THE FEFFERMAN-STEIN DECOMPOSITION OF SMOOTH FUNCTIONS AND ITS APPLICATION TO $H^p(\mathbb{R}^n)$

AKIHITO UCHIYAMA
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We show the "Fefferman-Stein decomposition" of smooth bump functions. As an application of this we get one result about the singular integral characterization of $H^p(\mathbb{R}^n)$. Our method does not use sub-harmonicity.

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1. Introduction. In this paper functions considered are complex-valued unless otherwise explicitly stated. Cubes considered have sides parallel to the coordinate axes. For a function $f(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$, let

$$\|f\|_{\text{BMO}} = \sup_I \int_I |f(x) - f_I| \, dx / |I|,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$, $|I|$ denotes the Lebesgue measure of $I$ and

$$f_I = \int_I f(x) \, dx / |I|.$$ 

A function $f(x)$ is said to belong to $\text{BMO}(\mathbb{R}^n)$ if $\|f\|_{\text{BMO}} < +\infty$.

Let $\theta_1(\xi), \ldots, \theta_m(\xi) \in C^\infty(S_{n-1})$, where

$$S_{n-1} = \{ \xi \in \mathbb{R}^n : |\xi| = 1 \}$$

and

$$|\xi| = |(\xi_1, \ldots, \xi_n)| = \left( \sum_{j=1}^n \xi_j^2 \right)^{1/2}.$$
For $h \in L^2(\mathbb{R}^n)$ let

$$K_j h = \left( \theta_j(\xi/|\xi|) \hat{h}(\xi) \right) \ast h, \quad j = 1, \ldots, m,$$

where $\hat{}$ and $\ast$ are the Fourier and inverse Fourier transforms. As is well known [see Stein [29] p. 75], there exist $\alpha_j \in \mathbb{C}$ and $\Omega_j(x) \in C^\infty(S_{n-1})$ such that

$$\int_{|x|=1} \Omega_j(x) = 0$$

and

$$K_j h(x) = \alpha_j h(x) + \text{P.V.} \int \Omega_j \left( \frac{x - y}{|x - y|} \right) |x - y|^{-n} h(y) \, dy$$

for any $h \in L^2(\mathbb{R}^n)$. For $g \in L^\infty(\mathbb{R}^n)$ let

$$K_j g(x) = \alpha_j g(x) + \text{P.V.} \int \Omega_j \left( \frac{x - y}{|x - y|} \right) |x - y|^{-n} - \Omega_j \left( \frac{-y}{|y|} \right) |y|^{-n} \chi_{\{ |y| > 1 \}} g(y) \, dy$$

where $\chi_E$ denotes the characteristic function of a set $E \subset \mathbb{R}^n$. In [32], the author showed

**Theorem A.** If

$$\text{rank} \begin{pmatrix} \theta_1(\xi) \cdots \theta_m(\xi) \\ \theta_1(-\xi) \cdots \theta_m(-\xi) \end{pmatrix} \equiv 2 \quad \text{on } S_{n-1},$$

then for any $f \in \text{BMO}(\mathbb{R}^n)$ there exist $g_1, \ldots, g_m \in L^\infty(\mathbb{R}^n)$ such that

$$f = \sum_{j=1}^m K_j g_j \quad (\text{modulo constants})$$

and

$$\sum_{j=1}^m \|g_j\|_\infty \leq C_{1,1} \|f\|_{\text{BMO}},$$

where $C_{1,1}$ is a constant depending only on $\theta_1, \ldots, \theta_m$.

**Remark 1.** The case when $K_1, \ldots, K_{n+1}$ are the Riesz transforms and the identity operator is the case considered by C. Fefferman [13] and C. Fefferman-Stein [14].
REMARK 2. In [32] we assumed that \( f \) has compact support. But this restriction can be removed.

Consequently, if (1.1) is satisfied, then the singular integral operators \( K_1, \ldots, K_m \) characterize \( H^1(\mathbb{R}^n) \). In this paper, we continue this research.

In the following, \( f(x) = (f_1(x), \ldots, f_m(x)) \) and \( g(x) \) denote \( C^m \)-valued functions. We use the following notations:

\[
|f(x)| = \left( \sum_{j=1}^{m} |f_j(x)|^2 \right)^{1/2},
\]

\[
K h(x) = (K_1 h(x), \ldots, K_m h(x)),
\]

\[
K \cdot f(x) = \sum_{j=1}^{m} K_j f_j(x),
\]

\[
K^* \cdot f(x) = \sum_{j=1}^{m} K_j^* f_j(x),
\]

where \( K_j^* h(x) = (\tilde{\theta}_j(\xi/|\xi|) \hat{h}(\xi))(x) \). \( I(x, t) \) denotes a cube in \( \mathbb{R}^n \) with center \( x \) and side length \( t \).

DEFINITION 1.1. Let

\[
S = \{ f \in L^2(\mathbb{R}^n, C^m) : K^* \cdot f(x) \equiv 0 \},
\]

where \( L^2(\mathbb{R}^n, C^m) \) denotes the set of \( C^m \)-valued functions \( f(x) \) with \( f_1, \ldots, f_m \in L^2(\mathbb{R}^n) \).

DEFINITION 1.A. [Coifman-Rochberg [9].] For a real-valued function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), let

\[
\| f \|_{\text{BLO}} = \sup_I \int_I f(x) - \inf_{y \in I} f(y) \, dx / |I|,
\]

where \( I \) is taken over all cubes in \( \mathbb{R}^n \). A function \( f(x) \) is said to belong to \( \text{BLO}(\mathbb{R}^n) \) if \( \| f \|_{\text{BLO}} < +\infty \). [Note that \( \| \cdot \|_{\text{BLO}} \) is not a norm.]

Our main result is the following.

THEOREM 1. Suppose that (1.1) holds. Let \( f \in C^1(\mathbb{R}^n, C^m) \),

\[
|f(x)| \leq (1 + |x|)^{-n-1},
\]
and

\[(1.3) \quad \left| \frac{\partial}{\partial x_j} f(x) \right| \leq (1 + |x|)^{-n-2}, \quad j = 1, 2, \ldots, n.\]

Let \( w(x) \) be a nonnegative function defined on \( \mathbb{R}^n \) such that

\[(1.4) \quad \| -\log w \|_{\text{BLO}} \leq c_0.\]

Then there exists \( g \in L^2(\mathbb{R}^n, \mathcal{C}^m) \) such that

\[(1.5) \quad f - g \in S\]

and that

\[(1.6) \quad |g(x)| \leq C_{1,2} w(x) \left( \int_{f(0,1)} w(y) \, dy \right)^{-1} (1 + |x|)^{-n-1/2},\]

where \( c_0 \) and \( C_{1,2} \) are positive constants depending only on \( \theta_1, \ldots, \theta_m \).

**Remark 3.** If \( f(x) \) is \( \mathbb{R}^m \)-valued and if \( \theta_j(\xi) \equiv \overline{\theta}_j(\xi) \) for \( j = 1, \ldots, m \), then we can take \( g(x) \) to be \( \mathbb{R}^m \)-valued.

**Remark 4.** If we apply Theorem 1 to the case when \( K_1 = \) the identity operator and \( f(x) = (f(x), 0, \ldots, 0) \), then (1.5) implies

\[f(x) = g_1(x) + \sum_{j=2}^{m} K_j^* g_j(x).\]

This is the reason why we call Theorem 1 the Fefferman-Stein decomposition of smooth bump functions. The point is the fact that we can dominate \( g_1, \ldots, g_m \) pointwise by a “function” on the right-hand side of (1.6).

The idea of this theorem comes from P. W. Jones’s recent work “\( L^\infty \) estimate for the \( \overline{\partial} \) problem in a half-plane” [25]. We explain the relation between Theorem 1 and Jones’s result in §3.

The proof of Theorem 1 is given in §5. The Main Lemma in §4 is crucial and is itself a partial result related to the Fefferman-Stein decomposition of certain weighted BMO spaces in terms of singular integral operators \( K_1, \ldots, K_m \). The Main Lemma is proved in §§6–9. Its proof is a refinement of the argument in [32].

As a corollary to Theorem 1, we get one result about the singular integral characterization of \( H^p(\mathbb{R}^n) \). Let \( \psi \in \mathcal{D}(\mathbb{R}^n) \) be a fixed real-valued function satisfying \( \int \psi(x) \, dx = 1 \). For \( h \in \mathcal{S}'(\mathbb{R}^n) \), let

\[h^+(x) = \sup_{t>0} |(h * \psi_t)(x)|,\]
where $\psi_i(x) = t^{-n}\psi(x/t)$. For $+\infty > p > 0$, let
\[
\|h\|_{H^p} = \|h^+\|_{L^p}.
\]
For $h = (h_1, \ldots, h_m) \in \mathcal{S}'(\mathbb{R}^n) \oplus \cdots \oplus \mathcal{S}'(\mathbb{R}^n)$, let
\[
h^+(x) = \sup_{t > 0} |(h*\psi_i)(x)| = \sup_{t > 0} \left| ((h_1*\psi_i)(x), \ldots, (h_m*\psi_i)(x)) \right|.
\]
It is known that $\|\cdot\|_{H^p}$ is essentially independent of the choice of $\psi$. [See C. Fefferman-Stein [14].]

**Definition 1.** For $q > 0$ and for a measurable function $f(x)$ let
\[
M_q f(x) = \sup_{I \ni x} \left( \int_I |f(y)|^q \, dy / |I| \right)^{1/q},
\]
where $I$ is taken over all cubes containing $x$.

**Theorem 2.** If (1.1) holds, then there exist $p_0 \in (0, 1)$ and $C_{13} \in \mathbb{R}$, depending only on $\theta_1, \ldots, \theta_m$, such that
\[
(Kh)^+(x) \leq C_{13} M_{p_0} \left( M_{1/2}(\|Kh\|) \right)(x)
\]
for any $x \in \mathbb{R}^n$ and any $h \in L^2(\mathbb{R}^n)$.

**Remark 5.** For $h \in L^2(\mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n, C^m)$, let
\[
h^{++}(x) = \sup_{t > 0, \ |x-z| < t} |(h*P_t)(z)|,
\]
where $P_t(x)$ is the Poisson kernel, that is,
\[
P_t(x) = c_n t^2 \left( |x|^2 + t^2 \right)^{(n+1)/2}, \quad c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}.
\]
Then in the above inequality, we can replace $(Kh)^+(x)$ by $(Kh)^{++}(x)$.

**Corollary 1.** If (1.1) holds and if $\max(1/2, p_0) < p \leq 1$, then
\[
c_{1,4} \|h\|_{H^p} \leq \sum_{j=1}^m \|K_jh\|_{L^p} \leq c_{1,5} \|h\|_{H^p}
\]
for any $h \in L^2(\mathbb{R}^n)$ and

\begin{equation}
(1.8) \quad c_{1.4} \|h\|_{H^p} \leq \sum_{j=1}^{m} \left( \int_{\mathbb{R}^n} \lim_{t \to +0} K_j(h \ast P_t)(x) \left| P_t(x) \right|^p \, dx \right)^{1/p} \leq c_{1.5} \|h\|_{H^p}
\end{equation}

for any $h \in H^p(\mathbb{R}^n)$, where $c_{1.4}$ and $c_{1.5}$ are positive constants depending only on $\theta_1, \ldots, \theta_m$ and $p$.

