AN ARTIN RELATION (MOD 2) FOR FINITE GROUP ACTIONS ON SPHERES

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Recently it has been shown that whenever a finite group $G$ (not a $p$-group) acts on a homotopy sphere there is no general numerical relation which holds between the various formal dimensions of the fixed sets of $p$-subgroups ($p$ dividing the order of $G$). However, if $G$ is dihedral of order $2q$ ($q$ an odd prime power) there is a numerical relation which holds (mod 2). In this paper, actions of groups $G$ which are extensions of an odd order $p$-group by a cyclic 2-group are considered and a numerical relation (mod 2) is found to be satisfied (for such groups acting on spheres) between the various dimensions of fixed sets of certain subgroups; this relation generalises the classical Artin relation for dihedral actions on spheres.

0. Introduction. When a $p$-group $P$ acts on a mod $p$ homology $n$-sphere $X$, the fixed point set, $X^H$, of any subgroup $H$ has the mod $p$ homology of an $n(H)$-sphere, for some integer $n(H)$. The function from subgroups of $P$ to integers defined by $H \mapsto n(H)$ is called the dimension function and any such function arising in this way is known to originate in a real representation of $P$ (see [2]). If $P$ is elementary abelian, the Borel identity holds (see [1, pg. 175]):

$$n - n(P) = \sum (n(H) - n(P))$$

(sum over all $H \leq P$ such that $P/H = \mathbb{Z}_p$). The motivation for this identity comes from consideration of representations of $P$.

Now suppose $G$ is the dihedral group $D_p$ ($p$ odd prime) (a semidirect product of $\mathbb{Z}_p$ and $\mathbb{Z}_2$ via the automorphism of $\mathbb{Z}_p$, $g \rightarrow g^{-1}$). If $V$ is a real representation of $G$, one can by considering the real irreducible representations of $G$, write down the following Artin relation,

$$\dim V^G = \dim V^\mathbb{Z}_2 - \left( \frac{\dim V - \dim V^{\mathbb{Z}_p}}{2} \right).$$

In [3], K. H. Dovermann and Ted Petrie show that for actions of $D_p$ (and more generally any non-$p$-group) on a homotopy sphere one cannot expect to find a numerical relation between the various dimensions of the fixed sets (in particular for smooth actions of $D_p$ one cannot expect the Artin relation to hold). However, in [8, Thm. 1.3], E. Straume has shown that
the Artin relation does hold, \((\text{mod } 2)\). Specifically,

**Theorem** ([8, Thm. 1.3.]): If \(X\) is a mod2 \(p\) homology \(n\)-sphere (i.e., \(X \sim_{2p} S^n\)) with an action of \(D_p = G\) and \(X^{Z_p} \sim_p S^i\), \(X^{Z_2} \sim_{2} S^m\) then \(\chi(X^G) = \chi(S^d)\) where

\[
d \equiv m - \left(\frac{n - l}{2}\right) \pmod{2}.
\]

In this paper we will generalize the Straume’s result, and hence the Artin relation, considerably. Suppose \(G\) is a finite group which is an extension of an odd order \(p\)-group \(P\) by a cyclic \(2\)-group \(Q = Z_{2^k}\); \(P \to G \to Q\). We will call such groups, “\(p\)-elementary”, though this is not quite standard. Such \(G\) are always semi-direct products (Schur-Zassenhaus Lemma) via a homomorphism \(Z_{2^k} \to \text{Aut}(P)\). If \(G\) acts on \(X \sim_{2p} S^n\), we have:

**Theorem 1.** There exists a sequence of subgroups \(e = P_m \leq P_{m-1} \leq \cdots \leq P_1 \leq P_0 = P\) and a corresponding sequence of non-negative integers \(k_1 \leq k_2 \leq \cdots \leq k_m\) such that \(\chi(X^G) = \chi(S^d)\) where

\[
d \equiv n(Z_{2^k}) - \left(\sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-i}}\right) \pmod{2}.
\]

