UNCONDITIONAL BASES AND FIXED POINTS OF NONEXPANSIVE MAPPINGS

Pei-Kee Lin
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OF NONEXPANSIVE MAPPINGS

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We prove that every Banach space with a 1-unconditional basis has
the fixed point property for nonexpansive mappings. In fact the argument works if the unconditional constant is \( < (\sqrt{33} - 3)/2 \).

1. Introduction. Let \( K \) be a weakly compact convex subset of a
Banach space \( X \). We say \( K \) has the fixed point property if every nonexpansive map \( T: K \to K \) (i.e. \( ||Tx - Ty|| \leq ||x - y|| \) for \( x, y \in K \)) has a fixed point. We say \( X \) has the fixed point property if every weakly compact convex subset of \( X \) has the fixed point property.

It is known that \( L_1 \) fails the fixed point property [A]. On the other
hand, Kirk [Ki 1] proved that every Banach space with normal structure
(for the definition see [D]) has the fixed point property. Karlovitz (see
[Ka 1] and [Ka 2]) extended Kirk’s work. Let us explain what Karlovitz
did.

Suppose \( K \) is weakly compact convex and \( T: K \to K \) is nonexpansive.
\( K \) contains a weakly compact convex subset \( K_0 \) which is minimal for \( T \).
This means \( T(K_0) \subseteq K_0 \) and no strictly smaller weakly compact convex subset of \( K_0 \) is invariant under \( T \). If \( K_0 \) contains only one point, then \( T \)
has a fixed point. Hence, we may assume that \( \text{diam } K_0 = \sup \{ ||x - y|| : x, y \in K_0 \} > 0 \). It is easy to see that \( K_0 \) contains a sequence \((x_n)\) with
\[
\lim_{n \to \infty} ||x_n - Tx_n|| = 0.
\]
We call such a sequence an approximate fixed point sequence for \( T \). Indeed, fixed \( y \in K_0 \), one can choose \( x_n \) to be the fixed point of the strict contraction, \( T_n: K_0 \to K_0 \), given by \( T_n x = (1 - n^{-1})Tx + n^{-1} \). Note we only need that \( K_0 \) is closed, bounded and convex for this argument. Karlovitz proved the following theorem.

THEOREM A. Let \( K \) be a minimal weakly compact convex set for a
nonexpansive map \( T \), and let \((x_n)\) be an approximate fixed point sequence.
Then for all \( x \in K \)
\[
\lim_{n \to \infty} ||x - x_n|| = \text{diam } K.
\]

Maurey [M] used the ultraproduct techniques to prove that \( c_0 \) and
every reflexive subspace of \( L_1 \) have the fixed point property. Odell and the
author [E-L-O-S] used Maurey's technique to prove that $T_s$ (the Tsireleson space of Figiel and Johnson [F-J]) and $T_s^*$ have the fixed point property.

In §II we give some examples of Banach spaces with an unconditional basis and discuss the fixed point property on those spaces.

In §III we introduce the ultraproduct technique and rewrite the Karlovitz Theorem in the ultraproduct language.

In §IV we prove that every Banach space with a 1-unconditional basis has the fixed point property. Indeed, our argument shows that if $X$ has an unconditional basis with unconditional constant (for definition see §II) $\lambda < 1.37$, then $X$ has the fixed point property. Also we prove the every superreflexive space (by Enflo [En] this is a space isomorphic to a uniformly convex space) with a suppression unconditional basis has the fixed point property.

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2. Examples of spaces with an unconditional basis. Let $X$ be a Banach space. A sequence $\{e_n\}_{n=1}^\infty$ in $X$ is called a Schauder basis of $X$ if for every $x \in X$ there is a unique sequence of scalars $\{a_n\}_{n=1}^\infty$ so that $x = \sum_{n=1}^\infty a_n e_n$. A Schauder basis $\{e_n\}_{n=1}^\infty$ is called an unconditional basis if for every choice of signs $\varepsilon_n$ (i.e. $\varepsilon_n = \pm 1$), $\sum_{n=1}^\infty \varepsilon_n a_n e_n$ converges whenever $\sum_{n=1}^\infty a_n e_n$ converges. If $\{e_n\}$ is an unconditional basis, then the number