**Remark 6.** For $h \in H^p(\mathbb{R}^n)$, $p < 1$, we define $h \ast P_t$ by $(\hat{h}(\xi) \hat{P}_t(\xi)) \hat{P}_t(\xi)$, which is known to belong to $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. It is also known that for any $h \in H^p(\mathbb{R}^n)$, $\lim_{t \to +0} K_j(h \ast P_t)(x)$ exists almost everywhere. [See Stein [29] p. 201.]

**Remark 7.** Inequality (1.7) with $p = 1$ holds for any $h \in \mathcal{S}'(\mathbb{R}^n)$, whose Fourier transform is an integrable function on some neighborhood of the origin, if we define $K_j h = (\theta_j \hat{h})$ in the sense of distributions and if we define

$$
\|K_j h\|_{L^1} = +\infty
$$

for the distribution $K_j h$ that does not belong to $L^1(\mathbb{R}^n)$. [In Corollary 1 of [32], we showed the above. But the statement in [32] was somewhat ambiguous.]

As another application of Theorem 2, we get the following extension of the results of Csereteli, Gundy and Varopoulos. [See [12], [18] and [34].]

**Corollary 2.** Let

\begin{equation}
(1.9) \quad \sum_{j=1}^{m} \left| \theta_j(\xi) - \theta_j(-\xi) \right| \neq 0 \quad \text{for any } \xi \in S_{n-1}.
\end{equation}

Let $h$ be a finite complex measure on $\mathbb{R}^n$ and let $dh = f \, dx + ds$, where $f \in L^1(\mathbb{R}^n)$ and $s$ is singular. Then

$$
\liminf_{\lambda \to +\infty} \lambda \left\{ \sum_{j=1}^{m} \left| \lim_{t \to +0} K_j(h \ast P_t)(x) \right| > \lambda \right\} \geq C_{1.6} \|s\|_M,
$$

where $C_{1.6}$ is a positive constant depending only on $\theta_1, \ldots, \theta_m$ and where $\|s\|_M$ is the total variation of $s$ on $\mathbb{R}^n$. 
Remark 8. It is known that for any finite measure \( h \)
\[
\lim_{t \to +0} K_j \left( h \ast P_t \right)(x)
\]
exists almost everywhere.

Proofs of Theorem 2 and corollaries are given in §2.

Notation. A dyadic cube is a cube of the form
\[
\prod_{j=1}^{n} \left[ k_j 2^{-k}, (k_j + 1)2^{-k} \right],
\]
where \( k_1, \ldots, k_n \) and \( k \) are integers. For a cube \( I \), \( x_I \), \( l(I) \) and \( Q(I) \) denote the center of \( I \), the side length of \( I \) and
\[
\left\{ (x, t) \in \mathbb{R}^{n+1} : x \in I, t \in (0, l(I)) \right\},
\]
respectively. For \( \alpha > 0 \), \( \alpha I \) denotes a cube concentric with \( I \) and with \( l(\alpha I) = \alpha l(I) \). \( \Sigma_{2m-1} \) denotes \( \{ \nu = (v_1, \ldots, v_m) \in \mathbb{C}^m : \Sigma_{j=1}^{m} |v_j|^2 = 1 \} \). \( |\nu| \) denotes \( (\Sigma_{j=1}^{m} |v_j|^2)^{1/2} \). For \( \nu \in \mathbb{C}^m \setminus \{0\} \), \( U(\nu) \) denotes \( \nu/|\nu| \). [For the sake of convenience, let \( U(0) = (1, 0, \ldots, 0) \).] For \( \nu \) and \( \mu \in \mathbb{C}^m \), \( \langle \nu, \mu \rangle \) denotes \( \Sigma_{j=1}^{m} (\text{Re } v_j \text{ Re } \mu_j + \text{Im } v_j \text{ Im } \mu_j) \), i.e., the inner product in \( \mathbb{R}^{2m} \). For \( \theta \in C^\infty(S_{n-1}) \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \), \( \theta(\xi) \) denotes \( \theta(\xi/|\xi|) \). The letter \( C \) denotes various positive constants depending only on \( \theta_1, \ldots, \theta_m \).

Acknowledgements. The author would like to express his deep gratitude to Professor P. W. Jones and the Department of Mathematics at the University of Chicago for their kindness during the academic year 1981–1982.

2. Proofs of Theorem 2 and Corollaries.

Lemma 2.A. [See Coifman-Rochberg [9].] If \( h(x) \equiv 0 \) and if \( M_1 h(x) \not\equiv +\infty \), then
\[
\| \log M_1 h \|_{\text{BLO}} \leq C_{2.1}.
\]

Proof of Theorem 2. By dilation and translation the proof of Theorem 2 can be reduced to the inequality
\[
(2.1) \quad \left| \int K h(x) \psi(x) \, dx \right| \leq C_{1,3} M_{p_0} \left( M_{1/2}(|K h|) \right)(0).
\]
Put $\epsilon = c_0/2C_2$. Take any $v \in \Sigma_{2m-1}$. Applying Theorem 1 to $f(x) = \psi(x)v$ and $w(x) = M_{1/2}(|Kh|(x))^{-\epsilon}$, we get $g(x)$ such that
\[ K^* \cdot (\psi v - g) \equiv 0 \]
and such that
\[
|g(x)| \leq CM_{1/2}(|Kh|(x))^{-\epsilon} \left( \int_{I(0,1)} M_{1/2}(|Kh|(y))^{-\epsilon} dy \right)^{-1} \\
\times (1 + |x|)^{-n-1/2} \\
\leq CM_{1/2}(|Kh|(x))^{-\epsilon} M_{1/2}(|Kh|(0))^{-\epsilon}(1 + |x|)^{-n-1/2}.
\]
Thus
\[
\left| \int Kh(x) \psi(x) \, dx \cdot v \right| = \left| \int Kh(x) \cdot g(x) \, dx \right| \leq |Kh(x)||g(x)| \, dx \\
\leq CM_{1/2}(|Kh|(0))^{-\epsilon} \int M_{1/2}(|Kh|(x))^{-\epsilon}(1 + |x|)^{-n-1/2} \, dx \\
\leq CM_{1-\epsilon}(M_{1/2}(|Kh|))(0).
\]
[In the first and the second formulae of the last string of inequalities, $\cdot$ denotes the inner product in $C^m$.] This concludes the proof of (2.1). Remark 5 follows from the same argument. \hfill \Box

Proof of Corollary 1. Let $h \in L^2$ and $\max(1/2, p_0) < p < 1$. From Theorem 2 and the Hardy-Littlewood maximal theorem, it follows that
\[
(2.2) \quad c_p \|Kh\|_{H^p} \leq \|Kh\|_{L^p} \leq \|Kh\|_{H^p},
\]
where
\[
\|Kh\|_{H^p} = \|(Kh)^+\|_{L^p}.
\]
From the boundedness of singular integral operators on $H^p$, it follows that
\[
(2.3) \quad \|Kh\|_{H^p} \leq c_p \|h\|_{H^p}.
\]
On the other hand, by (1.1) there exist multipliers homogeneous of degree zero
\[
\Theta_1(\xi), \ldots, \Theta_m(\xi) \in C^\infty(S_{n-1})
\]
such that
\[
\sum_{j=1}^m \theta_j(\xi) \Theta_j(\xi) \equiv 1 \quad \text{on } S_{n-1}.
\]
So
\[
(2.4) \quad \|h\|_{H^p} = \left\| \sum \Theta_j(\xi) (K_j h)(\xi) \right\|_{H^p} \leq c_p \sum \|K_j h\|_{H^p} \leq c_p \|K h\|_{H^p}.
\]

Thus, from (2.2)–(2.4), we get (1.7).

Let \( h \in H^p \). Applying (1.7) to \( h \ast P_t \), we get
\[
(2.5) \quad c_{1.4} \|h \ast P_t\|_{H^p} \leq \sum_{j=1}^m \|K_j (h \ast P_t)\|_{L^p} \leq c_{1.5} \|h \ast P_t\|_{H^p}.
\]

It is known that
\[
h \ast P_t \rightarrow h \quad \text{in } H^p \quad \text{as } t \rightarrow +0
\]
and that
\[
\sup_{t>0} \left| K_j (h \ast P_t)(x) \right| \in L^p.
\]

Thus by the Lebesgue dominated convergence theorem, we get
\[
\left\| K_j (h \ast P_t) \right\|_{L^p} \rightarrow \left( \int \left( \lim_{t \rightarrow +0} K_j (h \ast P_t)(x) \right)^p dx \right)^{1/p} \quad \text{as } t \rightarrow +0.
\]

Therefore, letting \( t \rightarrow +0 \) in (2.5), we get (1.8).

\[\square\]

**Lemma 2.1.** Let \( u(x, t) \) be a nonnegative function defined on \( \mathbb{R}^n \times [0, +\infty) \) and continuous on \( \mathbb{R}^n \times (0, +\infty) \). Let \( q > 1 \). If
\[
(2.6) \quad u(x, 0) = \lim_{t \rightarrow +0} u(x, t) \quad \text{a.e. } x
\]
and if
\[
(2.7) \quad \left| \left\{ x \in \mathbb{R}^n : \sup_{t \geq 0} u(x, t) > \lambda \right\} \right| \leq \lambda^{-q}
\]
for any \( \lambda > 0 \), then
\[
(2.8) \quad \lim_{t \rightarrow +0} M_1(u(\cdot, t))(x) = M_1(u(\cdot, 0))(x) \quad \text{a.e. } x.
\]

**Proof.** Take any \( \varepsilon > 0 \). By (2.6) and (2.7) there exists \( t_0 > 0 \) such that \( |G| < \varepsilon \), where
\[
G = \left\{ x \in \mathbb{R}^n : \sup_{t \in [0, t_0]} |u(x, t) - u(x, 0)| > \varepsilon \right\}.
\]

Since
\[
\int_G \sup_{t \geq 0} u(x, t) \, dx < C \varepsilon^{1-1/q}
\]
by (2.7), there exists a measurable set $E$ such that

$$|E| < C e^{(1-1/q)/2},$$

$$|M_1(u(\cdot, t))(x) - M_1(u(\cdot, 0))(x)| < C e^{(1-1/q)/2} + \varepsilon$$

for any $x \in E^c$ and any $t \in [0, t_0]$. Since $\varepsilon > 0$ is arbitrary, we get (2.8).

Proof of Corollary 2. Put $\theta_0 \equiv 1$ and $K_0 = \text{the identity operator. By}

the usual argument about maximal singular integral operators and the

Hardy-Littlewood maximal theorem, we get

$$\lambda \left\{ x \in \mathbb{R}^n : \sum_{j=0}^{m} \sup_{t > 0} |K_j(h \ast P_t)(x)| > \lambda \right\} \leq C \|h\|_M$$

for any $\lambda > 0$ and

(2.9) $\limsup_{\lambda \to +\infty} \lambda \left\{ x \in \mathbb{R}^n : \sum_{j=0}^{m} \sup_{t > 0} |K_j(h \ast P_t)(x)| > \lambda \right\} \leq C \|s\|_M$.

