GYSIN HOMOMORPHISM AND SCHUBERT CALCULUS

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Let $G$ be a connected reductive linear algebraic group over the field of complex numbers, and $B$ a fixed Borel subgroup of $G$. The study of the homological properties of $G/B$ can be carried out by two well-known methods. The first of these methods is due to A. Borel and involves the identification of the cohomology ring of $G/B$ with the quotient ring of the ring of polynomials on the Lie algebra $\mathfrak{h}$ of the Cartan subgroup $H \subset G$ by the ideal generated by the $W$-invariant polynomials (where $W$ is the Weyl group of $G$). The second method is classical, and based on the calculation of the homology with the aid of the partition of $G/B$ into cells, the so-called Schubert cells. The correspondence between these approaches has been studied in the paper by Bernstein, Gel'fand and Gel'fand, where in the quotient ring of the polynomial ring figuring in Borel's model of the cohomology, the authors have found a symmetrical basis dual to the Schubert cells. Moreover, they have given a formula (Intersection formula) which expresses the intersection of any Schubert cell with a cell of codimension one. In the same paper, the authors have also generalized these results, except the intersection formula, to the case when $B$ is replaced by an arbitrary parabolic subgroup $P \subset G$.

On the other hand, for any parabolic subgroup $P \subset G$ containing $B$, the cohomology Gysin homomorphism of $\pi: G/B \to G/P$ has been studied by Akyildiz and Carrell, where an explicit formula has been obtained between the rings figuring in Borel's model of the cohomologies. This formula also enables one to obtain some of the results mentioned above for $G/P$ from the corresponding results on $G/B$. In this note, we consider the case where $G = \text{GL}(n+1)$ and $G/P$ is the Grassmann manifold. By using the explicit description of the Gysin homomorphism and the intersection formula given in the cohomology ring of $G/B$ we obtain three main theorems of the symbolic formalism, known as Schubert Calculus, concerning the cohomology ring structure of the Grassmann manifold. Although there are several different approaches for proving these theorems (see Kleiman and Laksov), it seems that none of them uses the cohomology ring structure of $\text{GL}(n+1)/B$, where $B$ is the group of upper triangular matrices in $\text{GL}(n+1)$. We thus hope that this alternative point of view may be used to understand the generalized Schubert Calculus.

1. Gysin homomorphism of $\pi$: $G/B \to G/P$. In this section we first give the rings figuring in Borel's model of the cohomologies of $G/B$, $G/P$, and then we write the formula for the cohomology Gysin homomorphism of $\pi$: $G/B \to G/P$ between these rings. Here $G$ is a connected
reductive linear algebraic group over the field of complex numbers, $B$ is a fixed Borel subgroup of $G$, and $P$ is a parabolic subgroup of $G$ containing $B$. We adopt the following notation: $B_u$ the unipotent radical of $B$, $H$ a fixed maximal torus contained in $B$, $\mathfrak{g}$ the Lie algebra of $G$, $\mathfrak{h}$ and $\mathfrak{h}_u$ the Lie algebras of $H$ and $B_u$ respectively, $\Delta \subset \mathfrak{h}^*$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$, $\Delta_+$ the set of positive roots, namely the set of roots of $\mathfrak{h}$ in $\mathfrak{h}_u$, $\Sigma \subset \Delta_+$ the set of simple roots, $W$ the Weyl group of $G$, $\Theta$ is the subset of $\Sigma$ such that the parabolic subgroup $P_{\Theta}$ corresponding to $\Theta$ is equal to $P$, $W_{\Theta}$ the subgroup of $W$ generated by the reflections $\sigma_\alpha$, $\alpha \in \Theta$, $\Delta_{\Theta}$ the subset of $\Delta_+$ consisting of linear combinations of the elements of $\Theta$, $W_{\Theta}$ the set of $w \in W$ such that $w\Theta \subset \Delta_+$.