$$\sup \left\{ \left\| \sum_{i=1}^n \varepsilon_i a_i e_i \right\| : \left\| \sum_{i=1}^n a_i e_i \right\| = 1; \varepsilon_i = \pm 1 \right\}$$

is called the unconditional constant of $\{e_n\}_{n=1}^\infty$. If $\{e_n\}_{n=1}^\infty$ is an unconditional basis and $F$ is a subset of $\mathbb{N}$, then the projection

$$P \left( \sum_{n=1}^\infty a_n e_n \right) = \sum_{n \in F} a_n e_n$$

is called the natural projection associated with $F$ to the unconditional basis $\{e_n\}_{n=1}^\infty$. It is clear that the norm of any natural projection is smaller than the unconditional constant of the basis. We say an unconditional basis is suppression unconditional if every natural projection associated to the basis has norm 1.

Example 1. The natural basis $e_n = \{0, 0, 0, \ldots, 1, 0, \ldots\}$ is an unconditional basis in each of the spaces $c_0$ and $l_p$, $1 \leq p < \infty$. Browder [Br] proved that every uniformly convex space has the fixed point property. Since $l_p$, $1 < p < \infty$, are uniformly convex [C], they have the fixed point
property. Lim [Lm] proved that every weak* compact convex subset of $l_1$
has weak* normal structure. Hence, every nonexpansive mapping on
weak* compact convex subsets of $l_1$ has a fixed point. Maurey proved $c_0$
has the fixed point property.

**Example 2.** Let $X_M$ be $l_2$ with the new norm
\[
\|x\| = \max\{\|x\|_\infty, M^{-1}\|x\|_2\}.
\]
Then the natural basis is an unconditional basis with unconditional
constant $\lambda = 1$. It is known that $X_M$ fail to have normal structure
whenever $M \geq \sqrt{2}$. But $X_M$ still have the fixed point property ([Ka 1],
[B-S] and [E-L-O-S]).

**Example 3.** The norm on the sequence space $T_s$ is given implicitly by
\[
\|x\|_s = \sup\left\{\|x\|_\infty, \frac{1}{2} \sum_{k=1}^{n} \|E_k x\|_s\right\}
\]
where the “sup” is taken over all admissible set $(E_k)_{k=1}^n$ and $(Ex)(i)$
equals $x(i)$ for $i \in E$ and 0 otherwise. $(E_k)_{k=1}^n$ is admissible if the $E_k$’s are
finite subsets $N$ with $n < \min E_1 \leq \max E_1 < \min E_2 \leq \max E_2 < \cdots
< \min E_n$. $T_s$ is a reflexive Banach space with a 1-unconditional basis.
Hence, $T_s$ has the fixed point property ([E-L-O-S]).

**Example 4.** $(l_1, | \cdot |)$ is $l_1$ with norm
\[
|x| = \max(\|x^+\|_1, \|x^-\|_1)
\]
where $x^+$ and $x^-$ are the positive and negative parts of $x$. Then $(l_1, | \cdot |)$ is
isometrically isomorphic to the dual of $(c_0, \| \cdot \|)$ where the norm is given by
\[
\|x\| = \|x^+\|_\infty + \|x^-\|_\infty.
\]
The natural basis is a suppression unconditional basis of $(l_1, | \cdot |)$, and the
unconditional constant of this basis is 2. Lim [Lm] showed that there is a
weak* compact subset $K$ of $l_1$ and an isometry $T: K \rightarrow K$ such that $T$
has no fixed points. But every weakly compact subset of $l_1$ is compact. Hence,
$(l_1, | \cdot |)$ has the fixed point property.

**Example 5.** An *Orlicz function* $M$ is a continuous non-decreasing and
convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t \to \infty} M(t) =
\infty$. To any Orlicz function $M$ we associate the space $l_M$ of all sequences of
scalars $x = (a_1, a_2, \ldots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$. 
The space $l_M$ equipped with the norm

$$
\|x\| = \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M \left( \frac{|a_n|}{\rho} \right) \leq 1 \right\}
$$

is a Banach space called an \textit{Orlicz sequence space}. If

$$
\lim_{t \to 0} \sup M(2t)/M(t) < \infty,
$$

then $l_M$ has a 1-unconditional basis. In this case, $l_M$ has the fixed point property.