It is also known that

$$\kappa_j(x) = \lim_{t \to +0} K_j(h \ast P_t)(x)$$

exists almost everywhere and that $\kappa_0(x) = f(x)$ a.e. By (2.9)

(2.10) $\limsup_{\lambda \to +\infty} \lambda \left\{ x \in \mathbb{R}^n : \frac{1}{2} \left( \sum_{j=0}^{m} |\kappa_j(x)| \right) > \lambda \right\} \leq C \|s\|_M$.

Applying Lemma 2.1 to

$$u(x, t) = \begin{cases} \left( \sum_{j=0}^{m} |K_j(h \ast P_t)(x)| \right)^{1/2} & \text{if } t > 0, \\ \left( \sum_{j=0}^{m} |\kappa_j(x)| \right)^{1/2} & \text{if } t = 0, \end{cases}$$

and $q = 2$, we get

$$M_{1/2} \left( \sum_{j=0}^{m} |K_j(h \ast P_t)| \right)(x) \to M_{1/2} \left( \sum_{j=0}^{m} |\kappa_j| \right)(x) \text{ a.e. } x \text{ as } t \to +0.$$  

Similarly

(2.11) $M_{p_0} \left( M_{1/2} \left( \sum_{j=0}^{m} |K_j(h \ast P_t)| \right) \right)(x) \to M_{p_0} \left( M_{1/2} \left( \sum_{j=0}^{m} |\kappa_j| \right) \right)(x)$  

a.e. $x$ as $t \to +0$. 


Since \( \theta_1, \ldots, \theta_m \) satisfy (1.9), \( \theta_0, \ldots, \theta_m \) satisfy (1.1). By Remark 5,
\[
(h \ast P_t)^{++}(x) = (K_0(h \ast P_t))^{++}(x)
\]
\[
\leq CM_{p_0} \left( M_{1/2} \left( \sum_{j=0}^{m} |K_j(h \ast P_t)| \right) \right)(x)
\]
for any \( t > 0 \). Letting \( t \to +0 \), we get
\[
(2.12) \quad h^{++}(x) \leq CM_{p_0} \left( M_{1/2} \left( \sum |\kappa_j| \right) \right)(x) \quad \text{a.e. } x
\]
form (2.11).

On the other hand, [18] and [34] showed
\[
(2.13) \quad \liminf_{\lambda \to +\infty} \lambda \left| \{x \in \mathbb{R}^n : h^{++}(x) > \lambda \} \right| \geq c\|s\|_M,
\]
where \( c > 0 \) depends only on the dimension. Thus, for a sufficient large \( \lambda \), we have
\[
\|s\|_M \leq C\lambda \left| \left\{ x \in \mathbb{R}^n : M_{p_0} \left( M_{1/2} \left( \sum |\kappa_j| \right) \right)(x) > \lambda \right\} \right|
\]
\[
\leq C\lambda^{1-p_0} \int_{\{M_{1/2} \sum |\kappa_j| > \lambda /2\}} M_{1/2} \left( \sum |\kappa_j| \right)(x)^{p_0} dx
\]
\[
\leq C\lambda^{1-p_0} \left| \left\{ M_{1/2} \left( \sum |\kappa_j| \right)(x) > \lambda /2 \right\} \right|^{1-p_0} \|s\|_M^{p_0}
\]
by (2.10). Therefore
\[
\lambda \left| \left\{ M_{1/2} \left( \sum |\kappa_j| \right)(x) > \lambda \right\} \right| \geq C\|s\|_M \quad \text{as } \lambda \to +\infty.
\]
Repeating the same argument, we get
\[
\lambda \left| \left\{ \sum_{j=0}^{m} |\kappa_j(x)| > \lambda \right\} \right| \geq C\|s\|_M \quad \text{as } \lambda \to +\infty.
\]
Since \( \lambda \{|\kappa_0(x)| > \lambda \} \to 0 \) as \( \lambda \to +\infty \), we get Corollary 2.

3. Jones's formula. In this section, we explain the relation between Theorem 1 and Jones's recent work [25].

**Definition 3.A.** A complex measure on the upper half-plane \( \mathbb{R}^2_+ = \{(x, t) : x \in \mathbb{R}, t > 0\} \) is called a Carleson measure if
\[
\sup_I |\mu|(Q(I))/|I| = \|\mu\|_c < +\infty,
\]
where \( |\mu| \) is the total variation of \( \mu \), \( I \) is taken over all intervals.
Suppose that \( \|\mu\|_c \leq 1 \). It has been shown by Carleson [3] [see also Hörmander [19]] that there exists \( F \in L^\infty(\mathbb{R}) \) such that

\[
\|F\|_{L^\infty} \leq C
\]

and such that

\[
\int_R f(x)F(x) \, dx = \iint_{R^2_+} f(x, t) \, d\mu(x, t)
\]

for any \( f \in L^1(\mathbb{R}) \) with \( \text{supp} \hat{f} \subseteq [0, +\infty) \), where

\[
f(x, t) = \int_R P_t(y)f(x-y) \, dy,
\]

\[
P_t(y) = \frac{t}{(\pi(y^2 + t^2))}.
\]

Recently, Jones [25] gave an explicit formula for the construction of \( F \).

**Definition 3.B.** [Jones [25].] For a measure \( \mu \) on \( \mathbb{R}^2_+ \) let

\[
J(\mu, x, \xi) = \frac{1}{\pi} \frac{\text{Im} \xi}{(x - \xi)(x - \bar{\xi})} \times \exp \left( \iint_{0 < \text{Im} \eta \leq \text{Im} \xi} \frac{-i}{x - \eta} + \frac{i}{\xi - \eta} \, d|\mu|(\eta) \right)
\]

where \( i = (-1)^{1/2} \), and \( \xi \) and \( \eta \) are complex numbers. [We identify \( \eta \) with \( (\text{Re} \eta, \text{Im} \eta) \in \mathbb{R}^2 \).]

**Theorem 3.A.** [Jones [25].] Let \( \|\mu\|_c \leq 1 \). Set

\[
F(x) = \iint_{R^2_+} J(\mu, x, \xi) \, d\mu(\xi).
\]

Then,

\[
\|F\|_{L^\infty} \leq C
\]

and (3.2) holds.

Our Theorem 1 can be regarded as a generalization of the formula (3.3). In Jones’s argument, we can replace the formula (3.3) by Theorem 1. In the following, we sketch it.

Let \( H \) be the Hilbert transform, that is

\[
Hf = (-i(\text{sign} \xi)\hat{f}(\xi))'.
\]
For $t > 0$ set
\[
u_t(x) = \int_{\frac{0 < t}{y \in K}} s^{-1}(1 + |x - y|/s)^{-3/2} d|\mu|(y, s).
\]

LEMMA 3.A. [See [9] and [22].] Let $\|\mu\|_c \leq 1$. Then,
\[
\|\nu_t\|_{BLO} \leq C_{3,1}
\]
and
\[
\int_I \nu_t \, dx \leq C_{3,1}|I|
\]
for any cube $I$ with $l(I) \geq t$.

Set $\epsilon = c_0/C_{3,1}$. Then $e^{-\epsilon \nu_t(x)}$ satisfies (1.4) and
\[
\int_{y-t}^{y+t} e^{-\epsilon \nu_t(x)} \, dx/t \geq C
\]
for any $y \in \mathbb{R}$. So, by applying Theorem 1 and Remark 3 to $K_1 =$ the identity operator and $K_2 = -H$ and by using dilation and translation, for each $(y, t) \in \mathbb{R}_+^2$ we get real-valued functions $g_{1,(y,t)}(x)$ and $g_{2,(y,t)}(x)$ such that
\[
P_t(y - x) - g_{1,(y,t)}(x) - Hg_{2,(y,t)}(x) \equiv 0,
\]
\[
|g_{j,(y,t)}(x)| \leq Ce^{-\epsilon \nu_t(x)} t^{-1}(1 + |y - x|/t)^{-3/2} \quad (j = 1, 2).
\]

Set
\[
F(x) = \iint_{\mathbb{R}_+^2} g_{1,(y,t)}(x) + ig_{2,(y,t)}(x) \, d\mu(y, t).
\]

Then
\[
|F(x)| \leq C \iint_{\mathbb{R}_+^2} e^{-\epsilon \nu_t(x)} t^{-1}(1 + |y - x|/t)^{-3/2} d|\mu|(y, t)
\]
\[
= C \int \exp \left( -\epsilon \int_{0 < s \leq t, v \in K} s^{-1}(1 + |x - v|/s)^{-3/2} d|\mu|(v, s) \right) \cdot t^{-1}(1 + |y - x|/t)^{-3/2} d|\mu|(y, t)
\]
\[
\leq Ce^{-1} \left[ \exp \left( -\epsilon \int_{0 < s \leq t, v \in K} s^{-1}(1 + |x - v|/s)^{-3/2} d|\mu|(v, s) \right) \right]_{t=0}^{t=+\infty}
\]
\[
\leq Ce^{-1}
\]
and
\[
\int f(x)F(x) \, dx = \int \int_{\mathbb{R}^2_+} d\mu(y, t) \left( g_{1,(y,t)}(x) + ig_{2,(y,t)}(x) \right) f(x) \, dx \\
= \int \int_{\mathbb{R}^2_+} d\mu(y, t) \int P_t(y-x) f(x) \, dx = \int \int_{\mathbb{R}^2_+} f(y, t) \, d\mu(y, t)
\]
for any \( f \in L^1(\mathbb{R}) \) with \( \text{supp} \widehat{f} \subseteq [0, +\infty) \).

4. Weighted BMO. In the following, we assume (1.4).

**Definition 4.1.** For a measurable set \( E \) let
\[
m_w(E) = \int_E w(x) \, dx
\]
and
\[
w(E) = \sup_{x \in E} w(x).
\]

**Definition 4.2.** For \( f(x) \in L_{1,0}(\mathbb{R}^n, C^m) \), let
\[
\|f\|_{\text{BMO}_w} = \sup_I \int_I |f(x) - f_I| \, dx / m_w(I),
\]
where the supremum is taken over all cubes in \( \mathbb{R}^n \) and \( f_I = \int_I f \, dx / |I| \).

For the scalar-valued case, this definition is due to Muckenhoupt-Wheeden [26]–[27].

We prepare some easy lemmas.

**Lemma 4.A.** If \( \|f\|_{\text{BMO}_w} \leq 1 \), then for any cube \( I \) and any \( \lambda > 0 \),
\[
\left| \{ x \in I : |f(x) - f_I| > \lambda \} \right| / |I| \leq C_{4,1} e^{-C_{4,2} \lambda / w(I)}.
\]

**Lemma 4.B.** For any cube \( I \) and any \( \lambda > 0 \)
\[
\left| \{ x \in I : -\log w(x) > -\log w(I) + \lambda \} \right| / |I| \leq C_{4,1} e^{-C_{4,2} \lambda / c_0}.
\]

These follow from [21], (1.4) and [9].