Let $X(H)$ be the group of characters of $H$, and $R = \text{Sym}(C \otimes \mathbb{Z} X(H))$, the symmetric algebra of $C \otimes \mathbb{Z} X(H)$. We denote by the same symbol an element of $X(H)$ and the corresponding element of $\Delta$ when it can be done without any ambiguity. For the basic facts about algebraic groups and the Borel model of the cohomologies of homogeneous spaces, the reader is referred to [6] and [1], [3], [5] respectively. Since $W$ acts on $X(H)$ we get an action of $W$ on $R$ in the usual way. Let $R^W$ be the ring of invariants of $W$, and $I = \{ f \in R^W : f(0) = 0 \}$. For any character $\alpha$ of $H$, let $L_\alpha$ be the associated homogeneous line bundle on $G/B$: $L_\alpha = G \times \mathbb{C}/\sim$, $(g, z) \sim (g', z')$ if and only if $g' = gb$ for some $b$ in $B$ and $z' = \alpha(b^{-1})z$, where $\alpha$ is extended on $B$ with $\alpha(u) = 1$ for $u$ in $B_u$. It is shown in [1] that the characteristic homomorphism $c: R \to H^*(G/B, \mathbb{C})$ determined by $c(\alpha) = c_1(L_\alpha)$, the first Chern class of $L_\alpha$ for any $\alpha \in X(H)$, induces an isomorphism of graded algebras $\bar{c}: R/IR \isom H^*(G/B, \mathbb{C})$ (see also [5]). Moreover, if $R^W_{\Theta}$ is the ring of invariants of $W_{\Theta}$, then one also has the following commutative diagram between the graded algebras (cf. [1]):

\[
\begin{array}{ccc}
R/IR & \xrightarrow{\sim} & H^*(G/B, \mathbb{C}) \\
\uparrow & & \uparrow \pi^* \\
R^W_{\Theta}/IR^W_{\Theta} & \xrightarrow{\sim} & H^*(G/P, \mathbb{C}),
\end{array}
\]

where $\pi^*$ is the cohomology map of $\pi: G/B \to G/P$. This is the Borel model of the cohomologies of $G/B$ and $G/P$.

For the cohomology Gysin homomorphism of $\pi: G/B \to G/P$, we first recall the operators $A_\omega: R \to R$ for $\omega \in W$ (cf. [3], [4], [5]). For each $\alpha \in \Delta$ the element $f - \sigma_\alpha \cdot f$ is divisible by $\alpha$ for any $f \in R$, where $\sigma_\alpha$ is the reflection corresponding to $\alpha$. Thus $A_\alpha: R \to R$, $A_{\alpha} f = (f - \sigma_\alpha \cdot f)/\alpha$, is a well-defined $R^W$-linear operator on $R$. Let $\alpha_1, \ldots, \alpha_l \in \Sigma$, and let $\omega = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ be any element of $W$. Then
(i) if the length \( l(\omega) \) of \( \omega \) is less than \( l \), then \( A_{a_1} \cdots A_{a_l} = 0 \),
(ii) if \( l(\omega) = l \), then the operator \( A_{a_1} \cdots A_{a_l} \) depends only on \( \omega \) and not on the representation of \( \omega \) in the form \( \omega = \sigma_{a_1} \cdots \sigma_{a_l} \). In this case we put \( A_{\omega} = A_{a_1} \cdots A_{a_l} \).

We note that the operator \( A_{\omega} : R \to R \) preserves the ideal \( IR \), and thus induces an operator \( \overline{A}_{\omega} : R/IR \to R/IR \) of homogeneous degree \( -l(\omega) \). Moreover for \( \omega_1, \omega_2 \in W \), one has \( \overline{A}_{\omega_1}\overline{A}_{\omega_2} = \overline{A}_{\omega_1\omega_2} \) if \( l(\omega_1\omega_2) = l(\omega_1) + l(\omega_2) \), \( \overline{A}_{\omega_1}\overline{A}_{\omega_2} = 0 \) otherwise (cf. [4], [5]). It is shown in [2] that the cohomology Gysin homomorphism \( \pi_* : H^*(G/B, C) \to H^*(G/P, C) \) of the natural map \( \pi : G/B \to G/P \) is given by the following commutative diagram:

\[
\begin{align*}
\tilde{c} : R/IR & \quad \sim \quad H^*(G/B, C) \\
\overline{A}_{\tau_\Theta} \downarrow & \quad \downarrow \pi_* \\
\tilde{c} : R^{W_\Theta}/IR^{W_\Theta} & \quad \sim \quad H^*(G/P, C),
\end{align*}
\]

where \( \tau_\Theta \) is the unique element of \( W_\Theta \) of maximal length. Moreover, one has the following formula for \( \overline{A}_{\tau_\Theta} \) (cf. [2]):

\[
(2) \quad \overline{A}_{\tau_\Theta}(f) = \sum_{\tau \in W_\Theta} \det(\tau) \tau \cdot f / \prod_{\alpha \in \Delta_\Theta} \alpha.
\]

This is the explicit description of the cohomology Gysin homomorphism of \( \pi : G/B \to G/P \).

2. Schubert calculus. The correspondence between the Borel's model of the cohomology of \( G/P \) and the calculation of the homology with the aid of partition of \( G/P \) into Schubert cells has been studied in [3]. In the case of \( G/B \), the authors have also given a formula (intersection formula) in \( R/IR \) which expresses the intersection of any Schubert cell with a cell of codimension one. In this section, we first give this correspondence. Then we write the intersection formula only for the case \( G = GL(n + 1) \). Later, by using this formula and the results of §1. We obtain three main theorems, known as Schubert Calculus, concerning the cohomology ring structure of Grassmannians.

We keep the notation of §1. Let \( \omega_0 \) be the unique element of \( W \) of maximal length, and \( P_{\omega_0} = (1/|W|)\prod_{\alpha \in \Delta^+} \alpha \mod IR \), where \( |W| \) is the order of \( W \). For each \( \omega \in W \), let \( P_{\omega} = A_{\omega^{-1}\omega_0}(P_{\omega_0}) \). It is shown in [3] that \( \{ P_{\omega} : \omega \in W \} \) is a free \( Z \)-basis of \( R/IR \) which is dual to the Schubert cells. Namely, for each \( \omega \in W \), let \( X_\omega \) denote the cycle class of the closure of \( B(\omega_0\omega)x_0 \) in \( H_*(G/B, Z) \), \( x_0 = B \in G/B \). Then under the Poincare
duality map $\mathcal{P}$, $\mathcal{P}(X_{\omega}) = \bar{c}(P_{\omega})$. Similarly, for the parabolic subgroup $P = P_{\emptyset}$ of $G$ containing $B$, one has the following: For each $\sigma \in W_{\emptyset}^+$ let $Y_\sigma$ be the cycle class of the closure of $B(\omega_0^\sigma)\pi(x_0)$ in $H_*(G/P, Z)$, where $\pi: G/B \to G/P$ is the natural map. Then the Poincare dual $\mathcal{P}(Y_\sigma)$ of $Y_\sigma$, $\sigma \in W_{\emptyset}^+$, is equal to $\bar{c}(P_{\sigma} \mod IR_{W^\sigma})$. Moreover, $\{P_{\sigma} \mod IR_{W^\sigma}: \sigma \in W_{\emptyset}^+\}$ is a free $Z$-basis of $R_{W^\emptyset}/IR_{W^\emptyset}$. Although the proof of this fact is given in [3], for the sake of completeness of our approach we will give a different proof by using the Gysin homomorphism (cf. [2]). Let $\tau_\emptyset$ be the unique element of $W_{\emptyset}$ of maximal length. Since $l(\sigma \tau) = l(\sigma) + l(\tau)$ for any $\sigma \in W_{\emptyset}$ and $\tau \in W_{\emptyset}$, we have

$$\bar{A}_{\tau_\emptyset}(P_{\sigma \tau}) = \bar{A}_{\tau_\emptyset}S_{\sigma_1-1, \omega_0^\sigma} \bar{A}_{\tau_\emptyset}(P_{\omega_0}) = \delta_{\tau_\emptyset, \tau} A_{\sigma_1-1, \omega_0^\sigma} \bar{A}_{\tau_\emptyset}(P_{\omega_0}) = \delta_{\tau_\emptyset, \tau} P_{\sigma \tau}.$$  