It should be noted here that the sequence of subgroups can be selected so that each factor group \(P_{i-1}/P_i\) is an irreducible representation of \(Z_{2^k}\) over the field \(Z_p\). If this is done, then by a Jordan-Hölder type theorem the length \(m\) is unique. Also, the subgroups \(P_i\) and the integers \(k_i\) depend entirely on the group structure of \(G\). It isn’t difficult to verify that a \(p\)-group with an action of \(Z_{2^k}\) has a decomposition similar to the above; we have taken pains in Lemma 2 below to ensure that one exists of an especially nice type. Also, we should regard Theorem 1 as a generalization of the situation for linear representations (see the remark following §3).

I would like to express sincere thanks to the referee, whose comments resulted in substantial improvements.

1. **Irreducible representations of** \(Z_{2^k}\) **over** \(Z_p\) (**\(p\) odd prime).** In this section we want to determine the irreducible representations of \(Z_{2^k}\) over \(Z_p\). The necessary results are contained in Lemma 1.

From now on the cyclic group \(Z_{2^k}\) will be written \(C(j)\). If \(\lambda\) is a \(2^j\) root of \(-1\) in \(Z_p\), \(Z_p^\lambda\) denotes the one-dimensional representation of \(C(j + 1)\) given by multiplication by \(\lambda\) (this includes the case \(\lambda = -1\), corresponding to \(j = 0\) and \(C(1)\), in which case we write \(Z_p^1\)).
For any $m$ such that $1 \leq m \leq k$, one can consider the induced representation of $C(m)$ over $\mathbb{Z}_p$, $\text{Ind}^{C(m)}_{C(i)}(\mathbb{Z}_p)$, which we write as $\rho_m$. As a vector space over $\mathbb{Z}_p$, $\rho_m$ has dimension $2^{m-1}$ and a generator of $C(m)$ acts on a basis $\{a_i\}_{i=1}^{2^{m-1}}$ by $a_i \rightarrow a_i + 1$ if $i < 2^{m-1}$ while $a_{2^{m-1}} \rightarrow -a_1$. In general, if $G$ is any group, $H \leq K \leq G$ are subgroups and $V$ is a representation of $H$ over some field, then induction is transitive, i.e. $\text{Ind}^G_H(V) = \text{Ind}^G_K(\text{Ind}_H^K(V))$. Also if $V = V_1 + V_2$ then $\text{Ind}^G_H(V) = \text{Ind}^G_H(V_1) + \text{Ind}^G_H(V_2)$. (For more information on induced representations, see [7; Chapter 7]).

Now there is a one-to-one (up to similarity) correspondence between faithful irreducible representations of $C(k)$ over $\mathbb{Z}_p$ and the irreducible factors of $x^{2k-1+1}$. Our main concern here, therefore, will be to understand the factorisation of $x^{2k-1+1}$ over $\mathbb{Z}_p$. Given any irreducible factor of $x^{2k-1+1}$, note that if $\alpha$ is a root then the companion matrix in $\mathbb{Z}_p(\alpha)$ provides a representation of $C(k)$ which is faithful, irreducible, and such that the generator of $C(k)$ has the given factor as its characteristic polynomial. In the following lemma, evidently (a) is well-known—a proof is included for completeness.

**Lemma 1.** (a) The irreducible factors of $x^{2k-1+1}$ all have the same degree $d$ and that degree is the order of $p \pmod{2^k}$, i.e. $p^d \equiv 1 \pmod{2^k}$.

(b) If $k = 1$ or if $k > 1$ and $p \equiv 1 \pmod{4}$ then all the irreducible, faithful representations of $C(k)$ over $\mathbb{Z}_p$ are either 1-dimensional or are induced up from a 1-dimensional representation of a proper subgroup, all of the same dim = $2^{k-1}$.