**Lemma 4.1.** For any cubes \( I \) and \( J \) and for any \( t > 0 \),
\[
(4.1) \quad \left| \{ x \in I : w(x) \leq tw(I) \} \right| / |I| \leq C_{4,1} t^{C_{4,2} / c_0},
\]
(4.2) \[ \int_I w(I) - w(x) \, dx/m_w(I) \leq Cc_0 \quad \text{i.e.} \quad (1 + Cc_0)m_w(I) \geq w(I)|I|, \]

(4.3) \[ \text{if } J \supset I, \text{ then } \quad w(J)/w(I) \leq C(|J|/|I|)^{c_0/c_{4.2}}, \]

(4.4) \[ \text{if } |I| = |J|, \text{ then } \quad w(J)/w(I) \leq C(1 + |x_I - x_j|/l(I))^{c_0n/c_{4.2}}. \]

**Lemma 4.2.**

\[ \|w\|_{\text{BMO}_w} \leq Cc_0. \]

The above two lemmas are easy consequences of Lemma 4.2B.

**Definition 4.3.** For \(0 < \epsilon \leq 1\), let

\[ \|f\|_{\text{Lip}\epsilon} = \sup_{x, y: x \neq y} |f(x) - f(y)|/|x - y|^\epsilon, \]

\[ \|f\|_{\text{Lip}^2} = \sum_{j=1}^n \left\| \frac{\partial}{\partial x_j} f \right\|_{\text{Lip}^1}. \]

**Lemma 4.3.** If \(1 \geq \epsilon \geq c_0 n/C_{4.2}\) and if \(\text{supp} f \subset I(0, t)\), then

\[ \|f\|_{\text{BMO}_w} \leq Ct^\epsilon \|f\|_{\text{Lip}\epsilon/w(I(0, t))}. \]

**Proof.** We may assume \(t = 1\). Take any cube \(I\) in \(\mathbb{R}^n\). If \(l(I) > 1\) and \(I \cap I(0, 1) \neq \emptyset\), then

\[ \int_I |f(x) - f_I| \, dx/m_w(I) \leq C\|f\|_{L^1/m_w(I)} \leq C\|f\|_{\text{Lip}\epsilon/w(I(0, 1))}. \]

If \(l(I) \leq 1\) and \(I \cap I(0, 1) \neq \emptyset\), then

\[ \int_I |f(x) - f_I| \, dx/m_w(I) \leq Cl(I)^\epsilon \|f\|_{\text{Lip}\epsilon/w(I)} \leq C\|f\|_{\text{Lip}\epsilon/w(I(0, 1))} \]

by (4.3).

**Main Lemma.** Let \(c_0 > 0\) be small enough depending only on \(\theta_1, \ldots, \theta_m\). Let \(t > 0\). Suppose that (1.4),

(4.5) \[ \|f\|_{\text{BMO}_w} < c_0 \]

and

(4.6) \[ \text{supp} f \subset I(0, t) \]
hold. Then there exists \( g(x) \) such that

\[
|g(x)| \leq w(x)(1 + |x|/t)^{-n-1/2}
\]

and

\[
f - g \in S.
\]

We prove this Main Lemma in §9.

5. Proof of Theorem 1. Let \( h(t) \in C^\infty([0, +\infty)) \) be such that

\[
h(t) \geq 0, \quad \text{supp } h \subset [1/4, 1],
\]

and

\[
\sum_{k=1}^{\infty} h_k(t) = 1 \quad \text{on } [1, +\infty),
\]

where

\[
h_k(t) = h(2^{-k}t) \quad \text{for } k = 1, 2, 3, \ldots.
\]

Set

\[
h_0(t) = 1 - \sum_{k=1}^{\infty} h_k(t).
\]

Then

\[
f(x) = \sum_{k=0}^{\infty} h_k(|x|)f(x)
\]

and

\[
\|h_k(|x|)f(x)\|_\text{BMO}_w \leq C2^k\|h_kf\|_{\text{Lip}_1/w(I(0, 2^k))} \leq C2^{-k(n+1)/w(I(0, 1))}
\]

by (1.2), (1.3) and Lemma 4.3.

Applying the Main Lemma in §4 to each \( h_kf \), we get \( g_k \) such that

\[
h_kf - g_k \in S,
\]

\[
|g_k(x)| \leq c_0^{-1}C2^{-k(n+1)}w(x)(1 + 2^{-k}|x|)^{-n-1/2}/w(I(0, 1)).
\]

Set

\[
g(x) = \sum_{k=0}^{\infty} g_k(x).
\]

Then (1.5) is clear and (1.6) follows from

\[
\sum_{k=0}^{\infty} |g_k(x)| \leq c_0^{-1}Cw(x)\sum 2^{-k(n+1)}(1 + 2^{-k}|x|)^{-n-1/2}/w(I(0, 1))
\]

\[
\leq c_0^{-1}Cw(x)(1 + |x|)^{-n-1/2}/w(I(0, 1)).
\]
6. The property of the space $S$. The hard part in our argument is the problem, "What property does the space $S$ have?" Since $\theta_1, \ldots, \theta_m$ satisfy (1.1), $\tilde{\theta}_1, \ldots, \tilde{\theta}_m$ satisfy (1.1). Then by Lemma 2.2 of [32] there exist functions

$$\Theta_j(\xi, \nu) \in C^\infty(S_{n-1} \times \Sigma_{2m-1}), \quad 1 \leq j \leq m,$$

such that

$$\sum_{j=1}^m \hat{\Theta}_j(\xi) \Theta_j(\xi, \nu) \equiv 1,$$

$$\text{Re} \sum_{j=1}^m \tilde{\nu}_j(\Theta_j(\xi, \nu) + \Theta_j(-\xi, \nu)) \equiv 0,$$

$$\text{Im} \sum_{j=1}^m \tilde{\nu}_j(\Theta_j(\xi, \nu) - \Theta_j(-\xi, \nu)) \equiv 0.$$

This fact tells us that for any $\nu \in \Sigma_{2m-1}$ the set of real-valued functions

$$\{ \langle p(x), \nu \rangle : p \in S \}$$

is a sufficiently large class of functions. More precisely, we obtain

**Lemma 6.1.** Let $\nu \in \Sigma_{2m-1}$. Let $I$ be a cube. Let $b(x)$ be a real-valued function such that

(6.1) $\text{supp } b \subset 3I$,

(6.2) $\int b(x) \, dx = 0$,

(6.3) $\|b\|_{\text{Lip}^2} \leq l(I)^{-2}$.

Then there exists a $C^m$-valued function $p(x)$ such that

(6.4) $p \in S$,

(6.5) $\int p(x) \, dx = 0$,

(6.6) $\langle p(x), \nu \rangle \equiv b(x)$,

(6.7) $|p(x)| \leq C(1 + |x - x_j|/l(I))^{-n-1}$,

(6.8) $\left| \frac{\partial}{\partial x_j} p(x) \right| \leq Cl(I)^{-1}(1 + |x - x_j|/l(I))^{-n-2}, \quad j = 1, \ldots, n$.

**Proof.** Set

$$\tilde{p}_j(x) = - (\Theta_j(\xi, \nu)(\text{Re}(K^* \cdot (\nu)))(\hat{\xi}))(x)$$

$$- i(\Theta_j(\xi, \nu)(\text{Im}(K^* \cdot (\nu)))(\hat{\xi}))(x)$$
and

\[ \hat{p}(x) = (\hat{p}_1(x), \ldots, \hat{p}_m(x)). \]

By the properties of \( \{\Theta_j\} \) and by the same argument as in Lemma 2.3 of [32],

\[ K^* \cdot \hat{p} = -K^* \cdot (bv), \quad \int \hat{p}(x) \, dx = 0, \quad \langle \hat{p}(x), \nu \rangle \equiv 0. \]

Set

\[ p(x) = \hat{p}(x) + b(x)\nu. \]

Then (6.4)--(6.6) hold. Since \( \hat{p}_j(x) \) can be written in the form of a linear combination of \( b \) and its images by Calderon-Zygmund singular integral operators with smooth kernels [see Stein [29] p. 75], (6.7)--(6.8) follow from (6.1)--(6.3). See Lemma 2.3 of [32] for details.

**Lemma 6.2.** The function \( p(x) \) of Lemma 6.1 can be decomposed as follows:

\[ p(x) = \sum_{j=1}^{\infty} 2^{-j(\alpha+1)} \beta_j(x), \quad \text{supp } \beta_j \subset 2^j I, \]

\[ \|\beta_j\|_{\text{Lip}^1} \leq C/(2^j L(I)), \]

\[ \int \beta_j(x) \, dx = 0, \]

\[ \langle \beta_j(x), \nu \rangle \equiv 0 \quad \text{if } j > 4, \quad \langle \beta_4(x), \nu \rangle \equiv b(x). \]

**Proof.** Let \( h_k(x) \) be as in (5.2)--(5.3). Then

\[
\begin{align*}
p(x) &= \left\{ h_0(2^{-4}|x|)p(x) + h_4(|x|) \int \sum_{k=5}^{\infty} h_k(|y|)p(y) \, dy / \int h_4(|y|) \, dy \right\} \\
&\quad + \sum_{j=5}^{\infty} \left\{ h_j(|x|)p(x) - h_{j-1}(|x|) \int \sum_{k=j}^{\infty} h_k(|y|)p(y) \, dy / \int h_{j-1}(|y|) \, dy \\
&\quad + h_j(|x|) \int \sum_{k=j+1}^{\infty} h_k(|y|)p(y) \, dy / \int h_j(|y|) \, dy \right\}
\end{align*}
\]

gives the desired decomposition. See Lemma 3.5 of [32] for details. \( \square \)
7. Weighted Carleson measures. We continue to assume (1.4).

**Definition 7.1.** For a measure $\mu$ defined on $\mathbb{R}_+^n$, let

$$\|\mu\|_{c.w} = \sup_I |\mu|(Q(I))/m_w(I),$$

where $I$ is taken over all cubes in $\mathbb{R}^n$.

We prepare some easy lemmas.

**Lemma 7.1.** If $\|\mu\|_{c.w} \leq 1$, then for any cube $I$

$$\iint_{Q(I)} w(I(x,t))^{-1} d|\mu|(x,t) \leq C|I|.$$

*Proof.* [For the definition of $w(I(x,t))$ recall Definition 4.1.] We may assume that $I$ is a closed dyadic cube. Let $\{I_{k,j}\}_{j=1}^{\infty}$ be the maximal closed dyadic subcubes of $I$ such that

$$w(I_{k,j}) \leq 2^{-k} w(I).$$

By (4.1)

$$\sum_j |I_{k,j}| \leq C_4 2^{-k+c_0/|I|}.$$

So

$$\iint_{Q(I)} w(I(x,t))^{-1} d|\mu| \leq C \sum_{k=0}^{\infty} \sum_j \iint_{Q(I_{k,j})} 2^{k+1} w(I)^{-1} d|\mu|$$

$$\leq C w(I)^{-1} \sum_k \sum_j 2^{k+1} m_w(I_{k,j})$$

$$\leq C w(I)^{-1} \sum_k 2^{k+1} \sum_j 2^{-k} w(I) |I_{k,j}|$$

$$\leq C \sum_k C_4 2^{-k+c_0/|I|} |I| \leq C |I|. \quad \Box$$

**Definition 7.2.** For nonnegative real numbers $\{\lambda_I\}$ where $I$ is taken over all dyadic cubes, set

$$\eta_k(x) = \sum_{I: l(I) = 2^{-k}} \lambda_I (1 + 2^k |x - x_I|)^{-n-1},$$

$$\epsilon_k(x) = \sum_{j=0}^{\infty} \left( \frac{2}{3} \right)^j \eta_{k-j}(x).$$
LEMMA 7.2.