This shows that $\{P_{\sigma} \mod IR_{W^\emptyset}: \sigma \in W_{\emptyset}^+\}$ is a basis of $R_{W^\emptyset}/IR_{W^\emptyset}$, because the Gysin homomorphism $\bar{A}_{\tau_\emptyset}$ is surjective and any element $\omega$ of $W$ can be written uniquely in the form $\omega = \sigma \tau$ for some $\sigma \in W_{\emptyset}^+$ and $\tau \in W_{\emptyset}$. On the other hand, it is clear that for $\sigma \in W_{\emptyset}^+$, the cycle class $X_{\sigma \tau_\emptyset}$ goes under the homology Gysin homomorphism of $\pi: G/B \to G/P$ to the cycle class $Y_\sigma$ in $H_*(G/P, Z)$. Thus from the commutativity of the diagram (1) and $\bar{c}(P_{\sigma \tau_\emptyset}) = \mathcal{P}(X_{\sigma \tau_\emptyset})$, we get $\bar{c}(P_{\sigma} \mod IR_{W^\emptyset})) = \mathcal{P}(Y_\sigma)$ for any $\sigma \in W_{\emptyset}^+$.

There is a formula (intersection formula) given in $R/IR$ which expresses the product $P_{\sigma_1} \cdot P_{\sigma_2}$ in terms of $P_{\omega}$, where $\sigma_1$ is the reflection corresponding to the simple root $\alpha \in \Sigma$ (cf. [3]). We will now write this formula for the group of invertible matrices $GL(n + 1)$. Let $G$ be $GL(n + 1)$, $B$ the group of upper triangular matrices in $G$, and $H$ the group of diagonal matrices in $B$. Then

$$R = \text{Sym}(C \otimes Z X(H)) = C[x_0, \ldots, x_n],$$

$$\Delta_+ = \{\alpha_{i, j} = x_i - x_j: 0 \leq i < j \leq n\},$$

$$\Sigma = \{\alpha_{i, i+1}: 0 \leq i \leq n - 1\}.$$  

Let $S_{n+1}$ be the symmetric group in $0, 1, \ldots, n$, and for $\sigma \in S_{n+1}$ let $\sigma(I)$ be the permutation matrix obtained from $(n + 1) \times (n + 1)$ identity matrix $I$. The homomorphism $\sigma^{-1} \to \sigma(I)$ gives an isomorphism between $S_{n+1}$ and the Weyl group $W$ of $G$. Moreover it can be checked that the action of $W$ on $R$ is given by

$$\sigma(I) \cdot f(x_0, \ldots, x_n) = \sigma f(x_0, \ldots, x_n) = f(x_{\sigma_0}, \ldots, x_{\sigma_n}), \text{ for any } \sigma \in S_{n+1}.$$  

Thus $IR$ is the ideal generated by the elementary symmetric functions of $x_0, \ldots, x_n$. 
We will now write the operators $A_\omega$, and the elements $P_\omega$ of $R/IR$ in terms of the elements of $S_{n+1}$. Let $\sigma = (a_0 \cdots a_n)$ be any element of $S_{n+1}$, namely $\sigma i = a_i$ for $i = 0,1,\ldots,n$. Then the reflection $\sigma_{a_i,j}$ corresponding to the root $\alpha_{i,j}$ is given by $\sigma_{a_i,j} = (i,j)(I)$, where $(i,j) \in S_{n+1}$ is the transposition obtained by changing $i$ with $j$. Let $A_{(i,i+1)} = A_{\alpha_{i,i+1}}$, and $P_\sigma = P_{\sigma(I)}$ for $\sigma \in S_{n+1}$. Then for any $\sigma \in S_{n+1}$, we get

$$A_{\sigma^{-1}} = A_{\sigma(I)}, \quad \text{and} \quad P_\sigma = A_{\sigma_0}(P_{\sigma_0}),$$

where $\omega_0 = (n \ 0 \cdots 1 \ 0)$, and

$$P_{\omega_0} = \frac{1}{(n+1)!} \prod_{0 \leq i < j \leq n} (x_i - x_j) \pmod{IR}.$$

