REMARKS ON THE PAPER: "BASIC CALCULUS OF VARIATIONS"

JOHN MACLEOD BALL
REMARKS ON THE PAPER
‘BASIC CALCULUS OF VARIATIONS’

J. M. BALL

We show that a condition studied in E. Silverman’s paper is not, as claimed, necessary for lower semicontinuity of multiple integrals in the calculus of variations.

The purpose of this note is to show that a condition studied in [7] is not, as claimed, a necessary condition for lower semicontinuity of multiple integrals in the calculus of variations. To keep things simple we consider integrals of the form

$$I_F(y) = \int_G F(y'(x)) \, dx,$$

where $G \subset \mathbb{R}^k$ is a bounded domain, $y: G \to \mathbb{R}^N$, $y'(x) = (\partial y^i/\partial x^a)$, and $F: M^{N \times k} \to \mathbb{R}$ is continuous. Here $M^{N \times k}$ denotes the linear space of real $N \times k$ matrices. We suppose throughout that $K \geq 2$, $N \geq 2$. In [7] $F$ is called $T$-convex if there exists a convex function $f$, defined on $\mathbb{R}^r$, $r = (\binom{N+k}{k}) - 1$, such that

$$F(p) = f(\tau(p)) \quad \text{for all } p \in M^{N \times k},$$

where $\tau(p)$ denotes the minors of $p$ of all orders $j$, $1 \leq j \leq \min(k, N)$, arranged in some prescribed order. $T$-convexity of $F$ was studied in [1, 2, 3] under a different name, polyconvexity, which we shall use in the remainder of this note, and it is equivalent to a condition introduced earlier by Morrey [4, p. 49]. (These papers contain lower semicontinuity and existence theorems for polyconvex integrands of the same type as given in [7, §§4–7].) Let us say that $I_F$ is lsc if $I_F(y) \leq \liminf_{j \to \infty} I_F(y_j)$ whenever $y_j \to y$ uniformly on $G$ with $\sup_{x, \bar{x} \in G} |y_j(x) - y_j(\bar{x})| \leq C < \infty$ for all $j$. (Equivalently, if $G$ has sufficiently regular boundary then $I_F$ is lsc if and only if $I_F$ is sequentially weak* lower semicontinuous on the Sobolev space $W^{1, \infty}(G; \mathbb{R}^n)$.) A consequence of [7, Theorem 3.6] is that $I_F$ lsc implies $F$ polyconvex; that this conclusion is false was pointed out implicitly by Morrey [4, p. 26]. Morrey’s remark is based on an example due to Terpstra [8] of a quadratic form

$$Q(p) = \sum_{1 \leq i, j \leq N} a_{i \alpha, j \beta} p_{i \alpha} p_{j \beta}.$$
having constant coefficients $a_{\alpha\beta}$ and with the properties

(i) (rank 1 convexity) $Q(\lambda \otimes \mu) \geq 0$ for all $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^k$,

(ii) there is no linear combination $\tilde{Q}(p)$ of $2 \times 2$ minors of $p$ such that

$$Q(p) \geq \tilde{Q}(p) \quad \text{for all } p \in M^{N \times k}.$$  

Terpstra showed that such quadratic forms exist if and only if $k \geq 3$ and $N \geq 3$. By Morrey [4, Theorem 5.2] $I_Q$ is lsc if and only if $Q$ satisfies (i). But if $Q$ satisfies (ii) then $Q$ is not polyconvex; more generally, we have the following proposition.

**Proposition.** Let $F(p) = Q(p)$ in a neighbourhood of $p = 0$. If $Q$ satisfies (ii) then $F$ is not polyconvex.

**Proof.** Suppose $F$ is polyconvex. By the convexity of $f$ there exists $\theta \in \mathbb{R}^r$ such that

$$F(p) = f(\tau(p)) \geq f(0) + \langle \theta, \tau(p) \rangle \quad \text{for all } p \in M^{N \times k}.$$  

We write $\langle \theta, \tau(p) \rangle = \sum_{j=1}^{\min(k,N)} \tilde{Q}_j(p)$, where each $\tilde{Q}_j(p)$ is a linear combination of $j \times j$ minors of $p$. Note that $F(0) = f(0) = 0$. For any $p$ and for $|t|$ sufficiently small we thus have

$$F(tp) = t^2Q(p) \geq \sum_{j=1}^{\min(k,N)} t^j\tilde{Q}_j(p).$$

Dividing by $|t|$ and letting $t \to 0$ we see that $\tilde{Q}_1(p) \equiv 0$. Dividing by $t^2$ and letting $t \to 0$ we obtain $Q(p) \geq \tilde{Q}_2(p)$, contradicting (ii). $\square$

