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We give the following necessary and sufficient conditions for a $*$-derivation, A, on $C[0,1]$, to generate a continuous group of $*$-automorphisms: A must be equivalent to the closure of pD, where $(pD)f(x) \equiv p(x)f'(x)$, with $(1/p)$ not locally integrable at the zeroes of p. We give similar necessary and sufficient conditions for a well-behaved $*$-derivation to generate a positive contraction semigroup. We show that any $*$-derivation on $C[0,1]$ has an extension (possibly on a larger space) that generates a continuous group of $*$-automorphisms.

Introduction. Continuous groups of $*$-automorphisms of C^*-algebras are of interest in quantum-dynamical systems. Their generators are closed, well-behaved $*$-derivations. (See Definitions 2, 4 and 5.) Batty [**1**] has characterized all closed quasi well-behaved derivations on $C[0,1]$: they are equivalent, via a homeomorphism of $[0,1]$, to the closure of the operator pD, where $(pD)f(x) \equiv p(x)f'(x)$. In this paper, we characterize generators of continuous groups of $*$-automorphisms on $C[0,1]$ by presenting necessary and sufficient conditions on p, that make the closure of pD such a generator.

An unsolved problem is the following. Given an accretive operator (see Definition 3) on a Banach space, does it have an extension (possibly on a larger space) that is m-accretive (generates a contraction semigroup)? Trotter [**3**] constructs, for any real-valued continuous p, an extension of pD that is m-accretive. Combining this with Batty's result, we conclude that every well-behaved $*$-derivation on $C[0,1]$ has an m-accretive extension, giving a partial solution to the problem above. With a slight modification of Trotter's construction, we show that every well-behaved $*$-derivation on $C[0,1]$ has an extension that generates a continuous group of $*$-automorphisms. The extension we construct acts on $\mathrm{BC}(\mathbf{R}^3)$.

For easy future reference, we present all definitions and preliminaries here. These need not be read until they are referred to in the course of the paper.

Definitions and preliminaries.

1. $\mathscr{D}(A)$ is the domain of the operator A.
2. $\bar{A}$ is the closure of the operator A.

A is *closed* if $A = \bar{A}$.

3. A linear operator, A, on the Banach space, X, is *accretive* if, for all x in $\mathscr{D}(A)$, there exists ϕ_x in X^* such that $\|\phi_x\| = 1$, $\phi_x(x) = \|x\|$ and $\mathrm{Re}[\phi_x(Ax)] \geq 0$.

The operator, A, is *m-accretive* if it generates a 1-parameter conraction semigroup, $\{T_t\}_{t\geq 0}$, that is, $\lim_{t\to 0^+}(1/t)(T_t x - x) = (-Ax)$, for all x in $\mathscr{D}(A)$. A is m-accretive if and only if A is closed, accretive and densely defined and $(1 + A)$ is surjective. (See [**2**].)

4. A is *well-behaved* if A and $(-A)$ are accretive. In other terminology, this is equivalent to (iA) being Hermitian.

5. If A is a linear operator on a C^*-algebra, $\mathscr{A}$, then A is a $*$*-derivation* if $A(xy) = x(Ay) + (Ax)y$ and $A(x^*) = (Ax)^*$ for all x, y in $\mathscr{D}(A)$.

Whenever A is a derivation, we will assume that $\mathscr{D}(A)$ is a dense $*$-sub-algebra.

The operator A generates a continuous group of $*$-automorphisms, $\{T_t\}_{t\in\mathbf{R}}$ $(-Ax = \lim_{t\to 0}(1/t)(T_t x - x))$ if and only if A is a well-behaved closed $*$-derivation, with $(1 + A)$ and $(1 - A)$ surjective.

6. $C[a, b]$ is the set of all continuous, complex-valued functions on $[a, b]$ with the supremum norm.