The intersection formula given in [3] can now be written as follows:

**Theorem** (Intersection formula, cf. [3, p. 17]). For any $r = 0,1,\ldots,n-1$, and $\sigma = (a_0 \cdots a_n) \in S_{n+1}$, we have

$$P_{(r,r+1)} P_\sigma = \sum P_{(i,j)\sigma},$$

where the summation is over all $0 \leq i < j \leq n$ such that

(a) $i \leq r < j$,

(b) $a_i < a_j$,

(c) if $i < k < j$, then either $a_k < a_i$ or $a_k > a_j$.

We are now ready to discuss the cohomology ring structure of Grassmannians. For any $q = 0,1,\ldots,n-1$, let $\Theta = \Sigma \setminus \{a_{q,q+1}\}$. Then the parabolic subgroup $P = P_\Theta$ of $G$ corresponding to $\Theta$ consists of all matrices of the form $(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix})$, where $A \in \text{GL}(q+1)$, and $B \in \text{GL}(n-q)$. Thus $G/P$ is the Grassmann manifold $G_{q,n}$ of $(q+1)$-planes in $C^{n+1}$. Moreover it is easy to see that corresponding to $\Theta$, we have

$$W_\Theta = \{(a_0 \cdots a_q \cdot q+1 \cdots a_n) \in S_{n+1}: 0 \leq a_0, \ldots, a_q \leq q, \quad q+1 \leq a_{q+1}, \ldots, a_n \leq n\},$$

and

$$W_\Theta^\perp = \{(a_0 \cdots a_q \cdots a_n) \in S_{n+1}: a_0 < \cdots < a_q, a_{q+1} < \cdots < a_n\}.$$

A sequence of integers $0 \leq a_0 < \cdots < a_q \leq n$ corresponds to a unique element $(a_0 \cdots a_q \cdot q+1 \cdots a_n)$ of $W_\Theta^\perp$, and we denote this element by $\sigma(a_0,\ldots,a_q)$. For any sequence of integers $0 \leq a_0 < \cdots < a_q \leq n$, let $\Omega(a_0,\ldots,a_q)$ denote the Poincare dual of $Y_{\sigma(a_0,\ldots,a_q)}$ in $H^*(G_{q,n}, Z) \subset H^*(G_{q,n}, C)$. $\Omega(a_0,\ldots,a_q)$ is called the Schubert cycle corresponding to
0 ≤ a_0 < ⋯ < a_q ≤ n. The Schubert cycle corresponding to 0 < 1 < ⋯ < q - 1 < q + k ≤ n for k = 0, 1, ⋯, n - q is called the special Schubert cycle, and it is denoted by Ω(k).

**Theorem (The basis theorem).** The Schubert cycles Ω(a_0, ⋯, a_q), 0 ≤ a_0 < ⋯ < a_q ≤ n, form a basis of $H^*(G, n, Z)$.

**Proof.** It follows from the above observations because $Ω(a_0, ⋯, a_q) = P_{σ(a_0, ⋯, a_q)} (mod IR)$ for any sequence of integers $0 ≤ a_0 < ⋯ < a_q ≤ n$.

We will now compute $Ω(a_0, ⋯, a_q) = P_{σ(a_0, ⋯, a_q)} (mod IR)$ for any sequence of integers $0 ≤ a_0 < ⋯ < a_q ≤ n$. The Schubert cycle corresponding to $0 < 1 < ⋯ < q ~ 1 < q + k < n$ for $k = 0, 1, ⋯, n - q$ is called the special Schubert cycle, and it is denoted by $Ω(k)$.