(c) If $k > 1$ and $p \equiv 3 \pmod{4}$ then all the faithful irreducible representations of $C(k)$ over $\mathbb{Z}_p$ are either 2-dimensional or are induced up from a 2-dimensional representation of a proper subgroup, all of the same dim = $2^{k-1+1}$.

**Proof.** (a) Let $g(x)$ be an irreducible factor of degree $d$ of $x^{2k-1} + 1$. Then $g(x)$ is the minimal polynomial for a primitive $2^k$ root of 1, say $\alpha$. Consider the splitting field of $x^{p^d} - x$, which is just $\mathbb{Z}_p(\alpha)$ (since the degree of $g$ is $d$). Thus $\alpha^{p^d-1} = 1$, so that $2^k | p^d - 1$ (since $\alpha$ is a primitive root of 1). Now let $\hat{d}$ be any natural number such that $2^k | p^d - 1$. We claim that $d \leq \hat{d}$, establishing (a). Let $F$ be the splitting field of $x^{p^d} - x$ and let $\phi$: $F \rightarrow F$ be the generator of the Galois group over $\mathbb{Z}_p$ given by $y \rightarrow y^{p^d}$. Suppose $\alpha, \phi(\alpha), \ldots, \phi^{n}(\alpha)$ are all distinct where $1 \leq n \leq \hat{d} - 1$ and consider the polynomial $h(x) = \Pi_{i=0}^{n}(x - \phi^i(\alpha))$. The coefficients are symmetric functions in the $\phi^i(\alpha)$ and are fixed by $\phi$ hence belong to $\mathbb{Z}_p$. 


Since \( h(\alpha) = 0 \), it follows that \( d \leq n + 1 \leq \hat{d} \). Thus \( d = \) degree of \( g \) is the order of \( p \) (mod \( 2^k \)).

(b) If \( k = 1 \), the only faithful, irreducible representation of \( C(1) \) is \( \mathbb{Z}_p \).

So, we will assume that \( k > 1 \) and that \( p \equiv 1 \) (mod 4). Let \( l \) be the largest integer such that \( p \equiv 1 \) (mod \( 2^l \)). If \( k \leq l \), then \( p \equiv 1 \) (mod \( 2^k \)) and part (a) implies that any faithful irreducible representation of \( C(k) \) has dimension 1, and these are given by multiplication by a \( 2^{k-l} \) root of \(-1, \lambda\). These are the representations \( \mathbb{Z}_p^\lambda \). If \( k > l \) and \( f(x) \) is an irreducible factor of \( x^{2^{l-1}} + 1 \) (deg \( f \) is 1, say \( f(x) = x - \lambda \)) then \( g(x) = f(x^{2^{k-l}}) \) has degree the order of \( p \) (mod \( 2^k \)), is a factor of \( x^{2^{k-l}} + 1 \) and is irreducible. On the other hand the characteristic polynomial of Ind\(_{C(1)}^{C(k)}(\mathbb{Z}_p^\lambda) \) is \( x^{2^{k-l}} - \lambda \) (note that this representation has dimension \( 2^{k-l} \)).

(c) If \( k > 1 \) and \( p \equiv 3 \) (mod 4), let \( l \) be the largest integer such that \( p \equiv -1 \) (mod \( 2^{l-1} \)). If \( k \leq l \) then \( p \equiv -1 \) (mod \( 2^{k-1} \)) and \( p^2 \equiv 1 \) (mod \( 2^k \)). Thus any irreducible factor of \( x^{2^{k-1}} + 1 \) has degree 2 and so the dimension of the corresponding representation is 2. If \( k > l \) and \( f(x) \) is an irreducible factor of \( x^{2^{l-1}} + 1 \) (of degree 2) then \( g(x) = f(x^{2^{k-l}}) \) has degree the order of \( p \) (mod \( 2^k \)), is a factor of \( x^{2^{k-l}} + 1 \) and is irreducible. However, the characteristic polynomial of Ind\(_{C(1)}^{C(k)}(V) \) is \( g(x) \) where \( V \) is a two-dimensional representation corresponding to \( f(x) \) (note that this representation has dimension \( 2^{k-l+1} \)). This completes the proof of the lemma.