**Example 6.** Let $(T_s, | \cdot |_s)$ be the $T_s$ with the norm

$$
|x|_s = \max \{ \|x^+\|_s, \|x^-\|_s \}.
$$

Then $(T_s, | \cdot |_s)$ has a suppression unconditional basis. It is still open whether $(T_s, | \cdot |_s)$ has the fixed point property or not. (Note: $T_s$ is not superreflexive.)

**3. Ultraproducts.** Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $X$ be a Banach space. The ultraproduct space $\hat{X}$ of $X$ is the quotient space of

$$
\{ (x_n) : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \| (x_n) \| = \sup_n \| x_n \| < \infty \}
$$

by $\mathcal{N} = \{ (x_n) \in l_\infty(X) : \lim_{n \to \mathcal{U}} \| x_n \| = 0 \}$. (Note $\lim_{n \to \mathcal{U}} \| x_n \|$ is the limit of $\| x_n \|$ over the ultrafilter $\mathcal{U}$.) We shall not distinguish between $(x_n)$ and the coset $(x_n) + \mathcal{N} \in \hat{X}$. Clearly,

$$
\| (x_n) \|_{\hat{X}} = \lim_{n \to \mathcal{U}} \| x_n \|.
$$

It is also clear that $X$ is isometric to a subspace of $\hat{X}$ by the mapping $x \to (x, x, \ldots)$. So we may assume that $X$ is a subspace of $\hat{X}$. We will write $\hat{y}, \hat{z}, \hat{w}$ for the general elements of $\hat{X}$ and $\hat{f}, \hat{g}$ for the elements of the dual $\hat{X}^*$. If $S_n$'s are uniformly bounded operators (projections) on $X$, then $\hat{S} = (S_n)$ which is given by $\hat{S}(x_n) = (S_n x_n)$ is a bounded operator (projection) on $\hat{X}$, and $\| \hat{S} \| \leq \sup_n \| S_n \|$. Suppose $X$ has an unconditional basis $(e_n)$. We say $\hat{P}$ is a \textit{natural projection} with respect to $(e_n)$ if there exist natural projections $P_n$ on $X$ associated to $(e_n)$ such that $\hat{P} = (P_n)$. We say $\hat{x}, \hat{y} \in \hat{X}$ are \textit{disjoint} if there exist two natural projections $\hat{P}, \hat{Q}$ on $\hat{X}$ such that $\hat{P}\hat{x} = \hat{x}$, $\hat{Q}\hat{y} = \hat{y}$ and $\hat{P}\hat{Q} = \hat{Q}\hat{P} = 0$. In other words, $\hat{x}$ and $\hat{y}$ are disjoint if they have the representations $(x_n)$ and $(y_n)$ such that $x_n$ and $y_n$ are disjoint in $X$ for all $n$.

Now let us translate Theorem A into ultraproduct language. Let $K$ be a weakly compact convex subset of $X$ which is minimal for nonexpansive
map \( T \). Let \( K = \{ (x_n) : x_n \in K \text{ for all } n \} \) and define \( \tilde{T} : \tilde{K} \to \tilde{K} \) by \( \tilde{T}(x_n) = (Tx_n) \). Clearly, \( \tilde{K} \) is closed bounded and convex and \( \tilde{T} \) is nonexpansive on \( \tilde{K} \). Furthermore, \( \tilde{T} \) has fixed points in \( \tilde{K} \). Indeed, if \((x_n)_{n=1}^\infty \) is an approximate fixed point sequence for \( T \) in \( K \), then for \( \tilde{y} = (x_n) \)

\[
\|\tilde{T}\tilde{y} - \tilde{y}\| = \lim_{n \to \infty} \|Tx_n - x_n\| = \lim_{n \to \infty} \|Tx_n - x_n\| = 0,
\]

and hence \( \tilde{T}\tilde{y} = \tilde{y} \). On the other hand, \( \tilde{T}\tilde{y} = \tilde{y} \) for \( \tilde{y} = (x_n) \) then some subsequence of \((x_n)_{n=1}^\infty \) is an approximate fixed point sequence for \( T \). In ultraproduct language, Theorem A becomes

**Theorem A'.** Let \( K \) be a minimal weakly compact convex set for a nonexpansive map \( T \). If \( \tilde{y} \) is a fixed point of \( \tilde{T} \) in \( \tilde{K} \) and \( x \in K \), then \( \|\tilde{y} - x\| = \text{diam}(K) \). Moreover, suppose \( \text{diam } K = 1 \) and \( 0 \in K \). Then for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( \|\tilde{y}\| > 1 - \epsilon \) whenever \( \|\tilde{T}\tilde{y} - \tilde{y}\| < \delta \).