Of course any $Q$ satisfying (i) and (ii) is not bounded below. However, applying the proposition to $F(p) = \max\{-1, Q(p)\}$ we see that if $Q$ satisfies (i), (ii) then $G(p) = \max\{0, 1 + Q(p)\}$ is nonnegative, $I_G$ is lsc (it is the maximum of two lsc functionals), but $G$ is not polyconvex.

The proof of Theorem 3.6 in [7] consists of first showing (Lemma 3.4, Corollary 3.5) that $I_F$ lsc implies $F$ polyconvex in the special case when $N \geq k$ and $F$ depends only on minors of maximal order $k$. This part of the proof does not appear to be complete. The general case is then reduced to the special one by adjoining new variables $\xi: G \to \mathbb{R}^k$ such that

$$F(y') = h\left(\tau\left(\begin{array}{c} \xi' \\ y' \end{array}\right)\right)$$

for some function $h$ depending only on $k$th order minors of the $(N + k) \times k$ matrix $(\xi')$; however, such a function $h$ does not in general exist, since all $k$th order minors of $(\xi')$ can be zero without determining $y'$. 
The example of Terpstra is neither explicit nor elementary, and being written in German is inaccessible to some. Recently D. Serre [5,6] has provided an explicit example, namely
\[ Q_\varepsilon(p) = H(p) - \varepsilon \sum_{i, \alpha=1}^{3} (p_{i\alpha})^2, \]
\[ H(p) = (p_{11} - p_{23} - p_{32})^2 + (p_{12} - p_{31} + p_{13})^2 \]
\[ + (p_{21} - p_{13} - p_{31})^2 + (p_{22})^2 + (p_{33})^2, \]
where \( N = k = 3 \) and \( \varepsilon \) is sufficiently small. To keep this note self-contained we now give a direct proof, following Serre [6], that \( Q_\varepsilon \) satisfies (i) and (ii). First we note that \( H(\lambda \otimes \mu) = 0 \) implies that
\[ \lambda_1 \mu_1 - \lambda_2 \mu_3 - \lambda_3 \mu_2 = \lambda_1 \mu_2 - \lambda_3 \mu_1 + \lambda_1 \mu_3 = \lambda_2 \mu_1 - \lambda_1 \mu_3 - \lambda_3 \mu_1 \]
\[ = \lambda_2 \mu_2 = \lambda_3 \mu_3 = 0, \]
and hence that \( \lambda = 0 \) or \( \mu = 0 \). Thus \( \inf_{|\lambda| = |\mu| = 1} H(\lambda \otimes \mu) \stackrel{\text{def}}{=} \varepsilon_0 \) is positive and (i) follows for \( \varepsilon < \varepsilon_0 \). Suppose for contradiction that
\[ Q_\varepsilon(p) > \tilde{Q}(p) = - \sum_{1 \leq i, \alpha \leq 3} A_{i\alpha} (\text{adj } p)_{i\alpha} \quad \text{for all } p, \]
where \( A \in M^{3 \times 3} \) is constant. Consider \( p \) of the form
\[ p = \begin{pmatrix} b + d & a - c & c \\ a + c & 0 & d \\ a & b & 0 \end{pmatrix}, \]
so that
\[ \text{adj } p = \begin{pmatrix} -bd & bc & d(a - c) \\ ad & -ac & c(a + c) - d(b + d) \\ b(a + c) & a(a - c) - b(b + d) & c^2 - a^2 \end{pmatrix}. \]
For such \( p \) we have \( H(p) = 0 \) and thus
\[ \sum_{1 \leq i, \alpha \leq 3} A_{i\alpha} (\text{adj } p)_{i\alpha} - \varepsilon(a^2 + b^2 + c^2 + d^2) \geq 0. \]
The left-hand side is a quadratic form in \( a, b, c, d \) given explicitly by
\[ a^2(A_{32} - A_{33} - \varepsilon) + b^2(-A_{32} - \varepsilon) + c^2(A_{23} + A_{33} - \varepsilon) \]
\[ + d^2(-A_{23} - \varepsilon) + (\text{terms in } ab, ac, ad, bc, bd, cd). \]
For this sum to be nonnegative the coefficients of \( a^2, b^2, c^2, d^2 \) must be nonnegative. But the sum of these coefficients is \(-4\varepsilon\), a contradiction.
Acknowledgment. I am grateful to R. V. Kohn and D. Serre for their comments.