2. Normal chief series for \( p \)-elementary groups. A normal chief series for a \( p \)-group \( P \) is a normal series whose adjacent quotients are elementary abelian. When \( P \) comes equipped with an automorphism \( \phi \) of period \( 2^k \) (as in the present case, via conjugation) we would like to find a \( \phi \) invariant normal chief series. We will call a representation of \( C(k) \) over \( \mathbb{Z}_p \) “homocyclic” if it decomposes into irreducible subrepresentations each having the same kernel.

**Lemma 2.** A \( p \)-group \( P \) with an automorphism \( \phi \) of period \( 2^k \) has a \( \phi \) invariant normal chief series whose adjacent quotients \( P_{i-1}/P_i \) are homocyclic representations of \( C(k) \) with kernels \( C(k_i) \), and \( k_i \leq k_{i+1} \).

**Proof.** For any \( p \)-group, \( P \), the characteristic subgroup \( P'P^p \) (\( P' \) is the commutator subgroup, \( P^p \) is generated by all \( p \)th powers) is called the Frattini subgroup, \( \hat{P} \). \( P/\hat{P} \) is elementary abelian and representatives in \( P \) of generators of \( P/\hat{P} \) will generate \( P \). Moreover, \( \hat{P} = e \) iff \( P \) is elementary abelian (see [5; Ch. 5, Thm. 1.1]).
Set $P_o = P$, consider the projection $\pi: P_o \to P_o/\hat{P}_o$ and suppose that the representation of $C(k)$ on $P_o/\hat{P}_o$ decomposes into $V_1 \oplus \bar{V}_1$, where $V_1$ is the sum of all irreducible summands having the same, minimal kernel among the kernels appearing on $P_o/\hat{P}_o$, say $C(k_1)$. Now let $P_1 \leq P_0$ be $\pi^{-1}(\bar{V}_1)$. Then on $P_o/P_1$, $C(k)$ acts with kernel $C(k_1)$. Consider $P_1/\hat{P}_1$ and write $P_1/\hat{P}_1$ as $V_2 \oplus \bar{V}_2$, where again $V_2$ is the sum of all irreducible summands with minimal kernel, say $C(k_2)$. $k_2 \geq k_1$ because generators for $P_1/\hat{P}_1$ lift to generators for $P_1$ and $C(k_1)$ acts trivially on $P_0$ hence on $P_1$ by [5; Thm. 1.4]. Let $P_2 = \pi^{-1}(\bar{V}_2)$ where $\pi: P_1 \to P_1/\hat{P}_1$. This process can be continued until a $P_j$ is found such that $P_j = e$. But then $P_j$ is elementary abelian and certainly $P_j$ can continue to be decomposed in this way. Thus we have a normal series

$$e = P_m \triangleleft P_{m-1} \triangleleft \cdots \triangleleft P_1 \triangleleft P_0 = P$$

such that $C(k)$ acts on $P_{i-1}/P_i$ with $C(k_i)$ and $k_i \leq k_{i+1}$, $i = 1, 2, \ldots, m$.

3. Special cases and the Main Theorem. If $G$ acts on a mod-$p$ homology sphere $X$, we wish to compare the degree, $\delta_x^r$, of a generator of $C(k)$ acting on $X$ with the degree, $\delta_{x^r}$, of the generator on $X^r$ (the induced action since $P \leq G$). The following lemma is central and is a modification of a key result of [8, compare Prop. 1.1].

