A CONSTRUCTION OF INNER MAPS PRESERVING THE HAAR MEASURE ON SPHERES

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We show, for $n > m$, the existence of non-trivial inner maps $f$: $B^n \rightarrow B^m$ with boundary values $f_\partial: S^n \rightarrow S^m$ such that $f_\partial^{-1}(A)$ has a positive Haar measure for every Borel subset $A$ of $S^m$ which has a positive Haar measure. Moreover, if $n = m$, the equality $\sigma(f_\partial^{-1}(A)) = \sigma(A)$ holds, where $\sigma$ is the Haar measure of $S^m$.

In this paper $C^n$ is an $n$-dimensional complex space with inner product defined by $\langle z^1, z^2 \rangle = \sum z^1_i z^2_i$ where $z^j = (z^j_1, z^j_2, \ldots, z^j_n)$ for $j = 1, 2$, and the norm $|z| = \langle z, z \rangle^{1/2}$. Let us introduce some notation:

$$B^n = \{ z \in C^n : |z| < 1 \}, \quad S^n = \partial B^n;$$

let $d$ be the metric on $S^n$:

$$d(z, z^*) = (1 - \text{Re}\langle z, z^* \rangle)^{1/2} = \frac{1}{\sqrt{2}}|z - z^*| \quad \text{for } z, z^* \in S^n,$$

and finally

$$B(z, r) = \{ z^* \in S^n : d(z, z^*) < r \} \quad \text{for } z \in S^n \text{ and } r > 0.$$

For every complex function $h: X \rightarrow C$ we define $Z(h) = h^{-1}(0)$. A holomorphic map $f: B^n \rightarrow B^m$ is called inner if

$$f_\partial(z) = \lim_{r \rightarrow 1} f(rz) \in S^m \quad \text{for almost every } z \in S^n$$

with respect to the unique, rotation-invariant Borel measure $\sigma_n$ on $S^n$ such that $\sigma_n(S^n) = 1$. If a continuous function $g: \overline{B}^n \rightarrow C^m$, defined on the closure of $B^n$, is holomorphic on $B^n$, we write $g \in A_m(B^n)$ or $g \in A(B^n)$ when $m = 1$. The theorem stated below is a generalization of the result of Aleksandrov [1]. Corollary 1 answers the problem given by Rudin [3]. Corollary 4 is a result of Aleksandrov obtained independently by the author.

**THEOREM.** Let $n \geq m$ and let $g = (g_1, \ldots, g_m) \in A_m(B^n)$, $h \in A(B^n)$ be maps such that $|g(z)| + |h(z)| \leq 1$ and $h(z) \neq 0$ for some $z \in B^n$. Then there exists an inner map $f = (f_1, f_2, \ldots, f_m): B^n \rightarrow B^m$ such that $f(z) = g(z)$ for every $z \in Z(h)$ and $f_i(z) = g_i(z)$ for every $z \in B^n$ and $i = 1, 2, \ldots, m - 1.$
COROLLARY 1. For every \( n \geq m \) there exist inner maps \( f: B^n \to B^m \) such that for every Borel subset \( A \subset S^m \) the inequality \( \sigma_n(f^{-1}(A)) > 0 \) holds provided \( \sigma_m(A) > 0 \). Moreover, if \( m = n \), the equality \( \sigma_n(f^{-1}(A)) = \sigma_n(A) \) holds and \( f \) is not an automorphism of \( B^n \).

COROLLARY 2. For every \( n \geq 1 \) there exist inner maps \( f: B^n \to B^m \), not automorphisms of \( B^n \), such that

\[
\int_{S^n} (h \circ f) \, d\sigma_n = \int_{S^n} h \, d\sigma_n
\]

for every continuous function \( h \) on \( S^n \).

Corollary 2 is an immediate consequence of Corollary 1. Let us assume that \( n \geq m \) and \( n \geq 2 \). To deduce the assertion of Corollary 1 from the Theorem let us take a holomorphic function \( k \in A(B^1) \) and the map \( g \in A_m(B^n) \), \( g(z) = p(z) + \frac{1}{4}z_n^2r(z_n) \), where \( p(z) = (z_1, z_2, \ldots, z_{m-1}, 0) \), \( r(z) = (0, \ldots, 0, k(z_n)) \) for \( z \in B^n \). Define \( h(z) = \frac{1}{4}z_1^2z_n^2 \). Then

\[
|g(z)| + |h(z)| \leq |p(z)| + \frac{1}{4} |z_n^2| + \frac{1}{4} |z_n^2| \leq \sqrt{1 - z_n^2} + \frac{1}{2} |z_n^2| \leq 1.
\]

By virtue of the Theorem there exists an inner map \( f = (f_1, f_2, \ldots, f_m): B^n \to B^m \) such that

1. \( f_j(z_1, z_2, \ldots, z_n) = z_j \) for \( j = 1, 2, \ldots, m - 1 \),
2. \( f_m(0, 0, \ldots, 0, z_n) = \frac{1}{4}z_n^2r(z_n) \),
3. \( f(z_1, z_2, \ldots, z_{n-1}, 0) = (z_1, z_2, \ldots, z_{m-1}, 0) \).