(7.1) \[ \lambda_I \leq (1 + 2^k |x - x_I|)^{n+1} \eta_k(x) \text{ if } l(I) = 2^{-k}, \]

(7.2) \[ \eta_k(x) \leq (1 + 2^k |y - y_I|)^{n+1} \eta_k(y), \]

(7.3) \[ \varepsilon_k(x) \leq (1 + 2^k |y - y_I|)^{n+1} \varepsilon_k(y). \]

Since this is easy, we omit the proof.

LEMMA 7.3. Let \( c_0 > 0 \) be small enough in (1.4). Let

(7.4) \[ \left\| \sum \lambda_j^2 |I| \delta_{(x_I, l(I))} \right\|_{c, w^2} \leq 1. \]

Then

(7.5) \[ \eta_k(x) \leq \varepsilon_k(x) \leq Cw(I(x, 2^{-k})), \]

(7.6) \[ \left\| \sum_{k=\infty}^\infty \varepsilon_k(x) \delta_{t=2^{-k}} \right\|_{c, w^2} \leq C, \]

where \( \delta_{(x, t)} \) is the Dirac measure concentrated at the point \( (x, t) \in R_{+1} \) and \( \delta_{t=a} \) denotes the measure induced from n-dimensional Lebesgue measure on the hyperplane \( t = a \) in \( R_{+1} \).

Proof. Since \( \lambda_J \leq Cw(J), \)

\[ \eta_k(x) \leq C \sum_{J: l(J) = 2^{-k}} (1 + 2^k \text{dist}(x, J))^{-n-1} w(J) \]

\[ \leq C \sum (\cdots)^{-n-1+Ce_0} w(I(x, 2^{-k})) \text{ by (4.4)} \]

\[ \leq Cw(I(x, 2^{-k})). \]

So,

\[ \varepsilon_k(x) \leq C \sum_{j=0}^\infty \left( \frac{2}{3} \right)^j w(I(x, 2^{-k+j})) \]

\[ \leq C \sum_{j=0}^\infty \left( \frac{2}{3} \right)^j 2^{jCe_0} w(I(x, 2^{-k})) \text{ by (4.3)} \]

\[ \leq Cw(I(x, 2^{-k})). \]

Condition (7.6) follows from almost the same argument as Lemma 3.2 of [32] with slight additional estimates about the order of growth of \( w \) as in the proof of (7.5). We omit the proof. □
8. The decomposition of weighted BMO functions. We continue to assume (1.4).

Following Chang-R. Fefferman [7], we decompose a weighted BMO function $f(x)$ and the weight function $w(x)$.

**Lemma 8.1.** Suppose that $\text{supp } f \subset I(0, 1)$ and $\|f\|_{\text{BMO}_w} \leq 1$. Then there exist $C^\infty$-valued functions $\{b_I(x)\}_I$ and nonnegative real numbers $\{\lambda_I\}_I$, where $I$ is taken over all dyadic cubes in $\mathbb{R}^n$, such that

\begin{align}
(8.1) & \quad f = \sum_I \lambda_I b_I, \\
(8.2) & \quad \lambda_I = 0 \quad \text{if } 3I \cap I(0, 1) = \emptyset, \\
(8.3) & \quad \text{supp } b_I \subset 3I, \\
(8.4) & \quad \int b_I dx = 0, \\
(8.5) & \quad \|b_I\|_{\text{Lip}} \leq C(I)^{-2}, \\
(8.6) & \quad \left\| \sum_I \lambda_I^2 |I| \delta_{x_I, l(I)} \right\|_{c, w^2} \leq C.
\end{align}

**Proof.** We use the idea of Chang-R. Fefferman [7]. Take a real-valued function $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ such that

\[ \text{supp } \varphi \subset \{ x \in \mathbb{R}^n : |x| < 1 \}, \]
\[ \int_0^{+\infty} \varphi(\xi t)^2 t^{-1} dt \equiv 1 \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}. \]

Set

\[ \lambda_I = |I|^{-1/2} \left( \iint_{I(I)} |\varphi_t * f(y)|^2 t^{-1} dt dy \right)^{1/2} \]

and

\[ b_I(x) = \iint_{I(I)} \varphi_t(x - y)(\varphi_t * f)(y) t^{-1} dt dy / \lambda_I, \]

where we define $0/0 = 0$ and

\[ T(I) = \{(x, t) : x \in I, t \in (l(I)/2, l(I))\}. \]

Then (8.2) is clear. Conditions (8.3)–(8.5) follow from the same argument as in Lemma 3.1 and Remark 3.1 of [32]. See [32] for details.
Since
\[ \sum_{I \subset J} \lambda_I |I| = \iint_{Q(J)} |Q_t * f(y)|^2 t^{-1} \, dt \, dy \]
\[ \leq C \iint_{3J} |f(x) - f_J|^2 \, dx \leq C |J| w(J)^2 \]
for any dyadic cube \( J \) by Lemma 4.A, (8.6) holds. \( \square \)

**Lemma 8.2.** Let \( k > 0 \). In Lemma 8.1 set
\[ f_k(x) = \sum_{I: l(I) \geq 2^{-k}} \lambda_I b_I(x). \]
Then
\[ \text{supp } f_k \subset I(0,3), \]
\[ |f_k(x) - f_k(y)| \leq C_w(I(x,2^{-k})) 2^k |x - y| \]
provided \( |x - y| < 2^{-k} \). \( \square \)

**Proof.** Set
\[ \Phi = \left( \int_1^\infty \phi(t \xi)^2 t^{-1} \, dt \right)^\psi \]
\[ = \lim_{\epsilon \to 0+} \left( \delta_0 - \int_{\epsilon}^1 \phi_t * \phi_t t^{-1} \, dt \right), \]
where \( \delta_0 \) is the dirac measure concentrated at the origin. Since
\[ f_k = f * 2^{kn} \Phi(2^k \cdot), \]
(8.7) is clear. (8.8) follows from \( \|f\|_{BMO} \leq 1 \) and from
\[ f_k(x) - f_k(y) = \int f(z) 2^{kn} (\Phi(2^k(x - z)) - \Phi(2^k(y - z))) \, dz. \] \( \square \)

From Lemmas 8.1–8.2 we get

**Lemma 8.3.** Let \( \|f\|_{BMO} \leq c_0 \). Let \( \text{supp } f \subset I(0,1) \). Let \( M \) be a positive integer. Then there exist \( f_M(x), \{b_I(x)\}_{I: \text{dyadic}} \) and nonnegative real numbers \( \{\lambda_{f,I}\}_{I: \text{dyadic}} \) such that
\[ f = \sum_{I} \lambda_{f,I} b_I + f_M, \]
(8.9)
(8.10) \[ \lambda_{f,I} = 0 \quad \text{if } 3I \cap I(0,1) = \emptyset \text{ or if } l(I) \geq 2^{-M}, \]

\[ \text{Lemma 8.4. Let } c_0 > 0 \text{ be small enough in (1.4). Let } M \text{ be a positive integer. Then there exist real-valued functions } w_M(x), \{b_I(x)\}_I: \text{dyadic and nonnegative real numbers } \{\lambda_{w,I}\}_I: \text{dyadic such that} \]

\[ w = \sum_I \lambda_{w,I} b_I + w_M, \]

\[ \lambda_{w,I} = 0 \quad \text{if } l(I) \geq 2^{-M}, \]

\[ \supp b_I \subseteq 3I, \]

\[ \int b_I \, dx = 0, \]

\[ \|b_I\|_{\text{Lip}^2} \leq C l(I)^{-2}, \]

\[ \left\| \sum_I \lambda_{w,I}^2 |I| \delta_{(x_I, l(I))} \right\|_{c,w^2} \leq Cc_0^2, \]

\[ w_k(x) \geq 3w(I(x,2^{-k}))/4, \]

where \( k \geq M \) and

\[ w_k(x) = \sum_{I: 2^{-M} \geq l(I) \geq 2^{-k}} \lambda_{w,I} b_I(x) + w_M(x). \]

\[ \text{Proof. Take the same } \varphi(x) \text{ as in the proof of Lemma 8.1. If } l(I) < 2^{-M}, \text{ then set} \]

\[ \lambda_{w,I} = |I|^{-1/2} \left( \iint_{T(I)} |\varphi_t * w(y)|^2 t^{-1} \, dt \, dy \right)^{1/2} \]

and

\[ b_I(x) = \iint_{T(I)} \varphi_t(x - y)(\varphi_t * w)(y)t^{-1} \, dt \, dy / \lambda_{w,I}. \]
If \( l(I) \geq 2^{-M} \), then set \( \lambda_{w,I} = 0 \) and \( b_I(x) = 0 \). Set
\[
 w_M(x) = w(x) - \sum_{I: l(I)<2^{-M}} \lambda_{w,I} b_I(x).
\]

Then (8.11)–(8.12) are clear. Conditions (8.13)–(8.16) follow from Lemma 4.2 and the same argument as in the proof of Lemma 8.1.

Let \( k \geq M \). Take the same \( \Phi \) as in the proof of Lemma 8.2. Then
\[
w_k = w * 2^k \Phi(2^k).
\]

Put \( J = I(x, 2^{-k}) \). Since
\[
|w(J) - w_k(x)| \leq C \int |w(J) - w(y)| 2^{kn} \Phi(2^k(x - y)) \, dy
\]
\[
\leq C \int_{2J} |w(J) - w(y)| \, dy / |J| \leq C_0 w(J)
\]
by (4.2)–(4.3), we get (8.17).

**Lemma 8.5.** Let \( j \) be a positive integer. Assume that \( \{b_I(x)\}_{I: \text{dyadic}} \) and \( \{\lambda_I\}_{I: \text{dyadic}} \) satisfy (8.4), (8.6),
\[
\text{supp } b_I \subset 2^j I
\]
and
\[
\|b_I\|_{\text{Lip}^1} \leq (2^j l(I))^{-1}.
\]

Let \( \alpha > 0 \). Set
\[
f(x) = \sum_{I: l(I)<\alpha} \lambda_I b_I(x).
\]

Then
\[
\|f\|_{\text{BMO}_w} \leq C 2^{jn},
\]
where \( C \) is independent of \( \alpha \).

**Proof.** Take any cube \( J \) (not necessarily dyadic). Let \( 2^{-k} \leq l(J) < 2^{-k+1} \). Set
\[
\tilde{f} = \sum_{I: l(I) \geq 2^{-j-k+1}} \lambda_I b_I
\]
and
\[
\tilde{f} = \sum_{I: l(I) < 2^{-j-k+1}, I \cap 3J \neq \emptyset} \lambda_I b_I.
\]
Then

(8.21) \( f = \tilde{f} + \tilde{f} \) on \( J \).