**Lemma 3.** Suppose $G$ is a semidirect product of an elementary abelian $p$-group $P$ and a cyclic 2-group $C(k)$ such that the action of $C(k)$ on $P$ (by conjugation) has kernel $C(m)$ and is irreducible. If $G$ acts on a mod-$p$ homology n-sphere $X$ then the degrees $\delta_x$ and $\delta_{x^r}$ are related as follows:

$$\delta_x = (-1)^{\epsilon} \delta_{x^r} \quad \text{where} \quad \epsilon = \frac{n - n(P)}{2k-m}.$$  

**Proof.** Proceeding exactly as in [8, loc. cit.], we consider the relative fibration $(X, Z) \to (X_p, Z_p) \to BP$, where $Z = X^p \sim_p S^r$. There is the spectral sequence of this relative fibering with $E_2$-term given by $E_2^{i,j} = H^i(BP) \otimes H^j(X, Z)$ (coefficients in $Z_p$). If $d: E_2^{0,n} \to E_2^{n-r, r+1}$ (where $r = n(P)$) is the transgression then $d(x) = A \otimes \delta z$ where $x$ generates $H^n(X)$ and $z$ generates $H^r(Z)$. If rank $P$ is 1 then $A = t^{(n-r)/2}$, where $t$ generates $H^2(BP)$. If rank $P > 1$ then recall the Borel identity, $n - r = \Sigma (n(H) - r)$, with sum on all corank 1 subgroups $H$ in $P$. Suppose there are exactly $s$ corank 1 subgroups $H_1, \ldots, H_s$ such that $n(H_j) - r > 0$. Letting $r_i = n(H_i)$, there are elements $w_1, \ldots, w_s \in H^2(BP)$ and an $a \in H^0(BP)$ such that $A = aw_1^d w_2^d \cdots w_s^d$, where $d_i = (r_i - r)/2$ (see [6, Thm. 2]).
Since $P$ is an irreducible representation of $C(k)$ (let $\alpha$ be a generator) with kernel $C(m)$, $P$ has either dimension 1 (if either $k - m = 1$ or if $k - m > 1$ and $p \equiv 1 \pmod{4}$ with $k - m \leq l$, where $l$ is as defined in the proof of Lemma 1 (b) and depends only on $p$), or has dimension $2^{k-m-1}$ (resp. $2^{k-m-1+1}$) (if $p \equiv 1 \pmod{4}$, resp. $p \equiv 3 \pmod{4}$).

Now, just as in [8], $C(k)$ acting by conjugation on $P$ determines an action of $C(k)$ on the fibration (and so on the spectral sequence) as follows. Define

$$\phi: EG \times X \to EG \times X \quad \text{by} \quad \phi(e, x) = (e\alpha, \alpha^{-1}x)$$

where $\alpha$ generates $C(k)$. For $g \in P$, we have $\phi(g(e, x)) = \psi(g)\phi(e, x)$ ($\psi$ is the automorphism of $P$ defined by $\alpha^{-1}ga = \psi(g)$). Thus we have an action on the fibration (since $EG \simeq EP$):

