounds a 3-cell; it is $P^2$-irreducible if it is irreducible and if it does not contain 2-sided projective planes.)
(3.1) LEMMA. Suppose $M^3$ is a compact connected $P^2$-irreducible 3-manifold with $\pi_1(M) = \pi_1(R)$ for some compact flat space form $R$. Then $M$ is sufficiently large.

Proof. By (2.5) in §2 the first homology group $H_1(M)$ is infinite except when $R = M_6$. Thus $M$ is sufficiently large in these cases. If $R = M_6$ then $\pi_1(M)$ splits as a free product with amalgamation $G_1 \ast_{Z \oplus Z} G_2$ (see §2) and therefore there is a 2-sided incompressible surface in $M$ (see [7]). Hence $M$ is sufficiently large.

As a corollary it follows that the space forms $M_1, \ldots, M_6$, $N_1, \ldots, N_4$ are sufficiently large. They are $P^2$-irreducible because their universal coverings are $\mathbf{R}^3$.

(3.2) LEMMA. Let $M$ be a connected 3-manifold so that $\pi_2(M) = 0$ and $Z \oplus Z \oplus Z \subseteq \pi_1(M)$. Then $M$ is closed and $Z \oplus Z \oplus Z$ has finite index in $\pi_1(M)$.

Proof. We have coverings $M' \to M'' \to M$ where $M'$ is the universal covering of $M$ and $M''$ corresponds to $Z \oplus Z \oplus Z$. Now suppose that either $M$ is not closed or the index is infinite. Then $H_3(M'') = 0$. Now $M'$ is non-compact and hence $H_3(M') = 0$. But $\pi_1(M') = \pi_2(M') = 0$ and therefore $\pi_3(M') = 0$ by the Hurewicz isomorphism theorem. In other words $M'$ is contractible, and this implies that $M''$ is a $K(Z \oplus Z \oplus Z, 1)$. Thus $H_3(M'') = H_3(Z \oplus Z \oplus Z) = Z$. Contradiction.

(3.3) THEOREM. Let $M$ be a $P^2$-irreducible connected 3-manifold such that $Z \oplus Z \oplus Z$ is a subgroup of $\pi_1(M)$. Then $M$ is a compact flat space form.

Proof. By the sphere theorem we have $\pi_2(M) = 0$, and since $\pi_1(M)$ is infinite the universal covering space is contractible. In other words $M$ is a $K(G, 1)$. Lemma (3.2) now implies that $M$ is closed and $Z \oplus Z \oplus Z$ has finite index in $G$. Replacing $Z \oplus Z \oplus Z$ by the intersection of its conjugates gives us an extension $1 \to Z \oplus Z \oplus Z \to G \to \Psi \to 1$ with $\Psi$ a finite group. By [1] $G$ is torsion free and according to [6] there is a compact flat space form $R$ with $\pi_1(R) = G$. Hence there is a homotopy equivalence $M \simeq R$ because $M, R$ are both spaces of type $K(G, 1)$. But $M, R$ are sufficiently large by (3.1) and therefore we can deform the homotopy equivalence $M \simeq R$ into a homeomorphism.
4. Proof of the Main Theorem.

(4.1) **Lemma.** Let $M$ be a compact irreducible 3-manifold. Then there exists an integer $n(M)$ such that if $P_1, \ldots, P_k$ are pairwise disjoint 2-sided projective planes in $M$ and $k > n(M)$ then some pair $P_i, P_j$ must be parallel (i.e., cobound a product) in $M$.

**Proof.** A 2-sided projective plane in a 3-manifold is incompressible [2, Lemma (5.1)]. Then (4.1) is a special case of Lemma (3.2) in [2].

(4.2) **Lemma.** Let $M^3$ be a connected 3-manifold with $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M)$ and let $S^2 \times [-1, 1]$ be a bicollar of the 2-sphere $S^2 = S^2 \times 0 \subset \text{int } M$.