**4. The main result.**

**Theorem 1.** Every Banach space \( X \) with 1-unconditional basis \( (e_n) \) has the fixed point property.

**Proof.** Suppose it were not true. Then there is a weakly compact convex subset \( K \) which is minimal for a nonexpansive map \( T \). Moreover, we may assume \( \text{diam } K = 1 \). By translation of \( K \), then passing to subsequences, we may suppose that \( 0 \in K \) and there exist an approximate fixed point sequence \((x_n)_{n=1}^\infty \) for \( T \) and natural projections \( P_n \) on \( X \) (with respect to \( (e_n) \)) such that \( P_nP_m \neq 0 \) if \( n \neq m \) and

\[
\lim_{n \to \infty} \|P_nx_n\| = \lim_{n \to \infty} \|x_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \|(I - P_n)x_n\| = 0.
\]

Let \( \tilde{h} = (x_n) \) and \( \tilde{z} = (z_n) \) with \( z_n = x_{n+1} \). Then \( \tilde{y} \) and \( \tilde{z} \) are fixed points of \( \tilde{T} \) with \( \|\tilde{y} - \tilde{z}\| = 1 \). For any \( x \in K \), \( x, \tilde{y} \) and \( \tilde{z} \) are disjoint. Indeed, let \( \tilde{P} = (P_n) \) and \( \tilde{Q} = (Q_n) \) with \( Q_n = P_{n+1} \). Then \( \tilde{P}\tilde{y} = \tilde{y} \) and \( \tilde{Q}\tilde{z} = \tilde{z} \) and for any \( x \in K \),

\[
\tilde{P}x = \tilde{Q}x = \tilde{P}\tilde{z} = 0 = \tilde{Q}\tilde{y}.
\]

Also since \( (e_n) \) is 1-unconditional, \( \|\tilde{y} - \tilde{z}\| = 1 = \|\tilde{y} + \tilde{z}\| \). Let \( \tilde{W} = \{ \tilde{w} : \tilde{w} \in \tilde{K} \text{ such that there exists } x \in K \}

(\text{depending on } \tilde{w} \text{ with max}\{\|\tilde{w} - x\|, \|\tilde{w} - \tilde{y}\|, \|\tilde{w} - \tilde{z}\|\} \leq 1/2\} \).

Clearly, \( \tilde{W} \) is a nonempty bounded closed convex set. (Note \( \|(\tilde{y} + \tilde{z})/2 - 0\| = \|(\tilde{y} + \tilde{z})/2\| = \|(\tilde{y} - \tilde{z})/2\| = 1/2 \). So \( (\tilde{y} + \tilde{z})/2 \in \tilde{W} \). Since
\( y, z \) are fixed points of \( T \) and \( T \) is a nonexpansive mapping, if \( \tilde{w} \in \tilde{W} \),
\[
\max(\|\tilde{T}w - Tx\|, \|\tilde{T}w - y\|, \|\tilde{T}w - z\|) \\
\leq \max(\|\tilde{w} - x\|, \|\tilde{w} - y\|, \|\tilde{w} - z\|) \leq 1/2.
\]
Thus \( \tilde{W} \) is invariant under \( \tilde{T} \); hence, it contains an approximate fixed point sequence for \( \tilde{T} \). On the other hand, for any \( \tilde{w} \in \tilde{W} \) there exists \( x \in K \) so that \( \|\tilde{w} - x\| \leq 1/2 \). Hence if \( \tilde{I} \) is the identity map in \( \tilde{X} \),
\[
\|\tilde{w}\| = \frac{1}{2} \left\| (\tilde{P} + \tilde{Q})\tilde{w} + (\tilde{I} - \tilde{P})\tilde{w} + (\tilde{I} - \tilde{Q})\tilde{w} \right\| \\
\leq \frac{1}{2} \left\| (\tilde{P} + \tilde{Q})\tilde{w}\right\| + \|\tilde{I} - \tilde{P}\|\tilde{w}\right\| + \|\tilde{I} - \tilde{Q}\|\tilde{w}\right\| \\
= \frac{1}{2} \left\| (\tilde{P} + \tilde{Q})(\tilde{w} - x)\right\| + \|\tilde{I} - \tilde{P}\|\tilde{w} - y\right\| + \|\tilde{I} - \tilde{Q}\|\tilde{w} - z\right\| \\
\leq \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right] = \frac{3}{4}.
\]
By Theorem A', \( \tilde{W} \) cannot contain any approximate fixed point sequences for \( \tilde{T} \). We have a contradiction. \( \square \)

We note that the proof of the above Theorem has some leeway. More precisely we have the following more general result.