$$(X, Z) \xrightarrow{\alpha} (X, Z)$$

$$\downarrow \quad \downarrow$$

$$(X_P, Z_P) \xrightarrow{\alpha} (X_P, Z_P)$$

$$\downarrow \quad \downarrow$$

$$BP \xrightarrow{\alpha} BP$$

$\bar{\alpha}: BP \to BP$ is induced by $\psi: P \to P$. If $P$ has dimension 1, $\bar{\alpha}^x(t) = \lambda t$, $\psi: P \to P$ is multiplication by $\lambda$, a $2^{k-m-1}$ root of $-1$ and $t$ generates $H^2(BP)$. If the dimension of $P$ is larger than 1, the action of $C(k)$ on the collection of subgroups $\{H_1, \ldots, H_s\}$ must be considered (and the corresponding action on $w_1, \ldots, w_s$). First of all, if $p \equiv 1 \pmod{4}$, $\alpha^{2^{k-m-1}}$ acts on $P$ by multiplication on the basis elements by $\lambda$, a $2^{l-1}$ root of $-1$ and no smaller power of $\alpha$ leaves the $H_i$ invariant (smaller powers are represented by even dimensional irreducible subrepresentations). If $p \equiv 3 \pmod{4}$, since there are no roots of $-1$ in $Z_P$, the smallest power of $\alpha$ leaving the $H_i$ invariant is $\alpha^{2^{k-m-1}}$ (this is just multiplication by $-1$). Therefore the members of $\{H_1, \ldots, H_s\}$ are permuted, each one in a orbit of size $2^{k-m-1}$ (if $p \equiv 1 \pmod{4}$) or size $2^{k-m-1}$ (if $p \equiv 3 \pmod{4}$). This observation has several consequences. If $H_i$ and $H_j$ are in the same orbit, $(n(H_i) - r)/2 = (n(H_j) - r)/2$ and it follows from the Borel Identity that $2^{k-m-1}$ ($p \equiv 1 \pmod{4}$) or $2^{k-m-1}$ ($p \equiv 3 \pmod{4}$) divides $(n - r)/2$. Now consider the class $aw_1^{t_1} \cdots w_s^{t_s}$. It follows from [6; Thm. 2; Lemma 3] that if $w_1, \ldots, w_{t_2k-m-i}$ ($p \equiv 1 \pmod{4}$) or $w_1, \ldots, w_{t_2k-m-i}$ ($p \equiv 3 \pmod{4}$) are in the same orbit, the classes are permuted, say $w_{i_j} \to w_{i_{j+1}}$ and $w_{t_2k-m-i} \to \lambda w_{i_1}$ ($\lambda$ a $2^{l-1}$ root of $-1$ and $p \equiv 1 \pmod{4}$) (or $w_{t_2k-m-i} \to -w_{i_1}$).
if \( p \equiv 3 \pmod{4} \). Under \( \bar{\alpha}^* \) the class \( \omega \{ w_1^d_1 \cdots w_s^d_s \} \) is sent to \( \lambda \omega \{ w_1^d_1 \cdots w_s^d_s \} \) (or \( -1 \omega \{ w_1^d_1 \cdots w_s^d_s \} \)) where \( \varepsilon = (n - r)/2^{k - m - l + 1} \) (or \( (n - r)/2^{k - m} \) if \( p \equiv 3 \pmod{4} \)).

Consider now the commutative diagram (from the \( E_2 \)-term):

\[
\begin{array}{ccc}
H^n(X, Z) & \xrightarrow{\alpha^*} & H^n(X, Z) \\
\downarrow d & & \downarrow d \\
H^{n-r}(BP) \otimes H^{r+1}(X, Z) & \xrightarrow{\bar{\alpha}^* \otimes \alpha^*} & H^{n-r}(BP) \otimes H^{r+1}(X, Z).
\end{array}
\]

We have:

\[
d(\alpha^* x) = \delta_X(A \otimes \delta z) = (\bar{\alpha}^* \otimes \alpha^*)(A \otimes \delta z)
= \lambda \varepsilon \delta_{X^p}(A \otimes \delta z) \quad \text{(or} \ (-1)^\varepsilon \delta_{X^p}(A \otimes \delta z))
\]

where

\[
\varepsilon = \begin{cases} 
(n - r)/2^{k - m - l + 1} & \text{if} \ p \equiv 1 \pmod{4}, \\
(n - r)/2^{k - m} & \text{if} \ p \equiv 3 \pmod{4}.
\end{cases}
\]

Thus

\[
\delta_X = \lambda \varepsilon \delta_{X^p} \quad \text{(or} \ (-1)^\varepsilon \delta_{X^p}).
\]

Since each of \( \delta_X, \delta_{X^p} \) is \( \pm 1 \), it follows that if \( k - m \leq l \),

\[
2^{k - m - l} | (n - r)/2
\]

while if \( k - m > l \),

\[
2^{l - 1} | (n - r)/2^{k - m - l + 1}
\]

(all of this only when \( p \equiv 1 \pmod{4} \)).