The set of all continuously differentiable functions we write as $C^1[a, b]$.

$$C_0[a, b] \equiv \{f \text{ in } C[a, b] | f(a) = 0 = f(b)\},$$

$$C_0^1[a, b] \equiv \{f \text{ in } C^1[a, b] | f(a) = 0 = f(b)\}.$$

For any topological space Ω, $\mathrm{BC}(\Omega)$ is the set of all bounded, uniformly continuous, complex-valued functions on Ω.

7. If A is a derivation on $C[0, 1]$, then x in $C[0, 1]$ is *well-behaved for A* if $\mathrm{Re}(Af)(x) = 0$ whenever $f(x) = \pm\|f\|_\infty$. Note that A is well-behaved if and only if all points in $[0, 1]$ are well-behaved.

A is *quasi well-behaved* if the interior of the set of well-behaved points for A is dense in $[0, 1]$.

The differentiation operator, D $(Df(x) \equiv f'(x))$, is quasi well-behaved, but not well-behaved. Its set of well-behaved points is $(0, 1)$.

8. If A, B are derivations on $C[0, 1]$, then A is *equivalent* to B if there exists θ, a homeomorphism of $[0, 1]$, such that $\mathscr{D}(A) = \{f \circ \theta | f \in \mathscr{D}(B)\}$ and $A(f \circ \theta) = (Bf)$ for all f in $\mathscr{D}(B)$.

Being quasi well-behaved, well-behaved, or the generator of a continuous group of $*$-automorphisms are all properties that are invariant under equivalence.

9. The derivation pD, on $C[0,1]$, is defined by $(pD)f(x) \equiv p(x)f'(x)$, with $\mathscr{D}(pD) = C^1[0,1]$.

Batty [1] showed that A is closed and quasi well-behaved if and only if A is equivalent to $\overline{pD}$, for some real-valued p in $C[0,1]$.

10. Λ: $X \to Y$ is an *embedding* of X in Y if Λ is injective and bicontinuous. If B is a linear operator on Y and A is a linear operator on X, then B is an *extension* of A if $\Lambda(\mathscr{D}(A))$ is contained in $\mathscr{D}(B)$ and $B(\Lambda x) = \Lambda(Ax)$, for all x in $\mathscr{D}(A)$.

11. If f is a complex-valued (including ∞) function on $[0,1]$, then f is *locally integrable at* b^+ if there exists $c > b$ such that f is finite on (b, c) and $\int_b^c |f|$ is finite. f is *locally integrable at* b^- if there exists $a < b$ such that f is finite on (a, b) and $\int_a^b |f|$ is finite.

THEOREM 1. *Suppose* $A = pD$ *on* $C_0[a, b]$, *where* $p \in C_0[a, b]$, $p(x)$ *is greater than zero for all* x *in* (a, b), *and* $\mathscr{D}(A) = C_0^1(a, b)$. *Then*

(a) $(1 - A)$ *is surjective if and only if* $(1/p)$ *is not locally integrable at* a^+.

(b) $(1 + A)$ *is surjective if and only if* $(1/p)$ *is not locally integrable at* b^-.

Proof. Let q: $(a, b) \to \mathbf{R}$ be such that $q'(x) = 1/p(x)$ for all x in (a, b).

(a) Suppose $(1 - A)f = g$. Solving this differential equation gives

(1)

$$f(y) = -e^{q(y)} \int_y^b (e^{-q(t)})' g(t)\, dt, \quad \text{for all } y \text{ in } (a, b).$$

If $(1/p)$ is locally integrable at a^+, then $q(a)$ is finite, so that $\int_a^b (e^{-q(t)})' g(t)\, dt$ must equal zero whenever g is in the range of $(1 - A)$. This means $(1 - A)$ is not surjective. Conversely, if $(1/p)$ is not locally integrable at a^+, then $\lim_{y \to a} q(y) = -\infty$, so that equation (1) defines, for any g in $C_0[a, b]$, a function f in $C_0^1[a, b]$ such that $(1 - A)f = g$; that is, $(1 - A)$ is surjective.