For any \( z \in B^{m-1} \) and \( l \geq m \) let

\[
B_z^l = \{ z^* \in B^l : z^*_j = z_j \text{ for } j = 1, 2, \ldots, m - 1 \},
\]

\[
S_z^l = \{ z^* \in S^l : z^*_j = z_j \text{ for } j = 1, 2, \ldots, m - 1 \},
\]

let \( \sigma_z^l \) be the rotation-invariant measure on the sphere \( S_z^l \) such that \( \sigma_z^l(S_z^l) = 1 \) and let \( f_z, f_z^* \) be the restrictions of \( f, f_* \) to the sets \( B_z^n \) and \( S_z^n \) respectively. From (1) it follows that \( f_z: B_z^n \to B_z^m \) and (2) says that \( f_z(w_1) = w_2 \), where \( w_1, w_2 \) are the centers of the balls \( B_z^n, B_z^m \) respectively. Since \( B_z^m \) is a one-dimensional complex ball, the equality \( \sigma_z^m((f_z^*)^{-1}(C)) = \sigma_z^m(C) \) holds for every Borel subset \( C \) of \( S_z^m \) and every \( z \) for which \( f_z \) is an inner map (see [4] p. 405). The function \( f_z \) is inner for almost every \( z \in B^{m-1} \) (with respect to the usual Lebesgue measure \( \lambda \) on \( B^{m-1} \)) because the map \( f \) is inner. Let us notice that there are positive functions
$s_1, s_2: R^{m-1} \to R_+$ such that for all Borel subsets $C^1 \subset S^n$, $C^2 \subset S^m$ we have

$$
\sigma_n(C^1) = \int_{B^{m-1}} s_1(z) \cdot \sigma^1_z(C^1) \, d\lambda(z),
$$

$$
\sigma_m(C^2) = \int_{B^{m-1}} s_2(z) \cdot \sigma^m_z(C^2) \, d\lambda(z),
$$

where $C^1_z = C^1 \cap S^n$, $C^2_z = C^2 \cap S^m$. Substituting $C_1 = (f^*)^{-1}(C_2)$ and using the equality $\sigma^m_z(C_1^1) = \sigma^m_z(C_2^2)$ (which holds for almost every $z$), it is easy to see that both of the above integrals are positive or equal to 0. If $n = m$ then $s_1 = s_2$ and the equality holds. This ends the proof of Corollary 1.

The following proof of the assertion of the Theorem is based on Löw’s construction of inner functions [3]. Let $g$ and $h$ be maps satisfying the assumptions of the Theorem. Then $\sigma_n(F) = 0$, where $F = Z(h) \cap S^n$. (This fact can be proved by induction. For $n = 1$ it is well-known theorem.) For $\delta > 0$ let

$$
F_\delta = \{ z \in S^n : d(z, F) < \delta \} \quad \text{and} \quad \| s \|_\delta = \sup_{z \in F_\delta} |s(z)|,
$$

where $s: S^n \to C^m$ is a continuous map. Observe that there exist constants $A_1, A_2$ such that for every $0 < r < \sqrt{2}$,

$$
A_1 r^{2n-1} \leq A(r) \leq A_2 r^{2n-1},
$$

where $A(r) = \sigma_n(B(z, r))$ for any $z \in S^n$.

Let $S \subset S^n$ be any closed subset of $S^n$, $\sigma_n(S) > 0$. Assume that for some number $r > 0$,

$$
\sigma_n(S_r) \leq 2\sigma_n(S),
$$

where $S_r = \{ z \in S^n : d(z, S) < r \}$. Let $\{ B(z^j, r) \}_{j=1}^{N(r)}$ be a maximal family of disjoint balls with centers $z^j \in S$. Since $S_r \supset \bigcup_{j=1}^{N(r)} B(z^j, r)$ and $S \subset \bigcup_{j=1}^{N(r)} B(z^j, 2r)$, applying inequalities (4) and (5), we get

$$
2\sigma_n(S) \geq \sigma_n(S_r) \geq \sigma_n\left( \bigcup_{j=1}^{N(r)} B(z^j, r) \right) = \sum_{j=1}^{N(r)} \sigma_n(B(z^j, r)) = N(r) \cdot A(r) \geq A_1 r^{2n-1} \cdot N(r)
$$
\[ \sigma_n(S) \leq \sigma_n\left( \bigcup_{j=1}^{N(r)} B(z_j, 2r) \right) = \sum_{j=1}^{N(r)} A(2r) = N(r) \cdot A(2r) \leq N(r) \cdot A_2 \cdot (2r)^{2n-1} = N(r) \cdot A_2 \cdot 2^{2n-1} \cdot r^{2n-1}. \]