**Theorem 2.** If \( X \) has an unconditional basis with unconditional constant \( \lambda < (\sqrt{33} - 3)/2 \), then \( X \) has the fixed point property.

**Proof.** Let \( \tilde{y}, \tilde{z}, \tilde{P} \) and \( \tilde{Q} \) be as in Theorem 1, and let
\[
\tilde{W} = \{ \tilde{w} : \tilde{w} \in \tilde{K} \text{ such that there exists } x \in K \text{ with} \\
\|\tilde{w} - x\| \leq \lambda/2 \text{ and } \max(\|\tilde{w} - \tilde{y}\|, \|\tilde{w} - \tilde{z}\|) \leq 1/2 \}.
\]
Since \( \|\tilde{y} + \tilde{z}\|/2 \leq \lambda\|\tilde{y} - \tilde{z}\|/2 = \lambda/2 \), \( \tilde{W} \) is a nonempty bounded closed convex set invariant under \( \tilde{T} \). Hence, \( \tilde{W} \) contains an approximate fixed point sequence for \( \tilde{T} \). For easy calculation, we assume that \( \tilde{W} \) has an element \( \tilde{w} \) with \( \|\tilde{w}\| = 1 \). Let \( x \in K \) with \( \|x - \tilde{w}\| \leq \lambda/2 \) and let \( \tilde{f} \in X^* \) with \( \tilde{f}(\tilde{w}) = 1 = \|\tilde{f}\| \). Hence, \( 1 - \tilde{f}(\tilde{y}) = \tilde{f}(\tilde{w} - \tilde{y}) \leq \|\tilde{w} - \tilde{y}\| \leq 1/2 \), and so \( \tilde{f}(\tilde{y}) \geq 1/2 \). Similarly, we also have the inequalities \( \tilde{f}(\tilde{z}) \geq 1/2 \) and \( \tilde{f}(x) \geq 1 - \lambda/2 \). Let \( \alpha = \tilde{f}((\tilde{I} - \tilde{P} - \tilde{Q})\tilde{w}) \). Then
\[
1 - \alpha = \tilde{f}(\tilde{w}) - \tilde{f}((\tilde{I} - \tilde{P} - \tilde{Q})\tilde{w}) \\
= \tilde{f}((\tilde{P} + \tilde{Q})\tilde{w}) = \tilde{f}(\tilde{P}\tilde{w}) + \tilde{f}(\tilde{Q}\tilde{w}),
\]
and so either $\tilde{f}(\tilde{P}\tilde{w}) \leq (1 - \alpha)/2$ or $\tilde{f}(\tilde{Q}\tilde{w}) \leq (1 - \alpha)/2$, say $\tilde{f}(\tilde{P}\tilde{w}) \leq (1 - \alpha)/2$. Since $\tilde{I} - 2\tilde{P}$ and $\tilde{I} - 2\tilde{P} - 2\tilde{Q}$ are reflections, $\|\tilde{I} - 2\tilde{P}\| \leq \lambda$ and $\|\tilde{w}\tilde{P} + 2\tilde{Q} - \tilde{I}\| \leq \lambda$. Hence, we have

$$
(2 - 2\alpha) - \lambda/2 \leq 2\tilde{f}((\tilde{P} + 2\tilde{Q})\tilde{w}) - \tilde{f}(\tilde{w} - x)
$$

$$
= \tilde{f}((2\tilde{P} + 2\tilde{Q})\tilde{w}) - \tilde{f}(\tilde{w} - x)
$$

$$
= \tilde{f}((2\tilde{P} + 2\tilde{Q})(\tilde{w} - x)) - \tilde{f}(\tilde{w} - x)
$$

$$
= \tilde{f}((2\tilde{P} + 2\tilde{Q} - \tilde{I})(\tilde{w} - x))
$$

$$
\leq \|\tilde{f}\| \|2\tilde{P} + 2\tilde{Q} - \tilde{I}\| \|\tilde{w} - x\| \leq \lambda^2/2,
$$