(b) Suppose $(1 + A)f = g$. Then

(2)

$$f(x) = e^{-q(x)} \int_a^x (e^{q(t)})' g(t)\, dt, \quad \text{for all } x \text{ in } (a, b)$$

As in (a), $(1 + A)$ is surjective if and only if $\lim_{x \to b} q(x) = \infty$, which is equivalent to $(1/p)$ being not locally integrable at b^-.

THEOREM 2. *$\overline{pD}$ generates a continuous group of $*$-automorphisms on $C[0,1]$ if and only if p is real-valued and in $C_0[0,1]$ and $(1/p)$ is not locally integrable at a^+ or a^-, whenever $p(a) = 0$.*

Proof. Suppose $\overline{pD}$ generates a continuous group of $*$-automorphisms on $C[0,1]$. Then $\overline{pD}$ is well-behaved, so p must be real-valued and in $C_0[0,1]$ (see Definitions 4 and 5).

Suppose $p(a) = 0$.

If there exists a sequence $\{a_n\}_{n=0}^{\infty}$, converging to a, such that, for all n, $a_n > a$, and $p(a_n) = 0$, then $(1/p)$ is not locally integrable at a^+. (See Definition 11.)

So suppose there exists $b > a$ such that $p(b) = 0$, $p(x) \neq 0$ for any x in (a, b). Let T_t be the $*$-automorphism generated by $\overline{pD}$. Since $(pD)f(a) = 0 = (pD)f(b)$, $(T_t f)(a) = f(a)$, $(T_t f)(b) = f(b)$, for all f. Thus T_t takes $C_0[a, b]$ into itself. This implies that $(1 + pD)|_{C_0[a,b]}$ and $(1 - pD)|_{C_0[a,b]}$ are surjective. (See Definition 5.) By Theorem 1, this implies that $(1/p)$ is not locally integrable at a^+. The same argument shows that $(1/p)$ is not locally integrable at a^-. Conversely, suppose p has the desired form. One could show directly that $(1 + \overline{pD})$ and $(1 - \overline{pD})$ are surjective, by using Theorem 1. However, it is much more elegant to use the following construction, due to Trotter ([**3**]), that presents the exact group generated by $\overline{pD}$.

For any real number t, define h_t: $[0,1] \to [0,1]$ in the following way.

$$h_t(x) \equiv x, \quad \text{if } p(x) = 0.$$

If x is in (a, b), where $p(a) = 0 = p(b)$, $p(y) \neq 0$ for any y in (a, b), we let q be such that $q'(y) = 1/p(y)$, for all y in (a, b). By hypothesis, q takes (a, b) onto the entire real line.

$$h_t(x) \equiv q^{-1}(q(x) - t).$$

We then define T_t, on $C[0,1]$, by

$$(T_t f)(x) \equiv f(h_t x). \tag{3}$$

Since $\{h_t\}_{t\in\mathbf{R}}$ is a group of homeomorphisms of $[0,1]$, $\{T_t\}_{t\in\mathbf{R}}$ is a continuous group of $*$-automorphisms on $C[0,1]$.

We have

$$\begin{aligned}
\frac{d}{dt}(T_t f)(x) &= \frac{\partial}{\partial t} f(h_t x) \\
&= f'(h_t x) \cdot \left[\frac{-1}{q'(q^{-1}(q(x) - t))}\right] = -p(h_t x) f'(h_t x) \\
&= -(pD)(T_t f)(x), \quad \text{for all } f \text{ in } C^1[0,1].
\end{aligned}$$

Thus $\overline{pD}$ generates $\{T_t\}_{t\in\mathbf{R}}$.