So we have proved the existence of positive constants \( C_1 \) and \( C_2 \) (\( C_1 = 1/2^{2n-1}, C_2 = 2/A_1 \)) such that

\[ \frac{C_1}{r^{2n-1}} \cdot \sigma_n(S) \leq N(r) \leq \frac{C_2}{r^{2n-1}} \cdot \sigma_n(S). \]

Let us assume now that \( r > 0, z \in B^n, k \) is a natural number and \( M_k \) is the maximal number of disjoint balls of radius \( r \) and with centers in \( B(z, (k+1)r) \). Because these balls are included in \( B(z, (k+2)r) \), an argument similar to the above gives the estimate

\[ M_k \leq C_3 k^{2n-1} \]

for some constant \( C_3 \). Let \( \varphi: (0,1) \to \mathbb{R} \) be the continuous, positive function defined by

\[ \varphi(a) = \frac{1}{4\pi} \cdot C_1 \cdot A_1 \cdot \arccos(a) \cdot \left[ \log \frac{1}{a} \right]^{(2n-1)/2}. \]

**Lemma 1.** Let \( 0 < 2\varepsilon < a < b, 0 < \delta < 2C_3 \cdot a, \varepsilon < C_3 e^{-2n}, R < 1 \). Let \( P \) be a closed subset of \( F_\delta \) and let \( v \) be a continuous map \( v: S^n \to \mathbb{C}^m \) such that \( |v(z)| > a \) for \( z \in P \). There exists a closed subset \( K \) of \( F_\delta \) and a holomorphic map \( u: \mathbb{C}^n \to \mathbb{C}^m \) such that:

(a) \[ \|v + h \cdot u\|_{\delta/2} \leq \max(1, \|f\|_{\delta/2}) + 3\varepsilon; \]

(b) \[ \|u\|_R = \sup_{|z| \leq R} |u(z)| \leq \varepsilon; \]

(c) \[ |v(z) + h(z) \cdot u(z)| > a - 3\varepsilon \quad \text{for } z \in K \cup P; \]

(d) \[ K \subset F_\delta, \quad K \cap P = 0 \quad \text{and} \quad \sigma_n(K) \geq \varphi(a) \cdot \left[ \log(4C_3/\delta \varepsilon) \right]^{-(2n-1)/2} \cdot \sigma_n(F_\delta - P); \]

(e) \[ |g(z)| < \varepsilon \quad \text{for } z \in B^n - F_{\delta/2}; \]

(f) \[ u_j \equiv 0 \quad \text{for } j = 1, 2, \ldots, m - 1, \text{ where } u = (u_1, u_2, \ldots, u_m). \]

**Proof.** If \( \sigma_n(P) = \sigma_n(F_\delta) \) then the map \( u = (0,0,\ldots,0) \) and the set \( K = \emptyset \) satisfy conditions (a)–(e). Let us assume that \( \sigma_n(P) < \sigma_n(F_\delta). \)
There exists a positive number $\gamma$ such that $\gamma < \delta/2$ and
\[
\sigma_n(S) \geq \frac{1}{2} \cdot \sigma_n(F_\delta - P),
\]
where $S = S^n - [(S^n - F_\delta) \cup P]_\gamma$.

Since $v, h$ are uniformly continuous maps and $S$ is a closed subset, there exists a positive number $\gamma^*$ such that
\[
|g(z) - g(z')| < \varepsilon \delta, \quad |v(z) - v(z')| < \varepsilon, \quad \sigma_n(S_r) \leq 2 \cdot \sigma_n(S)
\]
for $z, z' \in S^n, d(z, z') < \gamma^*$ and $r < \gamma^*$.

Let $r, m$ be positive numbers such that $r \leq \frac{1}{2} \min(\gamma, \gamma^*)$, $m$ is an integer and $mr^2 = \log(2C_3/\delta \varepsilon)$. Moreover we assume $m$ is large so that
\[
C_2 \cdot m(2^{n-1})/2 \cdot e^{-m(1-R)} < \varepsilon.
\]

Choose a maximal family $\{ B(z^i, r) \}_{i=1}^{N(r)}$ of pairwise disjoint balls with centers $z^i \in S^n$. Because of (9), condition (5) is satisfied, so inequalities (6) also hold. For $k = 1, 2, \ldots, \left\lceil \sqrt{2}/r \right\rceil$ and $z \in S^n$ let
\[
V_k(z) = \{ z^i : kr \leq d(z, z^i) < (k + 1)r \}
\]
and let $N_k(z)$ be the number of elements of the set $V_k$. Since $V_k(z) \subset B(z, (k + 1)r)$, from the definition of $M_k$, we have $N_k(z) \leq M_k$ and (7) gives us
\[
N_k(z) \leq C_3 k^{2n-1}.
\]