Note that

(8.22) \[ |\tilde{f}(x) - \tilde{f}(y)| \leq \sum_{h=-\infty}^{j+k-1} \sum_{l(I)=2^{-h}} \lambda_I |b_{I}(x) - b_{I}(y)| \]

\[ \leq \sum \sum \lambda_I 2^{-j+h} |x - y| \chi_{2^{j+I}}(x) \]

\[ \leq 2^{k} |x - y| \sum \sum \lambda_I 2^{-j-k} \chi_{2^{j+I}}(x) \]

\[ \leq 2^{k} |x - y| 2^{jn} w(I(x, 2^{-k})) \]

provided \( |x - y| < 2^{-k} \). By (8.4), (8.6), (8.18), (8.19) and by Lemma 3.3 of [32] we get

(8.23) \[ ||\tilde{f}||_{L^2} \leq C 2^{jn} \left( \sum_{I: l(I) \leq 2^{-j-k+1}, I \cap J \neq \emptyset} \lambda_I^2 |I| \right)^{1/2} \]

\[ \leq C 2^{jn} w(J) |J|^{1/2}. \]

Thus by (8.21)–(8.23)

\[ \int_{J} |f(x) - \tilde{f}(x_I)| dx / m_w(J) \]

\[ \leq C \left( \int_{J} |f(x) - \tilde{f}(x_I)|^2 dx / |J| \right)^{1/2} / w(J) \leq C 2^{jn}. \]

9. Proof of the Main Lemma in §4. We may assume \( t = 1 \) in (4.6) and

(9.1) \[ w(I(0, 1)) = 1. \]

In this section \( C_{9.1} \) is a large constant depending only on \( \theta_1, \ldots, \theta_m \). Let \( M \) be a large integer depending only on \( \theta_1, \ldots, \theta_m \) and \( C_{9.1} \). Let \( c_0 > 0 \) be small enough depending only on \( \theta_1, \ldots, \theta_m, C_{9.1} \) and \( M \). In particular

(9.2) \[ C_{9.1} 2^{-M} < 1 \quad \text{and} \quad C_{9.1} 2^{M(n+2)c_0} < 1. \]

First, we give a rough explanation of the procedure to construct \( g(x) \). We construct a sequence \( \{g_k\}_{k=M}^{\infty} \) such that

(i) \( |g_k(x)| \leq w_k(x) \chi_{I(0,4)}(x) \),

(ii) \( f_k - g_k + \text{(small errors)} \in S \).
[For the definitions of $w_k$ and $f_k$, recall Lemmas 8.2 and 8.4.] Then by letting $k \to +\infty$, we get $\tilde{g}$ such that

$$|\tilde{g}(x)| \leq w(x) \chi_{I(0,4)}(x),$$

$$f - \tilde{g} + \text{(small errors)} \in S.$$

Next we estimate the weighted BMO norms of the error terms and repeat the same procedure for them.

In order to meet the condition (i), we must adjust the length of the vector-valued function $g_k$. We must do this adjustment under the restriction (ii). Here we use the property of the space $S$ that was proved in Lemma 6.1.

Now we go into details.

By Lemmas 8.3–8.4, we get

$$f_M(x), \{b_I(x)\}_I: \text{dyadic and } \{\lambda_{f,I}\}_I: \text{dyadic}$$

$$w_M(x), \{b_I(x)\}_I: \text{dyadic and } \{\lambda_{w,I}\}_I: \text{dyadic}$$

such that (8.9)–(8.10), (8.3)–(8.5), (8.6)’–(8.8)’ and (8.11)–(8.17) hold.

Set

$$\lambda_I = \lambda_{f,I} + \lambda_{w,I} \quad \text{if } l(I) < 2^{-M},$$

$$\lambda_I = c_0 w(I) \quad \text{if } l(I) = 2^{-M},$$

$$\lambda_I = 0 \quad \text{if } l(I) > 2^{-M}.$$

**Lemma 9.1.**

$$\left\| \sum_I \lambda_I^2 |I| \delta_{(x_I, l(I))} \right\|_{c,w^2} \leq C c_0^2.$$

This is clear from (8.6)’ and (8.16).

From these $\{\lambda_I\}_I$, we define $\eta_k(x)$ and $\varepsilon_k(x)$ by Definition 7.2. Then by Lemmas 9.1 and 7.3 we get

**Lemma 9.2.** If $x \in I$ and $l(I) = 2^{-k}$, then

$$C \lambda_I \leq \eta_k(x) \leq \varepsilon_k(x) \leq C' c_0 w(I).$$

**Lemma 9.3.**

$$\left\| \sum_{k=-\infty}^{\infty} \varepsilon_k(x)^2 \delta_{l=2^{-k}} \right\|_{c,w^2} \leq C c_0^2.$$
We inductively construct
\[ \{ \varphi_k(x) \}_{k=M+1}^{\infty} \quad \text{and} \quad \{ \beta_{I,j}(x) \}_{I: \text{dyadic}, I(0) < 2^{-M}; j=4,5,6, \ldots} \]
with the following properties (C.1)–(C.8). Put
\[
\begin{align*}
\label{eq:9.6} p_{I,1}(x) &= \sum_{j=4}^{M-1} 2^{-j(n+1)} \beta_{I,j}(x), \\
\label{eq:9.7} p_{I,2}(x) &= \sum_{j=M}^{\infty} 2^{-j(n+1)} \beta_{I,j}(x), \\
\label{eq:9.8} g_M(x) &= f_M(x), \\
\label{eq:9.9} g_k(x) &= f_k(x) + \sum_{h=M+1}^{k} \sum_{I: \ell(I) = 2^{-h}} \lambda_I p_{I,1}(x) - \sum_{h=M+1}^{k} \varphi_h(x)
\end{align*}
\]

for \( k = M + 1, M + 2, \ldots \).

(C.1) \( \text{supp } \beta_{I,j} \subset 2^j I, \quad \| \beta_{I,j} \|_{\text{lip} 1} \leq C_{9.1} 2^{-j \ell(I)^{-1}} \), \( \int \beta_{I,j} \, dx = 0 \),

(C.2) \( \beta_{I,j}(x) \equiv 0 \) if \( I \cap I(0,4) = \emptyset \),

(C.3) \( p_{I,1} + p_{I,2} \in S \),

(C.4) \(| \varphi_k(x) | \leq C_{9.1}^2 \epsilon_k(x)^2 2^{M(n+2)} / w(I(x, 2^{-k})) \),

(C.5) \(| \varphi_k(x) - \varphi_k(y) | \leq C_{9.1}^4 c_0 2^{M(n+2)} \epsilon_k(x) 2^k |x - y| \)

provided \( |x - y| < 2^{-k} \),

(C.6) \( \text{supp } \varphi_k \subset \text{supp } g_k \subset I(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-k+M}) \),

(C.7) \(| g_k(x) | \leq w_k(x) \),

(C.8) \(| g_k(x) - g_k(y) | \leq C_{9.1} \epsilon_k(x) 2^k |x - y| \) provided \( |x - y| < 2^{-k} \).

The construction of the above functions is explained at the end of this section. We accept this construction temporarily and prove the Main Lemma. By the same argument as [32], we can show that \( \bar{g} = \lim_{k \to \infty} g_k \) exists in \( L^2 \). By (C.6)–(C.7), we get
\[
\text{supp } \bar{g} \subset I(0,4)
\]

and
\[
\label{eq:9.11} | \bar{g}(x) | \leq w(x).
\]

By (9.9)
\[
\label{eq:9.12} \bar{g}(x) = f(x) + \sum_{I: \ell(I) < 2^{-M}} \lambda_I p_{I,1}(x) - \sum_{h=M+1}^{\infty} \varphi_h(x).
\]
Set
\begin{align}
(9.13) \quad p_2(x) &= \sum_{I: \ell(I) < 2^{-M}} \lambda_I p_{I,2}(x), \\
(9.14) \quad \varphi(x) &= \sum_{k = M+1}^{\infty} \varphi_k(x).
\end{align}

**Lemma 9.4.**
\[
f - (\tilde{g} + p_2 + \varphi) \in S.
\]

**Proof.** Since
\[
\tilde{g} = f + \sum_{I: \ell(I) < 2^{-M}} \lambda_I (p_{I,1} + p_{I,2}) - p_2 - \varphi
\]
by (9.12), the lemma follows from (C.3).

**Lemma 9.5.**
\begin{align}
(9.15) \quad \text{supp } \varphi &\subset I(0,4), \\
(9.16) \quad \|\varphi\|_{\text{BMO}_w} &\leq C C_{9,1}^4 c_0^2 2^{M(n+2)}.
\end{align}

**Proof.** Condition (9.15) is clear from (C.6). Take any $I$ (not necessarily dyadic). Then
\[
\int_I \sum_{k: 2^{-k} < \ell(I)} |\varphi_k(x)| \, dx / m_w(I)
\leq C_{9,1}^2 2^{M(n+2)} \int_I \sum_{k: 2^{-k} < \ell(I)} \epsilon_k(x)^2 w(I(x, 2^{-k}))^{-1} \, dx / m_w(I)
\]
by (C.4)
\leq C_{9,1}^2 2^{M(n+2)} \int_I \sum \epsilon_k(x)^2 w(I(x, 2^{-k}))^{-2} \, dx / |I|
\leq CC_{9,1}^2 c_0^2 2^{M(n+2)}
\]
by Lemmas 9.3 and 7.1. On the other hand,
\[
\left| \sum_{k: 2^{-k} \geq \ell(I)} (\varphi_k(x) - \varphi_k(y)) \right| / w(I) \leq CC_{9,1}^4 c_0^2 2^{M(n+2)}
\]
if \( x, y \in I \) by (C.5). Thus
\[
\int_I \left| \sum_{k=M+1}^{\infty} \varphi_k(x) - \sum_{2^{-k} \geq l(I)} \varphi_k(x_I) \right| dx / m_w(I) \leq CC_9 c_0^2 2^{M(n+2)}. \]

Set
\[
f_2 = \varphi,
\]
\[
f_3 = \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I: l(I) \leq 2^{-j}} \lambda_I \beta_{I,j},
\]
\[
f_k = \sum_{j, I: l(I)=2^{k-j} \text{ and } l(I) < 2^{-M}} 2^{-j(n+1)} \lambda_I \beta_{I,j}, \quad k \geq 4.
\]

**Lemma 9.6.**
\[
\sum_{k=2}^{\infty} f_k = \varphi + p_2.
\]

**Lemma 9.7.** For \( k \geq 3 \),
\[
(9.17) \quad \text{supp} f_k \subset I(0, 2^k),
\]
\[
(9.18) \quad \int f_k \, dx = 0,
\]
\[
(9.19) \quad \|f_k\|_{BMO_w} \leq CC_9 c_0 2^{-M} 2^{-k(n+1)}.
\]

**Proof.** We show only (9.19). If \( k \geq 4 \), then
\[
\|f_k\|_{BMO_w} \leq \sum_{j=M+k-2}^{\infty} \sum_{I: l(I) = 2^{k-j-3}} 2^{-j(n+1)} \lambda_I \|\beta_{I,j}\|_{BMO_w}
\]
\[
\leq \sum_j 2^{-j(n+1)} Cc_0 \leq \sum_j 2^{-j(n+1)} Cc_0 2^{(j-k)n}
\]
\[
\leq Cc_0 2^{-M} 2^{-k(n+1)}.
\]
If \( k = 3 \), then
\[
\|f_3\|_{BMO_w} = \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \left\| \sum_{I: l(I) \leq 2^{-j}} \lambda_I \beta_{I,j} \right\|_{BMO_w}
\]
\[
\leq \sum_{j=M+1}^{\infty} 2^{-j} Cc_0 \leq Cc_0 2^{-M} \quad \text{by Lemmas 8.5 and 9.1.} \quad \square
\]
From Lemmas 9.4–9.7, we obtain the following.