and

$$
\alpha + \frac{1}{2} = \frac{1}{2} + 1 - (1 - \alpha) \leq \tilde{f}(\tilde{y}) + \tilde{f}(\tilde{w}) - 2\tilde{f}(\tilde{P}\tilde{w})
$$

$$
= \tilde{f}(\tilde{w} - \tilde{y}) + 2\tilde{f}(\tilde{y}) - 2\tilde{f}(\tilde{P}\tilde{w})
$$

$$
= \tilde{f}(\tilde{w} - \tilde{y}) + 2\tilde{f}(\tilde{P}\tilde{y}) - 2\tilde{f}(\tilde{P}\tilde{w})
$$

$$
= \tilde{f}(\tilde{w} - \tilde{y}) + 2\tilde{f}(\tilde{P}(\tilde{y} - \tilde{w})) = \tilde{f}((\tilde{I} - 2\tilde{P})(\tilde{w} - \tilde{y}))
$$

$$
\leq \|\tilde{f}\| \|\tilde{I} - 2\tilde{P}\| \|\tilde{w} - \tilde{y}\| \leq \lambda/2.
$$

Therefore, $3 - 3\lambda/2 \leq \lambda^2/2$ and $\lambda \geq (\sqrt{33} - 3)/2$. \qed

If $X$ has a suppression unconditional basis, we have the following strong result.

**Theorem 3.** Suppose $X$ has a suppression unconditional basis $(e_i)$. Then $X$ has the fixed point property whenever $X$ is superreflexive.

**Proof.** Suppose not and, as usual, let $K$ be a minimal set of diameter 1 for a nonexpansive map $T$. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be disjoint fixed points for $\tilde{T}$ in $\tilde{K}$. We shall prove $(\tilde{x}_i)^\prime$ is 2-equivalent to the unit basis of $l''_1$. Indeed, if $\Sigma_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ and $0 < c < 1$, then the same argument as given in the proof of Theorem 1 shows that every element in $\tilde{W} = \{ \tilde{w} : \tilde{w} \in \tilde{K} \text{ such that } x \in K \text{ with } \|x - \tilde{w}\| \leq c \}

\text{and } \|\tilde{w} - \tilde{x}_i\| \leq 1 - \alpha_i \text{ for } i = 1, 2, \ldots, n

has norm less than or equal to $1 - (1 - c)/n$. $\tilde{W}$ is a closed convex set which is invariant under $\tilde{T}$; hence, $\tilde{W}$ is empty. But

$$
\|\tilde{x}_j - \sum_{i=1}^n \alpha_i \tilde{x}_i\| \leq \|\sum_{i\neq j} \alpha_i (\tilde{x}_j - \tilde{x}_i)\| \leq 1 - \alpha_j,
$$

for $j = 1, 2, \ldots, n$. So $\|\Sigma_{i=1}^n \alpha_i \tilde{x}_i\| > c$ and so $\|\Sigma_{i=1}^n \alpha_i \tilde{x}_i\| = 1$. \qed
REMARK 1. The disjoint fixed point sequence \((\tilde{x}_i, \tilde{y}_i)\) for \(\tilde{T}\) as given in the proof of Theorem 3 is 1-equivalent to the unit vector basis of \((l_1^n, \| \cdot \|)\).
Indeed, let \(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n-1}\) be disjoint fixed points of \(\tilde{T}\) and \(\tilde{y}_{2n} = 0\). Then for \(n\)
\[
1 = \left\| \sum_{i=1}^{n} \alpha_i \tilde{y}_{2i-1} \right\| \leq \left\| \sum_{i=1}^{n} \alpha_i (\tilde{y}_{2i-1} - \tilde{y}_{2i}) \right\| \leq \sum_{i=1}^{n} \alpha_i = 1.
\]
Hence, \(\left\| \sum_{i=1}^{n} \alpha_i (\tilde{y}_{2i-1} - \tilde{y}_{2i}) \right\| = 1\). In general, we have that
\[
\left\| \sum_{i=1}^{n} \beta_i \tilde{x}_i \right\| = \max(\| (\beta_i)^+ \|_1, \| (\beta_i)^- \|_1) = \|(\beta_i)\|.
\]

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