Let $g(z) = \sum_{j=1}^{N(r)} \beta_j e^{-m(1-\xi(z, z^i))}$, where $\beta_j = (0, 0, \ldots, 0, \alpha_j, \in C^n$ is defined by $\beta_j = (0, 0, \ldots, 0, 0)$ if $|f(z^i)| \geq b$. If $|f(z^i)| < b$, then let $\beta_j$ be of the previous form, such that
\[
|f(z^i) + h(z) \cdot \beta_j| = b \quad \text{and} \quad |f(z^i) + \alpha \cdot h(z) \cdot \beta_j| \leq b
\]
for every $\alpha \in C, |\alpha| = 1$. Let us notice that for every $j, |\beta_j| \leq 1/|h(z^i)| \leq 1/\delta$ and that
\[
g(z) = \tilde{k} \cdot \sum_{j=1}^{N(r)} |\beta_j| \cdot e^{-md^2(z, z^i)} \cdot e^{iQ_{m,j}(z)}
\]
\[
= \tilde{k} \cdot \sum_{k=0}^{\left\lceil \sqrt{2}/r \right\rceil} \sum_{z^i \in V_k(z)} |\beta_j| e^{-md^2(z, z^i)} e^{iQ_{m,j}(z)}
\]
for some real functions $Q_{m,j}$ and $\tilde{k} = (0, 0, \ldots, 0, 1) \in C^n$.  

\[
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\]
If $V_{0}(z) = \emptyset$ or $z \in B(z', r)$ with $\beta_{j} = 0$ then, because of (11) and the inequality $mr^{2} > 2n$, we have

$$|g(z)| \leq \sum_{k=1}^{\left[\frac{r}{\delta}\right]} \sum_{z' \in V_{k}(z)} \frac{1}{\delta} e^{-m d^{2}(z, z')} \sum_{k=1}^{\left[\frac{r}{\delta}\right]} \frac{1}{\delta} |V_{k}(z)| e^{-m k^{2}r^{2}}$$

$$\leq \sum_{k=1}^{\infty} \frac{C_{3}}{\delta} k^{2n-l} e^{-k^{2}mr^{2}} \leq \sum_{k=1}^{\infty} e^{-kmr^{2}} \leq 2 \frac{C_{3}}{\delta} e^{-mr^{2}} = \varepsilon.$$

This proves part (e) of Lemma 1. If $z \in B(z', r)$ with $\beta_{j} \neq 0$ then

$$|v(z) + h(z) \cdot u(z)|$$

$$\leq |v(z') + h(z') \cdot \beta_{j} \cdot e^{-m d^{2}(z, z')} \cdot e^{2Q_{m,j}(z)}|$$

$$+ |[h(z) - h(z')] \cdot \beta_{j} \cdot e^{-m d^{2}(z, z')} \cdot e^{2Q_{m,j}(z)}| + |v(z) - v(z')|$$

$$+ |h(z) \cdot \sum_{z' \notin V_{0}(z)} \beta_{j} \cdot e^{-m d^{2}(z, z')} \cdot e^{2Q_{m,m}(z)}|$$

$$= I + II + III + IV.$$}

Because of (9)

$$III \leq \varepsilon \quad \text{and} \quad II \leq |h(z) - h(z')| \cdot |\beta_{j}| < \delta \cdot \varepsilon \cdot \frac{1}{\delta} = \varepsilon.$$}

By the same argument as in (12) we can prove that $IV \leq \varepsilon$. Moreover, we have $I \leq |v(z')| + |h(z') \cdot \beta_{j}| = b$. This altogether gives us

$$|v(z) + h(z) \cdot u(z)| \leq b + 3\varepsilon.$$}

Inequalities (12) and (14) prove part (a) of Lemma 1. Now we shall determine a certain subset $V$ of $W = \bigcup_{j=1}^{N(r)} B(z', r)$. To do this let us fix $j$, $1 \leq j \leq N(r)$, and let $z = \alpha = |v(z_{j})|, s(z) = e^{-m d^{2}(z, z')}, Q(z) = \text{arg}(e^{-m(1-((z, z')))} = m \cdot \text{Im}(z, z').$

Let us assume at first that $\alpha < 1$. We define

$$V_{j} = \{z \in B(z', r): s(z) \geq \alpha \text{ and } \cos Q(z) \geq \alpha\}.$$}

Using the same notation as in (13) we can write

$$|v(z) + h(z) \cdot u(z)| \geq I - II - III - IV.$$
As before, II ≤ ε, III ≤ ε and IV ≤ ε. Assuming \( z \in V_j \), we have

\[
I = \left| \nu(z') + h(z') \cdot \beta_j \cdot e^{-m \cdot d^2(z, z')} \cdot e^{iQ(z)} \right| \\
\geq |\alpha + (1 - \alpha) \cdot s(z) \cdot e^{iQ(z)}| \\
= \sqrt{\alpha^2 + 2\alpha(1 - \alpha) \cdot s(z) \cdot \cos Q(z) + (1 - \alpha)^2} \geq a
\]

because of our assumption about \( s(z) \) and \( \cos Q(z) \), the definition of \( \beta_j \) and simple geometry.