REFERENCES


Received April 29, 1983 and in revised form September 9, 1983. Research supported by a U. K. Science and Engineering Research Council Senior Fellowship and by the Mathematical Sciences Research Institute, Berkeley, California.

HERIOT-WATT UNIVERSITY
EDINBURGH, EH 14 4AS
SCOTLAND
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)
University of California
Los Angeles, CA 90024

C. C. MOORE
University of California
Berkeley, CA 94720

J. DUGUNDJI
University of Southern California
Los Angeles, CA 90089-1113

ARTHUR OGUS
University of California
Berkeley, CA 94720

R. FINN
Stanford University
Stanford, CA 94305

HUGO ROSSI
University of Utah
Salt Lake City, UT 84112

HERMANN FLASCHKA
University of Arizona
Tucson, AZ 85721

H. SAMELSON
Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS  E. F. BECKENBACH  B. H. NEUMANN  F. WOLF  K. YOSHIDA
(1906–1982)

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA  UNIVERSITY OF OREGON
UNIVERSITY OF BRITISH COLUMBIA  UNIVERSITY OF SOUTHERN CALIFORNIA
CALIFORNIA INSTITUTE OF TECHNOLOGY  STANFORD UNIVERSITY
UNIVERSITY OF CALIFORNIA  UNIVERSITY OF HAWAII
MONTANA STATE UNIVERSITY  UNIVERSITY OF TOKYO
UNIVERSITY OF NEVADA, RENO  UNIVERSITY OF UTAH
NEW MEXICO STATE UNIVERSITY  WASHINGTON STATE UNIVERSITY
OREGON STATE UNIVERSITY  UNIVERSITY OF WASHINGTON
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>K. Adachi, Le problème de Lévi pour les fibrés grassmanniens et les variétés drapeaux</td>
<td>1</td>
</tr>
<tr>
<td>John MacLeod Ball, Remarks on the paper: “Basic calculus of variations”</td>
<td>7</td>
</tr>
<tr>
<td>John Kelly Beem and Phillip E. Parker, Whitney stability of solvability</td>
<td>11</td>
</tr>
<tr>
<td>Alberto Facchini, Decompositions of algebraically compact modules</td>
<td>25</td>
</tr>
<tr>
<td>S. S. Khare, Finite group action and equivariant bordism</td>
<td>39</td>
</tr>
<tr>
<td>Horst Leptin, A new kind of eigenfunction expansions on groups</td>
<td>45</td>
</tr>
<tr>
<td>Pei-Kee Lin, Unconditional bases and fixed points of nonexpansive mappings</td>
<td>69</td>
</tr>
<tr>
<td>Charles Livingston, Stably irreducible surfaces in $S^4$</td>
<td>77</td>
</tr>
<tr>
<td>Kevin Mor McCrimmon, Nonassociative algebras with scalar involution</td>
<td>85</td>
</tr>
<tr>
<td>Albert Milani, Singular limits of quasilinear hyperbolic systems in a bounded domain of $\mathbb{R}^3$ with applications to Maxwell’s equations</td>
<td>111</td>
</tr>
<tr>
<td>Takemi Mizokami, On $M$-structures and strongly regularly stratifiable spaces</td>
<td>131</td>
</tr>
<tr>
<td>Jesper M. Møller, On the homology of spaces of sections of complex projective bundles</td>
<td>143</td>
</tr>
<tr>
<td>Nikolaos S. Papageorgiou, Carathéodory convex integrand operators and probability theory</td>
<td>155</td>
</tr>
<tr>
<td>Robert John Piacenza, Transfer in generalized prestack cohomology</td>
<td>185</td>
</tr>
<tr>
<td>Lance W. Small and Adrian R. Wadsworth, Integrality of subrings of matrix rings</td>
<td>195</td>
</tr>
<tr>
<td>James Michael Wilson, On the atomic decomposition for Hardy spaces</td>
<td>201</td>
</tr>
</tbody>
</table>