**Lemma 9.8.** Assume the hypothesis of the Main Lemma. Then there exist \( \tilde{g}(x) \) and \( \{f_j(x)\}_{j=2}^{\infty} \) such that

\[
\begin{align*}
(9.20) & \quad f - \left( \tilde{g} + \sum_{j=2}^{\infty} f_j \right) \in S, \\
(9.21) & \quad \text{supp } f_j \subset I(0, 2^j/t), \\
(9.22) & \quad \|f_j\|_{\text{BMO}} \leq c_0 \alpha(M, c_0) 2^{-j(n+1)}, \\
(9.23) & \quad |\tilde{g}(x)| \leq w(x), \\
(9.24) & \quad \text{supp } \tilde{g} \subset I(0, 4^j/t),
\end{align*}
\]

where

\[
\alpha(M, c_0) = C(C_{9.1} 2^{-M} + C_{9.1}^4 c_0 2^{M(n+2)}).
\]

Since we have assumed \( t = 1 \) at the beginning of this section, we showed the above only for the case \( t = 1 \). But the general case follows easily from the case \( t = 1 \).

**Proof of the Main Lemma.** We continue to assume \( t = 1 \). Take \( M \) and \( c_0 \) so that

\[
(9.25) \quad 1 + \alpha(M, c_0) < 2^{1/4}.
\]

Applying Lemma 9.8 to \( f \), we obtain \( \tilde{g} \) and \( \{f_j\}_{j=2}^{\infty} \) with (9.20)–(9.24). Next, applying Lemma 9.8 to each \( f_j \), we obtain \( \tilde{g}_j \) and \( \{f_{j,k}\}_{k=2}^{\infty} \). Repeating this process, we obtain \( \{\tilde{g}_{j_1, \ldots, j_l}\} \) and \( \{f_{j_1, \ldots, j_l}\} \) such that

\[
\begin{align*}
\text{supp } f_{j_1, \ldots, j_l, k} & \subset I(0, 2^{j_1+\cdots+j_l+k}), \\
\|f_{j_1, \ldots, j_l, k}\|_{\text{BMO}} & \leq c_0 \alpha^{i+1} 2^{-(j_1+\cdots+j_l+k)(n+1)}, \\
|\tilde{g}_{j_1, \ldots, j_l}(x)| & \leq \alpha^i 2^{-(j_1+\cdots+j_l)(n+1)} w(x), \\
\text{supp } \tilde{g}_{j_1, \ldots, j_l} & \subset I(0, 4 \cdot 2^{j_1+\cdots+j_l}).
\end{align*}
\]

Set

\[
g^i = \tilde{g} + \sum_{s=1}^{i} \sum_{j_1, \ldots, j_s} \tilde{g}_{j_1, \ldots, j_s}.
\]
Then
\[ f = \left( g' + \sum_{j_1, \ldots, j_{i+1}} f_{j_1, \ldots, j_{i+1}} \right) \in S. \]

Set
\[ g = \lim_{i \to \infty} g^i. \]

Since \( \sum_{j_1, \ldots, j_i+1} f_{j_1, \ldots, j_i+1} \) tends to 0 in \( L^2 \) as \( i \to \infty \), \( g \) satisfies (4.8). On the other hand,
\[ g = \tilde{g} + \sum_{k=1}^{\infty} \sum_{s: 1 \leq s \leq k/2} \sum_{j_i: j_i + \cdots + j_s = k} \tilde{g}_{j_1, \ldots, j_s} \]
\[ = \tilde{g} + \sum_{k=1}^{\infty} (9.26)_k \]

and
\[ \text{supp}(9.26)_k \subset I(0, 4 \cdot 2^k), \]
\[ |(9.26)_k| \leq 2^{-k(n+1)}w(x) \sum_{s: 1 \leq s \leq k/2} \alpha^s \left( \frac{k + s - 1}{s - 1} \right) \leq 2^{-k(n+1/2)}w(x) \]
by (9.25). Thus, (4.7) holds.

Construction of \( \{ \beta_{i,j} \} \) and \( \{ \varphi_k \} \).

We construct these functions inductively. We define \( g_M \) by (9.8). Then
\[ (9.27) \quad \text{supp} g_M \subset I(0, 3) \]
by (8.7)''. Since
\[ w_M(x) \geq C2^{-C_0M}w(I(0, 1)) \]
by (4.3) and (8.17) and since
\[ |f_M(x)| \leq Cc_0M, \]
we get
\[ (9.28) \quad |g_M(x)| \leq w_M(x) \]
if \( c_0 > 0 \) is small enough depending on \( M \). By (8.8)' and (9.4)
\[ (9.29) \quad |g_M(x) - g_M(y)| \leq C\varepsilon_M(x)2^M|x - y| \]
provided \( |x - y| < 2^{-M} \). [Recall that \( \varepsilon_k(x) \) is defined by Definition 7.2 from \( \{ \lambda_I \} \) defined by (9.3)-(9.5).]
Let $k > M$. Suppose that
\[
\{ \beta_{I,j} \}_{I(1) > 2^{-k}, j = 4, 5, 6, \ldots} \quad \text{and} \quad \{ \varphi_h \}_{h = M+1, \ldots, k-1}
\]
have been constructed and that $g_{k-1}$ defined by (9.8)-(9.9) satisfies
\[
(C.6)' \quad \text{supp } g_{k-1} \subset \{ 0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-(k-1-M)} \},
\]
\[
(C.7)' \quad |g_{k-1}(x)| \leq w_{k-1}(x),
\]
\[
(C.8)' \quad |g_{k-1}(x) - g_{k-1}(y)| \leq C_{9.1} \epsilon_{k-1}(x) 2^{k-1} |x - y| \quad \left( \text{provided } |x - y| < 2^{-k+1} \right).
\]

Notice that by (9.27)-(9.29) $g_M$ satisfies the above (C.6)-(C.8).

**Lemma 9.9.** If $|x - y| \leq 2^{M-k}$, then
\[
|g_{k-1}(x) - g_{k-1}(y)| \leq C_{9.1} 2^{M(n+1)} \epsilon_{k-1}(x) 2^k |x - y|.
\]

This follows from (C.8)' and (7.3).

Set
\[
(9.30) \quad \tilde{w}_k(x) = |g_{k-1}(x)| + \sum_{I: (9.31)} \lambda_{w, I} b_I(x)
\]
where $\Sigma$ is taken over all dyadic cubes $I$ such that
\[
(9.31) \quad I(I) = 2^{-k} \quad \text{and} \quad I \cap \{ 0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-(k-1-M)} \} \neq \emptyset.
\]
[Recall that $b_I$ and $b_I$ are defined by Lemmas 8.3–8.4.]

**Lemma 9.10.** If $|x - y| < 2^{-k}$, then
\[
|\tilde{w}_k(x) - \tilde{w}_k(y)| \leq C C_{9.1} \epsilon_k(x) 2^k |x - y|.
\]

This follows from (C.8)', the first two inequalities in Lemma 9.2 and (8.13)-(8.15).

From now we explain how to construct
\[
\{ \beta_{I,j} \}_{I(1) = 2^{-k}, j = 4, 5, 6, \ldots} \quad \text{and} \quad \varphi_k.
\]

For each $I$ with (9.31) we apply Lemma 6.1 to
\[
\nu = U(g_{k-1}(x_I)),
\]
\[
b(x) = \lambda_{w, I} b_I(x) - \inner{\lambda_{j, I} b_I(x), \nu}.
\]
where
\[
\lambda'_w, I = \lambda_{w,I}/\lambda_I,
\]
\[
\lambda'_f, I = \lambda_{f,I}/\lambda_I.
\]

[For the sake of convenience, we define \( \mathbf{1}/(0) = (1,0,...,0) \) and \( 0/0 = 0 \).]
Then we get \( p_I(x) \) satisfying (6.4)–(6.5), (6.7)–(6.8) and
\[
(6.6)' \quad \left\langle p_I(x), U(g_{k-1}(x_I)) \right\rangle = \lambda'_w, I b_I(x) - \left\langle \lambda'_f, I b_I(x), U(g_{k-1}(x_I)) \right\rangle.
\]

Applying Lemma 6.2 to \( p_I(x) \), we get \( \{\beta_{I,j}\}_{J=4}^\infty \). Define \( p_{I,1}(x) \) and \( p_{I,2}(x) \) by (9.6)–(9.7). Then (C.1)–(C.3) are clear.

Set
\[
q_I(x) = p_{I,1}(x) + \lambda'_f, I b_I(x),
\]
\[
h(x) = \sum_{I: (9.31)} \lambda_I q_I(x),
\]
\[
k(x) = g_{k-1}(x) + h(x).
\]

Then
\[
\text{supp } q_I \subset 2^{M-1}I,
\]
\[
|q_I(x)| \leq C(1 + 2^k|x - x_I|)^{-n-1},
\]
\[
|q_I(x) - q_I(y)| \leq C2^k|x - y|(1 + 2^k|x - x_I|)^{-n-2}
\]
provided that \( |x - y| < 2^{-k} \),
\[
|h(x)| \leq \sum_{I: (9.31)} \lambda_I |q_I(x)| \leq C\eta_k(x) \quad \text{by (9.36)},
\]
\[
|h(x) - h(y)| \leq \sum \lambda_I |q_I(x) - q_I(y)| \leq C\eta_k(x)2^k|x - y|
\]
provided \( |x - y| < 2^{-k} \) by (9.37),
\[
\text{supp } k \subset I(0, 3 + 2^{-1} + 2^{-2} + \cdots + 2^{-k+M})
\]
by (C.6)' and (9.35),
\[
|k(x) - k(y)| \leq |g_{k-1}(x) - g_{k-1}(y)| + |h(x) - h(y)|
\]
\[
\leq (C_{9.1}2^{-1}(x)/2 + C\eta_k(x))2^k|x - y|
\]
\[
\leq \frac{3}{4} C_{9.1}2^{-1}(x)2^k|x - y|
\]
provided \( |x - y| < 2^{-k} \) by (C.8)' and (9.39)
since \( C_{9.1} \) is large enough.
Set
(9.42) \[ k_1(x) = \langle k(x), U(g_{k-1}(x)) \rangle U(g_{k-1}(x)) \]
and
(9.43) \[ k_2(x) = k(x) - k_1(x) = h(x) - \langle h(x), U(g_{k-1}(x)) \rangle U(g_{k-1}(x)). \]

Then \( k_1 \) and \( k_2 \) are orthogonal. Set
(9.44) \[ v_I(x) = \langle q_I(x), U(g_{k-1}(x)) - U(g_{k-1}(x_I)) \rangle. \]

Then
(9.45) \[ \langle q_I(x), U(g_{k-1}(x)) \rangle = \langle q_I(x), U(g_{k-1}(x_I)) \rangle + v_I(x) \]
\[ = \langle p_{I,1}(x), U(g_{k-1}(x_I)) \rangle \]
\[ + \langle \lambda'_{I,1} b_I(x), U(g_{k-1}(x_I)) \rangle + v_I(x) \]
\[ = \langle p_I(x), U(g_{k-1}(x_I)) \rangle + \cdots + \cdots \]
\[ = \lambda'_{w,1} b_I(x) + v_I(x) \]
by (9.32) and (6.6)', Thus
(9.46) \[ k_1(x) = g_{k-1}(x) + \langle h(x), U(g_{k-1}(x)) \rangle U(g_{k-1}(x)) \]
\[ = \tilde{w}_k(x) U(g_{k-1}(x)) + \sum_{I: (9.31)} \lambda_I v_I(x) U(g_{k-1}(x)) \]
by (9.33), (9.45) and (9.30).