THEOREM 3. *Suppose A is a $*$-derivation on $C[0,1]$. Then A generates a continuous group of $*$-automorphisms if and only if A is equivalent to $\overline{pD}$, for some real-valued p in $C_0[0,1]$, where $(1/p)$ is not locally integrable at a^+ or a^-, whenever $p(a)=0$.*

Proof. Suppose A is a generator. Then A is well-behaved and closed, so Batty's result (see [**1**], and Definition 9) implies that there exists p such that A is equivalent to $\overline{pD}$. By Theorem 2, p must have the desired properties. The converse also follows from Theorem 2.

The following theorem weakens the hypotheses of Theorem 3, and has the weaker conclusion that A generates a continuous semigroup on $C[0,1]$.

THEOREM 4. *Suppose A is a well-behaved $*$-derivation on $C[0,1]$. Then A is m-accretive if and only if A is equivalent to $\overline{pD}$, where p is real-valued and in $C_0[0,1]$ and has the following property. Suppose that a, b are two consecutive zeroes of p. Then if $p>0$ on (a,b), $1/p$ is not locally integrable at b^-; if $p<0$ on (a,b), $1/p$ is not locally integrable at a^+.*

Proof. This is identical to the proofs of Theorems 2 and 3, with the following modifications. To get the desired properties of p, we used only part (b) of Theorem 1. Equation (3) is changed by defining $h_t(x) \equiv q^{-1}(\min(q(a), q(x)-t))$.

EXAMPLE. If we do not require that pD be well-behaved, then the condition that $(1/p)$ not be locally integrable at the zeroes of p is no longer a necessary condition, for pD to be m-accretive. This is shown by the following example. Let $p(x) \equiv -2x^{1/2}$. $(1/p)$ is locally integrable at 0^+, but pD is m-accretive on $C[0,1]$, for the following reasons. pD is accretive, because p is real-valued, and $p(0)=0$. If

$$f(x) \equiv e^{p(x)/2}\left[g(0) + \int_0^x (e^{-p(t)/2})' g(t)\,dt\right],$$

then $(1+pD)f = g$. Thus $(1+pD)$ is surjective. (See Definition 3.) It would be interesting to have necessary and sufficient conditions on p that would make $\overline{pD}$ m-accretive.

Chernoff has observed (unpublished) that any m-accretive operator may be thought of as differentiation, in the following way. Embed X in $\mathrm{BC}(\mathbf{R}^+, X)$ by $(\Lambda x)(t) \equiv e^{-tA}x$. Then $\Lambda(e^{-sA}x)(t) = (\Lambda x)(s+t)$, so that A generates the translation semigroup. For the following theorem, we make a similar construction.

THEOREM 5. *Suppose A and $(-A)$ both have m-accretive extensions on X. Then A has an extension that generates a continuous group of isometries, on* BC($\mathbf{R}$, X), *the space of bounded uniformly continuous functions from the real line into X.*

Proof. Let B be the m-accretive extension of A and C be the m-accretive extension of $(-A)$. Define $\Lambda: X \to \mathrm{BC}(\mathbf{R}, X)$ by

$$\begin{aligned} \Lambda x(t) = e^{-tB}x \quad & \text{if } t \geq 0, \\ e^{tC}x \quad & \text{if } t \leq 0. \end{aligned}$$

Let D equal differentiation, the generator of the translation group. I claim that D is an extension of $(-A)$.

$\Lambda(\mathscr{D}(B))$ contains only functions that are differentiable for $t > 0$, with a right-derivative at zero. $\Lambda(\mathscr{D}(C))$ contains only functions that are differentiable for $t < 0$, with a left-derivative at zero. Since $\mathscr{D}(A)$ is contained in $\mathscr{D}(B) \cap \mathscr{D}(C)$, $\Lambda(\mathscr{D}(A))$ will be contained in $\mathscr{D}(D)$ if

$$\frac{d^+}{dt}(\Lambda x)(0) = \frac{d^-}{dt}(\Lambda x)(0),$$

whenever x is in $\mathscr{D}(A)$.