Combining (15) and (16) we get

\[
|\nu(z) + h(z) \cdot u(z)| > a - 3\epsilon \quad \text{for } z \in V_j.
\]

Let \( \rho > 0 \) be defined by \( m\rho^2 = \log(1/a) \). Then \( \rho \leq r \) because \( mr^2 = 2C_2/\delta \epsilon \) and \( 2C_2/\delta \geq 1/a \). So \( B(z', \rho) \subset B(z', r) \), and if \( z \in B(z', \rho) \) then \( s(z) \geq a \). The set \( \{ z \in B(z', \rho) : \cos Q \geq a \} \) consists of certain strips in the ball \( B(z', \rho) \). An easy geometric argument shows that these strips have a total area at least

\[
\frac{1}{2\pi} \cdot \arccos a \cdot \sigma_n(B(z', \rho)) = \frac{1}{2\pi} \cdot \arccos a \cdot A(\rho).
\]

Moreover \( V_j \subset B(z', r) \subset F_\delta \). Using inequality (4) and the fact that the above strips are included in \( V_j \), we get

\[
\sigma_n(V_j) \geq \frac{1}{2\pi} \cdot \arccos a \cdot A(\rho) \geq \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1}.
\]

If \( \alpha \geq 1 \), we define \( V_j = B(z', \rho) \). Because \( \beta_j = 0 \), it follows from (12) that

\[
|\nu(z) + h(z) \cdot u(z)| \geq |\nu(z')| - |\nu(z) - \nu(z')| - |h(z) \cdot u(z)| \\
\geq a - \epsilon - |u(z)| \geq a - 2\epsilon
\]

for \( z \in V_j \).

Finally, we define \( K = \bigcup_{j=1}^{N(c)} V_j \). We observe that inequality (17) holds for \( z \in K \). If \( z \in P \), then \( V_0(z) = \emptyset \) and inequality (12) gives us

\[
|\nu(z) + h(z) \cdot u(z)| \geq |\nu(z)| - |u(z)| \geq a - \epsilon.
\]
This altogether proves part (c) of Lemma 1. It is easy to check that $K \cap P = \emptyset$. Inequalities (18), (6), (9) and the definitions of $\rho$ and $mr^2$ yield

$$\sigma_n(K) \geq \sigma_n \left( \bigcup_{j=1}^{N(r)} V_j \right) = \sum_{j=1}^{N(r)} \sigma_n(V_j)$$

$$\geq N(r) \cdot \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1}$$

$$\geq \frac{C_1}{r^{2n-1}} \cdot \sigma_n(S) \cdot \frac{1}{2\pi} \cdot A_1 \cdot \arccos a \cdot \rho^{2n-1}$$

$$\geq \frac{1}{4\pi} \cdot C_1 \cdot A_1 \cdot \arccos a \cdot (mr^2)^{-(2n-1)/2} \cdot (mp^2)^{2n-1} \cdot \sigma_n(F_\delta - P)$$

$$= \varphi(a) \cdot \log(4C_3/(\delta\varepsilon))^{-(2n-1)/2} \cdot \sigma_n(F_\delta - P).$$

This proves part (d) of Lemma 1. Finally, if $|z| \leq R$ then $\Re(1 - \left< z, z' \right>) \leq 1 - R$ for $j = 1, 2, \ldots, N(r)$. Because of the inequalities $mr^2 \geq 1$, (10) and (6), we have

$$|u(z)| \leq N(r) \cdot e^{-m(1-R)} \leq C_2 \cdot \frac{1}{r^{2n-1}} \cdot e^{-m(1-R)}$$

$$= C_2 \cdot m^{(2n-1)/2} \cdot e^{-m(1-R)} \cdot (mr^2)^{-(2n-1)/2}$$

$$\leq C_2 \cdot m^{(2n-1)/2} \cdot e^{-m(1-R)} \leq \varepsilon.$$

This proves part (d) of Lemma 1 and ends the proof.

**Lemma 2.** Let $v$ be a continuous map $v: S^n \to \mathbb{C}^m$ such that $\|v\|_\delta < b < 1$ for some $\delta < C_3$. Let $\frac{1}{4} > \varepsilon > 0$, $R < 1$. Then there exists a holomorphic map $u: \mathbb{C}^n \to \mathbb{C}^m$ and a closed set $K \subset F_\delta$ such that:

(a') $\|v + h \cdot u\|_\delta < b + \varepsilon$;

(b') $\|u\|_R \leq \varepsilon$;

(c') $|v(z) + h(z) \cdot u(z)| > b - \varepsilon$;

(d') $\sigma_n(K) \geq \sigma_n(F_\delta) - \varepsilon$;

(e') $|u(z)| \leq \varepsilon$ for $z \in S^n - F_\delta$;

(f') $u_j \equiv 0$ for $j = 1, 2, \ldots, m - 1$, where $u = (u_1, u_2, \ldots, u_m)$.