**Proposition.** Let $σ = (a_0 ⋯ a_q ⋯ a_n) ∈ W_Θ$. Then we have $P_{σ(σ)} = x_0^a ⋯ x_q^a (mod IR)$.
We note that the fundamental dominant weight $\chi_r$ corresponding to the simple root $\alpha_{r,r+1}$ is given by $\chi_r = x_0 + \cdots + x_r$. By [3, p. 18], since $P_{(r,r+1)} = \chi_r \pmod{IR}$, we have $P_{(r,r+1)} - P_{(r-1,r)} = x_r \pmod{IR}$ for $r = 0, 1, \ldots, n - 1$.

We prove the proposition by induction on $q$. If $q = 0$, then $\tau_0 \sigma = (b_0 \cdots b_n)$, $b_1 < \cdots < b_n$. In this case we can have either $\tau_0 \sigma = e$ (the identity) or $\tau_0 \sigma = \omega(p + 1) = (p + 0 \cdots p p + 2 \cdots n)$ for some $p = 0, 1, \ldots, n - 1$. If $\tau_0 \sigma = e$, then the claim follows from the identity. If $\tau_0 \sigma = \omega(p + 1)$ for some $p = 0, 1, \ldots, n - 1$, then we claim that $P_{\omega(p+1)} = x_{b_{p+1}}^b \pmod{IR}$. We prove this by induction on $p$. If $p = 0$, then $\omega(1) = (0,1)$. Since $P_{(0,1)} = x_0 = x_0 \pmod{IR}$, we have the claim for $p = 0$. We now assume $P_{\omega(p)} = x_{b_p}^b \pmod{IR}$. But by the intersection formula we have $P_{\omega(p+1)} = P_{\omega(p)} \omega(p+1) \pmod{IR}$. This finishes the induction argument for $q = 0$. We now assume the claim for $q - 1$. Let $\tau_0 \sigma = (a_q \cdots a_0 a_{q+1} \cdots a_n) = (b_0 \cdots b_n)$. If $a_0 = b_q = 0$, then the claim follows from the induction hypothesis for $q - 1$ because $b_0 > \cdots > b_{q-1}$, $0 = b_q < \cdots < b_n$. If $b_q > 0$, then we claim that $P_{\tau_0 \sigma} = x_{b_0}^b \cdots x_{b_q}^b \pmod{IR}$. This will be proved by induction on $b_q$. If $b_q = 1$, then $b_{q+1} = 0$. In this case $\omega = (q, q + 1) \tau_0 \sigma = (b_0 \cdots b_{q-1} 0 1 b_{q+2} \cdots b_n)$, and thus (by the induction hypothesis for $q - 1$) $P_{\omega} = x_{b_0}^b \cdots x_{b_{q-1}}^b \pmod{IR}$. Since $P_{(q,q+1)} - P_{(q-1,q)} = x_q \pmod{IR}$, we have from above

\[
\omega = (q, q + 2 + s) \tau_0 \sigma = (b_0 \cdots b_{q-1} b_q - 1 b_{q+1} \cdots b_{q+1+s} b_q b_{q+3+s} \cdots b_n).
\]

Since $b_0 > \cdots > b_q - 1, b_{q+1} < \cdots < b_{q+1+s} < b_q < b_{q+3+s} < \cdots < b_n$, by the induction hypothesis we have $P_{\omega} = x_{b_0}^b \cdots x_{b_{q-1}}^{b_{q-1}} \pmod{IR}$. But from above and the induction hypothesis, we get

\[
(P_{(q,q+1)} - P_{(q-1,q)}) P_{\omega} = P_{(q,q+2+s)} P_{\omega} = P_{\tau_0 \sigma} = x_q x_{b_0}^b \cdots x_{b_{q-1}}^{b_{q-1}} \pmod{IR},
\]

which is the claim. This completes the proof of the proposition.
**Corollary.** For any sequence of integers $0 < a_0 < \cdots < a_q < n$, we have

$$P_{\sigma(a_0, \ldots, a_q)} = \begin{vmatrix} h(a_0) & \cdots & h(a_0 - q) \\ \vdots & \ddots & \vdots \\ h(a_q) & \cdots & h(a_q - q) \end{vmatrix} \pmod{IR},$$

where $h(m)$ is the $m$th complete homogeneous symmetric function of $x_0, x_1, \ldots, x_q$. In particular $P_{\sigma(0,1,\ldots,q-1,q+k)} = h(k) \pmod{IR}$ for any $k = 0, 1, \ldots, n - q$.