If x is in $\mathscr{D}(A)$, then

$$\begin{aligned} \frac{d^+}{dt}(\Lambda x)(0) &= \lim_{t \to 0^+} \frac{1}{t}(\Lambda x(t) - x) = \lim_{t \to 0^+} \frac{1}{t}(e^{-tB}x - x) \\ &= -Bx = -Ax = Cx = -\lim_{t \to 0^+} \frac{1}{t}(e^{-tC}x - x) \\ &= \lim_{t \to 0^-} \frac{1}{t}(x - e^{tC}x) = \frac{d^-}{dt}(\Lambda x)(0). \end{aligned}$$

Thus $\Lambda(\mathscr{D}(A))$ is contained in $\mathscr{D}(D)$; a similar calculation shows that $\Lambda(Ax) = -D(\Lambda x)$.

In the following theorem, we use Trotter's construction ([3]), and the previous theorem, to show that every well-behaved $*$-derivation on $C[0,1]$ has an extension (possibly on a larger space) that generates a continuous group of $*$-automorphisms.

THEOREM 6. *Suppose A is a well-behaved $*$-derivation on $C[0,1]$. Then there exists an extension of A, on $\mathrm{BC}(\mathbf{R}^3)$, that generates a continuous group of $*$-automorphisms.*

Proof. Since A is well-behaved, A is closable, with well-behaved closure. (See [**1**].) Without loss of generality, we may assume $A = pD$, with p real-valued and in $C_0[0,1]$. (See Definitions 8 and 9.) Let $E \equiv \{x \mid p(x) = 0\}$. There exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of disjoint open intervals such that E^c, the complement of E, equals $\bigcup_{k=1}^{\infty}(a_k, b_k)$. For fixed k, we construct $h_{k,t}$ and $g_{k,t}$ from (a_k, b_k) into itself, as follows.

Let q_k be such that $q_k'(x) = 1/p(x)$, for all x in (a_k, b_k). Let $[c_k, d_k] \equiv q_k[a_k, b_k]$. ($c_k$ or d_k may equal $\pm\infty$.)

$$h_{k,t}(x) \equiv q_k^{-1}(\max(c_k, q_k(x) - t)),$$

$$g_{k,t}(x) \equiv q_k^{-1}(\min(d_k, q_k(x) + t)).$$

As in the proof of Theorem 2, we have

$$\frac{d}{dt} f(h_{k,t}x) = -Af(h_{k,t}x), \tag{4}$$

$$\frac{d}{dt} f(g_{k,t}x) = Af(g_{k,t}x) \quad \text{for } x \text{ in } (a_k, b_k).$$

Let $\Omega \equiv [(Ex\{0\})\bigcup_{k=1}^{\infty}[[a_k, b_k]x(k)]]$, the disjoint union of E and the closure of the components of the complement of E.

We embed $C[0,1]$ in $\mathrm{BC}(\Omega)$, by $(\Lambda f)(x, k) = f(x)$, for all x and k.

We define semigroups R_t and S_t, on $\mathrm{BC}(\Omega)$, by

$$(R_t f)(x, k) \equiv f(h_{k,t}x, k),$$

$$(S_t f)(x, k) \equiv f(g_{k,t}x, k), \quad \text{if } k > 0,$$

$$(R_t f)(x, 0) = f(x, 0) = (S_t f)(x, 0), \quad \text{for all } t.$$

By (4), the generator of R_t is an extension of A, and the generator of S_t is an extension of $(-A)$. By Theorem 5, A has an extension that generates a continuous group of isometries, on $\mathrm{BC}(\mathbf{R}, \mathrm{BC}(\Omega))$, which is contained in $\mathrm{BC}(\mathbf{R}^3)$. If we were to go through the construction in the proof of Theorem 5, we would get translation in one of the variables, on $\mathrm{BC}(\mathbf{R}^3)$, as the group generated by A; this is a group of $*$-automorphisms.

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