**Proof.** Let $a = b - \frac{1}{2}\varepsilon$ and choose a sequence $\{\varepsilon_j\}$ satisfying the assumptions of Lemma 1 and such that $6\Sigma_{j=1}^{2} \varepsilon_j < \varepsilon$. We can assume $\varepsilon_j = A \cdot \exp\{-(\tau \cdot j)^{2/(2n-1)}\}$, $A = 2C_3/\delta$ and $\tau$ is some large number.
Apply Lemma 1 to the data $a, \epsilon, R, \nu, P = 0$ to produce a holomorphic map $u_\lambda: \mathbb{C}^n \to \mathbb{C}^m$ and a closed set $K_1 \subset F_\delta$ such that:

(a) $\|v + h \cdot u_1\|_{\delta} \leq b + 3\epsilon_1$;
(b) $\|v_1\|_{R} \leq \epsilon_1$;
(c) $|v(z) + h(z) \cdot u_1(z)| \geq a - 3\epsilon_1$ for $z \in K_1$;
(d) $\alpha_1 = \sigma_n(K_1) \geq \varphi(a) \cdot \left(\log\left(A/\epsilon_1\right)\right)^{(2n-1)/2} \cdot \sigma_n(F_\delta)$;
(e) $|u_1(z)| \leq \epsilon_1$ for $z \in S^n - F_\delta$;
(f) $u_j \equiv 0$ for $j = 1, 2, \ldots, m - 1$, where $u_1 = (u_1^1, u_2^1, \ldots, u_m^1)$.

Suppose that holomorphic maps $u_1, u_2, \ldots, u_{p-1}$ ($u_j: \mathbb{C}^n \to \mathbb{C}^m$ for $j = 1, 2, \ldots, p - 1$) have been chosen together with closed sets $K_1, K_2, \ldots, K_{p-1}$ such that if $W_i = \bigcup_{j=1}^{i} K_j$ then $K_{i+1} \cap W_i = \emptyset$ and $\sigma_n(K_i) = \alpha_i, K_i \subset F_\delta$. A map $u_p: \mathbb{C}^n \to \mathbb{C}^m$ and a closed set $K_p$ is then obtained by applying Lemma 1 to the data $a - 3\sum_{i=1}^{p-1} \epsilon_i, \epsilon_p, R, v + h(z) \cdot (u_1 + u_2 + \cdots + u_{p-1}), W_{p-1}$. This produces a sequence $\{v_k\}$ of holomorphic maps $v_k: \mathbb{C}^n \to \mathbb{C}^m$ for $k = 1, 2, \ldots$ and a sequence $\{K_k\}$ of disjoint closed sets such that $K_k \subset F_\delta, \sigma_n(K_k) = \alpha_k$ and:

(a) $\|v + h \cdot \sum_{k=1}^{p} u_k\|_{\delta} \leq b + 3 \cdot \sum_{k=1}^{p} \epsilon_k < b + \epsilon$;
(b) $\left\|\sum_{k=1}^{p} u_k\right\|_{R} \leq \sum_{k=1}^{p} \|u_k\|_{R} \leq \sum_{k=1}^{p} \epsilon_k < \epsilon$;
(c) $|v(z) + h(z) \cdot \sum_{k=1}^{p} u_k(z)| \geq a - 3 \cdot \sum_{k=1}^{p} \epsilon_k$

$\geq a - \frac{1}{2} \epsilon = b - \epsilon$ for $z \in W_p$;
(d) $\alpha_p = \sigma_n(K_p)$

$\geq \varphi\left(a - 3 \cdot \sum_{k=1}^{p-1} \epsilon_i\right) \cdot \left(\log\left(A/\epsilon_p\right)\right)^{(2n-1)/2} \cdot \left(\sigma_n(F_\delta) - \sum_{k=1}^{p-1} \alpha_k\right)$

$\geq \varphi(a) \cdot \left(\log\left(A/\epsilon_p\right)\right)^{(2n-1)/2} \cdot \left(\sigma_n(F_\delta) - \sum_{k=1}^{p-1} \alpha_k\right)$;
(e) $\left|\sum_{k=1}^{p} u_k(z)\right| \leq \sum_{k=1}^{p} |u_k(z)| \leq \sum_{k=1}^{p} \epsilon_k < \epsilon$ for $z \in S^n - F_\delta$;
(f) $u_j^k \equiv 0$ for $k = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m - 1$,

where $u_k = (u_1^k, u_2^k, \ldots, u_m^k)$. 
If $\Sigma_{k=1}^{\infty} a_k < \sigma_n(F_\delta)$, (d) shows that there is a constant $C_4$ such that for every positive integer $k$,

$$\alpha_p \geq C_4 \cdot \left[\log \frac{A}{\epsilon_p}\right]^{-(2n-1)/2} = \left[C_4 \cdot (\tau p)^{2/(2n-1)}\right]^{-(2n-1)/2} = \frac{C_4}{\tau p}. $$

This is impossible, because then $\Sigma_{p=1}^{\infty} \alpha_p = \infty$ and $\alpha_p$ are the measures of the disjoint sets. Hence, we may assume that $\Sigma_{k=1}^{\infty} a_k = \sigma_n(F_\delta)$. It follows that for $p$ sufficiently large and $P = W_p$ we have $\sigma_n(P) = \Sigma_{k=1}^{p} \alpha_k > 1 - \epsilon$, which is part (d)' of Lemma 2. Letting $h = \Sigma_{k=1}^{p} u_k$, parts (a)', (b)', (c)', (e)', (f)' are just (a)$_p$, (b)$_p$, (c)$_p$, (e)$_p$, (f)$_p$. So we have proved the assertion of Lemma 2.