**Proof.** By the proposition we have $P_{\sigma(a_0, \ldots, a_q)} = x_0^{a_q} \cdots x_q^{a_0} \pmod{IR}$. On the other hand, from formula (2) we get

$$P_{\sigma(a_0, \ldots, a_q)} = \overline{A}_{\sigma}(P_{\sigma(a_0, \ldots, a_q)}) = \frac{\sum_{\tau \in W_{\sigma}} \det(\tau)x_0^{a_q} \cdots x_q^{a_0}}{\prod_{0 \leq i < j \leq q}(x_i - x_j)} \pmod{IR}.$$  

But this bialternant is equal to (cf. [8, p. 92])

$$\begin{vmatrix} h(a_q - q) & \cdots & h(a_q) \\ \vdots & \ddots & \vdots \\ h(a_0 - q) & \cdots & h(a_0) \end{vmatrix} = \begin{vmatrix} h(a_0) & \cdots & h(a_0 - q) \\ \vdots & \ddots & \vdots \\ h(a_q) & \cdots & h(a_q - q) \end{vmatrix},$$

which gives the claim.

**Theorem (The determinantal formula).** For all sequence of integers $0 < a_0 < \cdots < a_q < n$ the following formula holds in the cohomology ring $H^*(G_q, n, Z)$:

$$\Omega(a_0, \ldots, a_q) = \begin{vmatrix} \Omega(a_0) & \cdots & \Omega(a_0 - q) \\ \vdots & \ddots & \vdots \\ \Omega(a_q) & \cdots & \Omega(a_q - q) \end{vmatrix},$$

where we agree to put $\Omega(k) = 0$ for $k > n - q$ or $k < 0$.

**Proof.** It follows from the Corollary above the fact that; if $k > n - q$, then the $k$th complete homogeneous symmetric function of $x_0, \ldots, x_q$
(being equal to ± the $k$th elementary symmetric function of $x_{q+1}, \ldots, x_n$ (mod $IR$)) is equal to zero (mod $IR^w$), (cf. [8]).

This theorem, together with the basis theorem, implies that the special Schubert cycles generate the cohomology ring $H^*(G_{q,n}, Z)$ as a $Z$-algebra. Moreover, it reduces the problem of determining the product of two arbitrary Schubert cycles to the case where one (or for that matter, each) is a special Schubert cycle. This case is handled by the following theorem.

**Theorem (Pieri’s formula).** For all sequences of integers $0 \leq a_0 < \cdots < a_q \leq n$ and $k = 0, 1, \ldots, n - q$, the following formula holds in the cohomology ring $H^*(G_{q,n}, Z)$:

$$
\Omega(k) \Omega(a_0, \ldots, a_q) = \sum \Omega(b_0, \ldots, b_q),
$$

where the sum ranges over all sequences of integers $b_0 < \cdots < b_q$ satisfying $0 \leq a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_q \leq b_q \leq n$ and $\sum_{i=0}^q (b_i - a_i) = k$.

**Proof.** For any sequence of integers $0 \leq a_0 < \cdots < a_q \leq n$ and for $k = 0, 1, \ldots, n - q$, by the proposition and its corollary we have

$$
P_{\tau_0 \sigma(a_0, \ldots, a_q)} = x_0^{a_0} \cdots x_q^{a_q} \pmod{IR}
$$

and $P_{\sigma(0,1,\ldots,q-1,q+k)} = h(k) = \sum x_0^{l_0} \cdots x_q^{l_q} \pmod{IR}$, where the sum ranges over all integers $l_i \geq 0$ such that $\sum_{i=0}^q l_i = k$. Thus we have

$$
P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\tau_0 \sigma(a_0, \ldots, a_q)} = \sum x_0^{a_0+l_0} \cdots x_q^{a_q+l_0}
$$