(I) If $S^2$ does not separate $M$ then $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M_0)$, where $M_0 = M - \text{int } S^2 \times [-1, 1]$.

(II) If $S^2$ separates $M$ into $M_1, M_2$ then at least one of $\pi_1(M_1), \pi_1(M_2)$ contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** (II) follows from the Kurosh subgroup theorem in a standard way. Thus consider (I).

Let $D^2$ be a 2-cell such that $D^2 \times [-1, 1] \subseteq M_0$, $D^2 \times [-1, 1] \cap \partial M_0 = D^2 \times \{-1\} \cup D^2 \times \{1\}$, $D^2 \times \{-1\} \subseteq S^2 \times \{-1\}$ and $D^2 \times \{1\} \subseteq S^2 \times \{1\}$. Thus $M = M_0' \cup T$ and $M_0' \cap T = \partial T$ is a 2-sphere. By the van Kampen theorem $\pi_1(M) = \pi_1(M_0') \ast \mathbb{Z}$ and so $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \subseteq \pi_1(M_0')$ by the Kurosh subgroup theorem. Since $\pi_1(M_0') = \pi_1(M_0)$ the lemma follows. 

Now we complete the proof of the main theorem. There is a compact submanifold in $M$ whose fundamental group contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. Therefore we may assume that $M$ is compact. If the prime decomposition of $M$ is $M = M_1 \# \cdots \# M_n$ then by the Kurosh subgroup theorem $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is a subgroup of some $\pi_1(M_i)$. Also $M_i$ must be irreducible since the fundamental group of a 2-sphere bundle over $S^1$ is $\mathbb{Z}$. Thus, without loss of generality, assume $M$ is already irreducible. If $M$ does not contain 2-sided projective planes then $M$ is $P^2$-irreducible, and hence case (I) of the theorem now follows from (3.3).

Now assume that $P_1, \ldots, P_k \subseteq \text{int } M$ is a maximal collection of pairwise disjoint 2-sided projective planes such that no two are parallel and none are boundary parallel. Let $P_i \times [-1, 1] \subseteq \text{int } M$ be disjoint bicollars
and let $Q_1, \ldots, Q_m$ be the components of $M - \bigcup_i P_i \times (-1, 1)$. If $p: \tilde{M} \to M$ is the orientable covering then the $\tilde{Q}_j = p^{-1}(Q_j)$ are the connected pieces left after cutting $\tilde{M}$ along the thickened 2-spheres $p^{-1}(P_j) \times (-1, 1)$. Since $\pi_i(\tilde{M})$ is a subgroup of index 2 in $\pi_i(M)$ we conclude that $\pi_i(\tilde{M})$ has $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ for a subgroup. By (4.2) at least one of the pieces $\tilde{Q}_j$, say $\tilde{Q} = p^{-1}(Q)$, must contain $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ in its fundamental group. Next set $\hat{Q}$ = the manifold obtained from $Q$ by capping all the 2-sphere boundary components with 3-cells. Then $\hat{Q}$ is irreducible (see Theorem F of [5]) and so is a space form by (3.3).

If $i: \tilde{M} \to \hat{M}$ is the deck transformation of $\tilde{M} \to M$ then $i(\tilde{Q}) = \hat{Q}$ and so by radial extension we obtain an involution $\hat{i}: \hat{Q} \to \hat{Q}$ having fixed points but only finitely many. By (1.1) it follows that $\hat{i}$ is canonical. Thus $Q$ is one of $M^*_c$, $M^*_s$ or $M^*_b$.

To conclude the proof we need only analyze the way in which the pieces are sewn back together. Thus let $R^*$ be the projectively flat space form obtained from $Q$ by reattaching those $P_i \times [-1, 1]$ which were removed to produce $Q$. If $R^* = M$ we are in case (II) and if $R^* \neq M$ this is case (III).

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