Assume now that $g$ and $h$ satisfy the assumptions of the Theorem. Then $\|g\|_\delta \leq 1 - \delta$. To prove the Theorem, take a sequence $\delta_1, \delta_2, \ldots$ of positive numbers such that $\delta_1 < C_3$ and $\delta_{i+1} < \delta_i/2$ and let $a_1 = b_1 = 1 - \frac{1}{2}\delta_1$, $\epsilon_1 = \min(\frac{1}{10}, \frac{1}{4}\delta_1)$, $R_1 = \frac{1}{2}$. Apply Lemma 2 to the data $g_1 = g$, $b_1$, $\delta_1$, $R_1$ to get a map $u_1$ and a set $K_1 \subset F_\delta$ such that, for $p = 1$ and $g_1 = g$:

(i)$_p$ $\|g_p + h \cdot u_p\|_{\delta_p} < b_p + \epsilon_p < 1$;
(ii)$_p$ $\|u_p\|_{R_p} \leq \epsilon_p$;
(iii)$_p$ $|g_p(z) + h(z) \cdot u_p(z)| > b_p - \epsilon_p$ for $z \in K_p$;
(iv)$_p$ $\sigma_n(K_p) \geq \sigma_n(F_\delta) - \epsilon_p$;
(v)$_p$ $1 - |g_p(z) + h(z) \cdot u_p(z)|$

$$\geq \left(1 - \sum_{i=1}^{p} \epsilon_i\right)|h(z)| \quad \text{for } z \in S^n - F_\delta;$$
(vi)$_p$ $u_j^p \equiv 0$ for $j = 1, 2, \ldots, m - 1$ where $u_p = (u_1^p, u_2^p, \ldots, u_m^p)$.

Inequality (v) follows from (e)' of Lemma 2, because for $z \in S^n - F_\delta$, we have $|u_1(z)| < \epsilon_1$, so

$$1 - |v(z) + h(z) \cdot u_1(z)| \geq 1 - |v(z)| - |u_1(z) \cdot h(z)|$$

$$\geq |h(z)| - \epsilon_1 \cdot |h(z)| = (1 - \epsilon_1) \cdot |h(z)|.$$ 

Since $g_1 + h \cdot u_1$ is a continuous map on $\overline{B}^n$, there exists an $R_2$ such that $\frac{1}{2} + \frac{1}{2}R_1 < R_2 < 1$ and, for $p = 1$,

(vii)$_p$ $|g_p(R_{p+1} \cdot z) + h(R_{p+1} \cdot z) \cdot u_p(R_{p+1} \cdot z)| > b_p - 2\epsilon_p$

for $z \in K_p$. 
Suppose we have inductively found holomorphic maps $u_1, u_2, \ldots, u_p$, closed sets $K_1, K_2, \ldots, K_p$, real numbers $R_1, R_2, \ldots, R_{p+1}$, $b_1, b_2, \ldots, b_p$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ such that $\frac{1}{2} + \frac{1}{2} R_i < R_{i+1}$, $\varepsilon_i > 0$ for $i = 1, 2, \ldots, p$ and $\sum_{i=1}^p \varepsilon_i < 1/8$. Let us assume $g_{j+1} = g + h \cdot \sum_{i=1}^j u_i$ and conditions (i) $\ldots$ (vii) are satisfied for $j = 1, 2, \ldots, p$. We also assume that $1 - 1/j \leq b_j + \varepsilon_j < 1$. If $z \in (F_{\delta_{p+1}} - F_{\delta_p})$ then according to (v) $p$, we have

$$1 - |g_{p+1}(z)| \geq \left(1 - \sum_{i=1}^p \varepsilon_i\right) \cdot |h(z)| \geq \frac{1}{2} \cdot \delta_{p+1},$$

since $|h(z)| \geq \delta_{p+1}$. This, together with (i) $p$, shows that $\|g_{p+1}\|_{\delta_{p+1}} < 1$. Take any $b_{p+1} > 1 - 1/(p + 1)$ and $\varepsilon_{p+1}$ satisfying the inequalities $1 > b_{p+1} + \varepsilon_{p+1} > b_{p+1} > \|g_{p+1}\|_{\delta_{p+1}}$ and $\sum_{i=1}^{p+1} \varepsilon_i < 1/8$. Since the map $g_{p+1}$ is continuous on $B^n$, we can find a number $R_{p+2}$ such that $\frac{1}{2} + \frac{1}{2} R_{p+1} < R_{p+2} < 1$ and such that condition (vii) $p+1$ is satisfied. Now we can apply Lemma 2 to the data $g_{p+1}, b_{p+1}, \varepsilon_{p+1}, R_{p+1}$. We get some map $u_{p+1}$ and a set $K_{p+1}$. It follows from Lemma 2 that conditions (i) $p+1$ $\ldots$ (iv) $p+1$ and (vi) $p+1$ are satisfied. For $z \in S^n - F_{\delta_{p+1}}$, by the virtue of (e)' and (v) $p$, we have