$$
= f(x_0, \ldots, x_q) \pmod{IR}
$$

where the sum ranges over all integers $l_i \geq 0$ such that $\sum_{i=0}^q l_i = k$. We will now apply the operator $\tilde{\alpha}_{\tau_0}$ to equation (3). Since $\tilde{\alpha}_{\tau_0}$ is $R^w_{\Theta_1}$-linear and $h(k) \in R^w_{\Theta_1}$ (where $R^w_{\Theta_1}$ is the ring of invariants of the group $W_{\Theta_1}$), we have

$$
\tilde{\alpha}_{\tau_0} \left( P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\tau_0 \sigma(a_0, \ldots, a_q)} \right) = P_{\sigma(0,1,\ldots,q-1,q+k)} \tilde{\alpha}_{\tau_0} \left( P_{\tau_0 \sigma(a_0, \ldots, a_q)} \right)
$$

$$
= P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\sigma(a_0, \ldots, a_q)}.
$$

Let $J_{\Theta_1}(g) = \sum_{\tau \in W_{\Theta_1}} \det(\tau) \tau g$ for $g \in R$. Since $J_{\Theta_1}(\tau g) = \det(\tau) J_{\Theta_1}(g)$ for any $\tau \in W_{\Theta_1}$, we have in particular

(i) $J_{\Theta_1}(X_0^{m_0} \cdots X_q^{m_q}) = \det(\tau) J_{\Theta_1}(X_0^{m_0} \cdots X_q^{m_q})$ for any $\tau \in W_{\Theta_1}$,

(ii) $J_{\Theta_1}(X_0^{m_0} \cdots X_q^{m_q}) = 0$ if $m_i = m_j$ for some $i \neq j$.

By using (i), (ii), and the fact that $X_0^{n+1} = 0 \pmod{IR}$, it is to see that

$$
J_{\Theta_1}(f(x_0, \ldots, x_q)) = \sum J_{\Theta_1}(x_0^{a_0+\lambda_0} \cdots x_q^{a_q+\lambda_0}),
$$
where the sum ranges over all integers \( \lambda_i \geq 0 \) satisfying \( a_0 + \lambda_0 < \cdots < a_q + \lambda_q \leq n \) and \( \sum_{i=0}^{q} \lambda_i = k \). This gives by the proposition

\[
\tilde{J}_{\Theta_i}(\tilde{f}) = \sum \tilde{J}_{\Theta_i}(P_{\tau_0 \sigma(b_0, \ldots, b_q)}),
\]

where \( \tilde{f} = f(x_0, \ldots, x_q) \) (mod \( IR \)), and the sum ranges over all sequences of integers \( b_0 < b_1 < \cdots < b_q \) satisfying \( 0 \leq a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_q \leq b_q \leq n \) and \( \sum_{i=0}^{q} (b_i - a_i) = k \). Thus from formula (2) we get

\[
\tilde{A}_{\tau_0}(\tilde{f}) = \sum \tilde{A}_{\tau_0}(P_{\tau_0 \sigma(b_0, \ldots, b_q)}) = \sum P_{\sigma(b_0, \ldots, b_q)},
\]

where the sum is as above. By comparing (3), (4) and (5) we obtain

\[
P_{\sigma(0,1,\ldots,q-1,q+k)} P_{\sigma(a_0,\ldots,a_q)} = \sum P_{\sigma(b_0,\ldots,b_q)},
\]

where the sum ranges over all sequences of integers \( b_0 < b_1 < \cdots < b_q \) satisfying \( 0 \leq a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_q \leq b_q \leq n \) and \( \sum_{i=0}^{q} (b_i - a_i) = k \). By considering equation (6) (mod \( IR^{W_{\Theta}} \)), we get the claim, because \( \tilde{c} \) is a graded algebra isomorphism and \( \tilde{c}(P_{\sigma(a_0,\ldots,a_q)} \mod IR^{W_{\Theta}}) = \Omega(a_0,\ldots,a_q) \).

REFERENCES


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