Take any dyadic cube \( J \) with \( l(J) = 2^{-k} \).

**Lemma 9.11.** (i) If
(9.47) \[ |g_{k-1}(x_J)| \leq 3w_{k-1}(x_J)/4, \]
then
\[ |k(x)| \leq 7w_k(x)/8 \quad \text{on } J. \]
(ii) If
(9.48) \[ |g_{k-1}(x_J)| \geq w_{k-1}(x_J)/2, \]
then
\[ |g_{k-1}(x)| \geq w(J)/4 \quad \text{on } 2^M J. \]
Proof. By (8.17) and Lemma 9.2,
\[ w_k(x) > (1 - Cc_0)w_{k-1}(x). \]
By (9.38) and Lemma 9.2,
\[ |h(x)| \leq Cc_0w_k(x). \]
Thus (i) holds since \( c_0 \) is small enough.
Let \( x, y \in 2^M \). Then, by Lemmas 9.9 and 9.2
\[ |g_{k-1}(x) - g_{k-1}(y)| \leq CC_{9.1}2^{M(n+2)}c_0w(J). \]
Since \( c_0 \) is small enough depending on \( M \) and \( C_{9.1} \), (ii) follows from (8.17).

\textbf{Lemma 9.12.} If (9.48) holds and if \(|x - x_f|, |y - x_f| \leq 2^{M-k} \), then
\begin{align*}
(9.49) & \quad |U(g_{k-1}(x)) - U(g_{k-1}(y))| \\
& \leq CC_{9.1}2^{M(n+1)}\varepsilon_{k-1}(x)2^{k-1}|x - y|/w(J),
\end{align*}
\begin{align*}
(9.50) & \quad |U(k(x)) - U(k(y))| \leq CC_{9.1}2^{M(n+1)}\varepsilon_k(x)2^k|x - y|/w(J).
\end{align*}

The first inequality follows from Lemma 9.9 and part (ii) of Lemma 9.11. The second inequality follows from (9.41) and part (ii) of Lemma 9.11.

\textbf{Lemma 9.13.} If (9.48) holds, then
\begin{align*}
(9.51) & \quad |v_I(x)| \leq CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\left(1 + 2^k|x - x_I|\right)^{-n-1}/w(J) \quad \text{on } J,
\end{align*}
\begin{align*}
(9.52) & \quad |v_I(x) - v_I(y)| \\
& \leq CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)2^k|x - y|\left(1 + 2^k|x - x_I|\right)^{-n-1}/w(J) \quad \text{on } J,
\end{align*}
\begin{align*}
(9.53) & \quad \left| \sum_{I: (9.31)} \lambda_I v_I(x) \right| \leq CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\eta_k(x)/w(J) \quad \text{on } J,
\end{align*}
\begin{align*}
(9.54) & \quad \left| \sum_{I: (9.31)} \lambda_I v_I(x) - \sum_{I: (9.31)} \lambda_I v_I(y) \right| \\
& \leq CC_{9.1}2^{M(n+2)}\varepsilon_{k-1}(x)\eta_k(x)2^k|x - y|/w(J) \quad \text{on } J.
\end{align*}

Proof. (9.51) follows from (9.35)-(9.36) and (9.49). Note that
\[ v_I(x) - v_I(y) = \left\langle q_I(x) - q_I(y), U(g_{k-1}(x)) - U(g_{k-1}(x_I)) \right\rangle \\
+ \left\langle q_I(y), U(g_{k-1}(x)) - U(g_{k-1}(x_I)) \right\rangle. \]
Condition (9.35), (9.37) and (9.49) take care of the first term and conditions (9.36) and (9.49) take care of the second term. Thus, (9.52) holds. Conditions (9.53)–(9.54) follow from (9.51)–(9.52).

**Lemma 9.14.** If (9.48) holds, then

\[
(9.55) \quad |k_2(x)| \leq C\eta(x) \quad \text{on } J, \\
(9.56) \quad |k_2(x) - k_2(y)| \leq C\eta(x)2^k|x - y| \quad \text{on } J.
\]

*Proof.* (9.55) follows from the last formula of (9.43) and (9.38). Note that

\[
(9.57) \quad |U(g_{k-1}^+(x)) - U(g_{k-1}^+(y))| \leq 2^{k-1}|x - y|
\]

by (9.49), Lemma 9.2 and (9.2). So, (9.56) follows from the last formula of (9.43), (9.39), (9.38) and (9.57).

**Lemma 9.15.** If (9.48) holds, then

\[
(9.58) \quad |k(x)| - \tilde{w}_k(x) \leq C_{9.1}2^{M(n+2)}\varepsilon_k(x)^2/w(J) \quad \text{on } J, \\
(9.59) \quad \left| (|k(x)| - \tilde{w}_k(x)) - (|k(y)| - \tilde{w}_k(y)) \right| \\
\quad \quad \leq C_{9.1}2^{M(n+2)}\varepsilon_k(x)^22^k|x - y|/w(J) \quad \text{on } J.
\]

*Proof.* Set \( r_1(t) = (1 + t)^{1/2} - 1. \) Then

\[
(9.60) \quad |k(x)| - \tilde{w}_k(x) = \left( |k_1(x)|^2 + |k_2(x)|^2 \right)^{1/2} - \tilde{w}_k(x) \\
\quad = \left\{ \tilde{w}_k(x) + \sum_{I: (9.31)} \lambda_I v_I(x) \right\}^2 + |k_2(x)|^2 \quad \text{by (9.46)} \\
\quad = \tilde{w}_k(x)r_1 \left( 2\sum\lambda_I v_I(x) + \tilde{w}_k(x) \right) \\
\quad \quad + \left( \sum\lambda_I v_I(x)/\tilde{w}_k(x) \right)^2 + \left( |k_2(x)/\tilde{w}_k(x)|^2 \right) \\
\quad = \tilde{w}_k(x)r_2(x).
\]

Then by (9.53) and (9.55)

\[
(9.61) \quad |r_2(x)| \leq C_{9.1}2^{M(n+2)}\varepsilon_k(x)^2/w(J)^2.
\]

So, (9.58) holds.
By (9.60), the left-hand side of (9.59)
\[ \leq |\tilde{w}_k(x) - \tilde{w}_k(y)|r_2(x) + \tilde{w}_k(y)|r_2(x) - r_2(y)| \]
\[ \leq |\tilde{w}_k(x) - \tilde{w}_k(y)|r_2(x) + C\tilde{w}_k(y) \left\{ \left| \sum \lambda_f v_f(x) \right| - \left| \sum \lambda_f v_f(y) \right| \right\} / \tilde{w}_k(x) \]
\[ + \left| \sum \lambda_f v_f(y) \right| \tilde{w}_k(x)^{-1} - \tilde{w}_k(y)^{-1} \]
\[ + \left| k_2(x) \right|^2 - \left| k_2(y) \right|^2 \right\} / \tilde{w}_k(x)^2 \]
\[ + \left| k_2(y) \right|^2 \right\} \tilde{w}_k(x)^{-2} - \tilde{w}_k(y)^{-2} \}

Lemma 9.10 and (9.61) take care of the first term. Conditions (9.54), (9.53), Lemma 9.10, (9.56) and (9.55) take care of the second term.

Let \( t_k(x) \geq 0 \) be such that
\[ t_k(x) = 0 \quad \text{if} \quad |g_{k-1}(x)| \leq w_{k-1}(x)/2, \]
\[ t_k(x) = 1 \quad \text{if} \quad |g_{k-1}(x)| \geq 3w_{k-1}(x)/4, \]
\[ |t_k(x) - t_k(y)| \leq 2^k|x - y|. \]

Set
\[ \varphi_k(x) = t_k(x)(|k(x)| - \tilde{w}_k(x))U(k(x)), \]
\[ g_k(x) = k(x) - \varphi_k(x). \]

By (9.32)-(9.34) this definition of \( g_k \) coincides with (9.9).

Condition (C.4) follows from (9.62) and (9.58). Condition (C.5) follows from the inequality
\[ |\varphi_k(x) - \varphi_k(y)| \leq |t_k(x) - t_k(y)| \left| |k(x)| - \tilde{w}_k(x) \right| \]
\[ + t_k(y) \left| \left| k(x) \right| - \tilde{w}_k(x) \right| - \left( \left| k(y) \right| - \tilde{w}_k(y) \right) \right| \]
\[ + \left| \varphi_k(y) \right| \left| U(k(x)) - U(k(y)) \right|, \]

when combined with (9.62), (9.64), (9.58), (9.59), (C.4), (9.50), Lemma 9.2 and \( c_02^{M(n+1)} < 1 \). Condition (C.6) follows from (C.6)', (9.62) and (9.40). Condition (C.7) is clear from the definition of \( \varphi_k(x) \), part (i) of Lemma 9.11 and (9.58). Condition (C.8) follows from (9.41) and (C.5) if \( c_0 \) is small enough depending on \( M \) and \( C_{9,1} \).
10. Proof of Remark 3. In the proof of Main Lemma, if \( f \) is \( \mathbb{R}^m \)-valued, then \( f_M \) and \( \{b_j\} \) are \( \mathbb{R}^m \)-valued. By Remark 2.2 of [32], if \( \nu \in \mathbb{R}^n \cap \Sigma_{2m-1} \) and \( \theta_j(\xi) = \bar{\theta}_j(-\xi) \) for \( j = 1, \ldots, m \), then we can take \( p(x) \) in Lemma 6.1 to be \( \mathbb{R}^m \)-valued. Thus, if \( f \) is \( \mathbb{R}^m \)-valued, then we can take \( \bar{g} \) and \( \{f_j\} \) in Lemma 9.8 to be \( \mathbb{R}^m \)-valued. Thus we can take \( g \) in Main Lemma to be \( \mathbb{R}^m \)-valued.

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Received January 11, 1983. Supported in part by NSF MCS-8203319 and Grant-in-Aid for Scientific Research.

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