$$1 - |g_{p+1}(z) + h(z) \cdot u_{p+1}(z)|$$

$$\geq 1 - |g_{p}(z) + h(z) \cdot u_{p}(z)| - |h(z) \cdot u_{p+1}(z)|$$

$$\geq \left(1 - \sum_{i=1}^p \varepsilon_i\right) \cdot |h(z)| - |h(z)| \cdot \varepsilon_{p+1}$$

$$= \left(1 - \sum_{i=1}^{p+1} \varepsilon_i\right) \cdot |h(z)|.$$

So we have also proved that condition (v) $p+1$ is satisfied. Conditions (ii) $p$ $(p = 1, 2, 3 \ldots)$ and the definition of $g_{p}$ say that the sequence $\{g_{p}\}$ is convergent uniformly on every ball $R_p \cdot B^n$, and since $\lim_{p \to 1} R_p = 1$, this sequence is pointwise convergent to some holomorphic map $f$ on the ball $B^n$. From conditions (i) $p$ and (v) $p$ it follows that each map $g_{p}$ is bounded by 1 on $B^n$. So, also $\|f\|_{\infty} \leq 1$. For $\delta > 0$ let $L_p = F_{\delta} \cap \bigcap_{j>p} K_j$. Then, for $q$ large enough, $F_{\delta} \subset F_{\delta_p}$ for $p > q$. We have

$$\sigma_n(F_{\delta}) - \sigma_n(L_q) = \sigma_n\left(\bigcup_{j>q} (F_{\delta} - (F_{\delta} \cap K_j))\right)$$

$$\leq \sum_{j>q} \sigma_n(F_{\delta} - (F_{\delta} \cap K_j)) \leq \sum_{j>q} \sigma_n(F_{\delta_j} - K_j) < \sum_{j>q} \varepsilon_j.$$
Hence \( \lim_{q \to \infty} \sigma_n(L_q) = \sigma_n(F_\delta) \). It is obvious from \((iii)_p\) and the equality \( \lim_{p \to \infty} b_p = 1 \) that \( \lim_{R \to 1} f(Rz) = 1 \) for \( z \in L_q \), provided this limit exists. Since \( \delta \) was arbitrary, this proves that the map \( f \) is inner, since \( \sigma_n(\cap_p (S^n - F_{\delta_p})) = 0 \). Now it is easy to check that \( f \) satisfies the Theorem.

**COROLLARY 3.** Let \( m < n \) and let \( g \in A_m(B^m), \|g\|_\infty \leq 1 \). There exists an inner map \( f: B^n \to B^m \) such that

\[
 f(z_1, z_2, \ldots, z_m, 0, 0, \ldots, 0) = g(z_1, z_2, \ldots, z_m).
\]

**Proof.** Let \( \Phi: B^m \to B^m \) be an automorphism of \( B^m \) such that \( \Phi(g(0, \ldots, 0)) = (0, \ldots, 0) \). Take \( \tilde{g}: B^m \to B^m, \tilde{g}(z) = \Phi(g(z_1, z_2, \ldots, z_m)), \) \( h(z) = \frac{1}{2} \cdot z_n^2 \). By virtue of Schwartz’s lemma,

\[
 |\tilde{g}(z)| \leq \left( |z_1|^2 + |z_2|^2 + \cdots + |z_m|^2 \right)^{1/2}.
\]

So we have

\[
 |\tilde{g}(z)| + |h(z)| \leq \left( 1 - |z_n|^2 \right)^{1/2} + \frac{1}{2} \cdot |z_n|^2 \leq 1.
\]

We can apply the Theorem for \( g \) and \( h \) to get an inner map \( \tilde{f} \). The inner map \( f = \Phi^{-1}(\tilde{f}) \) will satisfy Corollary 3.

**COROLLARY 4.** There exists an inner function \( f: B^n \to D \) such that

\[
 \frac{\partial f}{\partial z_1}(0, 0, \ldots, 0) = 1.
\]

**Proof.** Take \( m = 1 \) in Corollary 3 and a function \( g: B^1 \to D, g(z) = z \).

**Remark.** The assumption \( g \in A_m(B^m) \) in Corollary 3 is not necessary: we can take any holomorphic map \( g: B^m \to B^m \). Then the map \( \tilde{g} \), defined as before, can be prolonged to a continuous map on \( \bar{B}^n - A \), where \( A \subset S^n \) and \( \sigma_n(A) = 0 \). One can check that the Theorem is still valid for such maps.
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