Finally we have,

\[
\delta_X = (-1)^\varepsilon \delta_{X^p} \quad \text{where} \ \varepsilon = (n - r)/2^{k - m}.
\]

This completes Lemma 3.

We can now prove an analogue of [8, Thm. 1.3]. Suppose \( G \) is a semidirect product of a \( p \)-group \( P \) and \( C(k) \). Also, suppose that \( G \) acts on a \( \mathbb{Z}_p \)-homology \( n \)-sphere \( X \).

**Lemma 4.** There is a sequence of subgroups \( e = P_m \lhd P_{m-1} \lhd \cdots \lhd P_1 \lhd P_0 = P \) and a corresponding sequence of non-negative integers \( k_1 \leq k_2 \leq \cdots \leq k_m \) such that if \( \delta_X \) and \( \delta_{X^p} \) denote, respectively, the degrees of a
generator $\alpha$ of $C(k)$ on $X$, $X^p$ then

$$\delta_X = (-1)^s \delta_{X^p}$$

where

$$\epsilon = \sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-k_i}}.$$ 

Proof. This now follows directly from Lemmas 2 and 3 applied to the $P_{i-1}/P_i$ action on $X^p$, where a normal series is obtained as in Lemma 2 and a refinement made so that adjacent quotients are irreducible.

The proof of the following is now clear.

**Theorem 1.** If $G$ is a semidirect product as above, acting on a mod-$2^p$ homology $n$-sphere $X$, then $\chi(X^G) = \chi(S^d)$ where

$$d \equiv n(\mathbb{Z}_{2^k}) - \left( \sum_{i=1}^{m} \frac{n(P_i) - n(P_{i-1})}{2^{k-k_i}} \right) \quad (\text{mod } 2)$$

where the $P_i$ and $k_i$ are as in Lemma 4.

**Proof.** From a well-known result of Floyd ([4]), $\chi(X^G)$ is the Lefschetz number of a generator of $\mathbb{Z}_{2^k}$ acting on $X^p$. One can easily verify that (from Lemma 4),

$$\delta_{X^p} = (-1)^{n-n(\mathbb{Z}_{2^k})+\epsilon}.$$ 

Since $n + n(P)$ is even,

$$\chi(X^G) = 1 + (-1)^{n(\mathbb{Z}_{2^k})-\epsilon}.$$ 

This completes the proof of Theorem 1.

**Corollary.** If $G$ and $X$ are as in Theorem 1 and, moreover, $G$ is a direct product then

$$\chi(X^G) = \chi(X^{\mathbb{Z}_{2^k}}).$$

**Proof.** The reader may check that in this case the sum term appearing in the conclusion is 0 mod 2 (this is easy to see via Lemma 3). Note that this corollary is also easily obtained from a well-known result of Floyd (see [1; Ch. III, Th. 4.4.]).

**Remark.** Suppose $G$ is an extension of an elementary abelian $p$-group $P$ by a cyclic 2-group $\mathbb{Z}_{2^k}$, $P \rightarrow G \rightarrow \mathbb{Z}_{2^k}$ and $\psi: \mathbb{Z}_{2^k} \rightarrow \text{Aut}(P)$ has kernel
If $V$ is a real representation of $G$ then we have:

$$\dim V^G \equiv \dim V^{Z_{2^m}} - \left( \frac{\dim V - \dim V^P}{2^{k-m}} \right) \pmod{2}.$$  

This can be verified by considering the real irreducible representations of $G$, which originate from complex irreducible representation which in turn are induced up from complex irreducible representations of the subgroup $P \times Z_{2^m}$. If those complex irreducible representations of $G$, for which both $P$ and $Z_{2^m}$ act nontrivially, are compared with those for which $P$ acts nontrivially but $Z_{2^m}$ acts trivially, the congruence above can be derived. It should also be noted that if $m = 0$ then the above congruence is actually an equality (for more information see [7; Chapters 7, 8 and 13]).

**REFERENCES**


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