

# Pacific Journal of Mathematics

**ON THE ATOMIC DECOMPOSITION FOR HARDY SPACES**

JAMES MICHAEL WILSON

## ON THE ATOMIC DECOMPOSITION FOR HARDY SPACES

J. MICHAEL WILSON

We give an extremely easy proof of the atomic decomposition for distributions in  $H^p(\mathbf{R}_+^{n+1})$ ,  $0 < p \leq 1$ . Our proof uses only properties of the nontangential maximal function  $u^*$ . We then adapt our argument to give a "direct" proof of the Chang-Fefferman decomposition for  $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ .

**I. Introduction.** Let  $\mathbf{R}_+^{n+1} = \{(x, y): x \in \mathbf{R}^n, y > 0\}$ . For  $u(x, y)$  harmonic on  $\mathbf{R}_+^{n+1}$  and  $A > 0$  define

$$u_A^*(x) = \sup_{|x-t| < Ay} |u(t, y)|.$$

We say that  $u \in H^p$  if  $u_A^* \in L^p$ , for any  $A$ , and set  $\|u\|_{H^p} = \|u_1^*\|_{L^p}$ . If  $u \in H^p$ ,  $0 < p < \infty$ , then  $f = \lim_{y \rightarrow 0} u(\cdot, y)$  exists (in  $\mathcal{S}'$ ) and is said to be in  $H^p$ . We set  $\|f\|_{H^p} = \|u\|_{H^p}$  (see [6]).

For  $0 < p \leq 1$ , a  $p$ -atom is a function  $a(x) \in L^2(\mathbf{R}^n)$  satisfying:

( $\alpha$ )  $\text{supp } a \subset Q$ ,  $Q$  a cube.

( $\beta$ )  $\|a\|_2 \leq |Q|^{1/2-1/p}$  ( $|Q|$  = the volume of  $Q$ ).

( $\gamma$ )  $\int a(x)x^\alpha dx = 0$  for all monomials  $x^\alpha$  with  $|\alpha| \leq [n(p^{-1} - 1)]$ .

The following theorem is well known [4] [7] [10]:

**THEOREM A.** *Let  $f \in H^p$ ,  $0 < p \leq 1$ . There exist  $p$ -atoms  $a_k$  and numbers  $\lambda_k$  such that*

$$(1) \quad f = \sum \lambda_k a_k \quad \text{in } \mathcal{S}'.$$

The  $\lambda_k$  satisfy  $\sum |\lambda_k|^p \leq C(p, n) \|f\|_{H^p}^p$ . Conversely, every sum (1) satisfies

$$\|f\|_{H^p}^p \leq C(p, n) \sum |\lambda_k|^p.$$

Now let  $u$  be biharmonic on  $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ . Define

$$u_A^*(x_1, x_2) = \sup_{\substack{|x_i - t_i| < Ay_i \\ i=1,2}} |u(t_1, y_1, t_2, y_2)|.$$

As before, we say that  $u \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$  if  $u_A^* \in L^p(\mathbf{R}^2)$ , and we set  $\|u\|_{H^p} = \|u_A^*\|_{L^p}$ . Such  $u$  give rise to boundary distributions  $f$ , which are said to be in  $H^p$ . (See [2].)

For  $0 < p \leq 1$ , a *Chang-Fefferman  $p$ -atom* is a function  $a \in L^2(\mathbf{R}^2)$  satisfying:

( $\alpha'$ )  $\text{supp } a \subset \Omega$ ,  $\Omega$  open,  $|\Omega| < \infty$ .

( $\beta'$ )  $\|a\|_2 \leq |\Omega|^{1/2-1/p}$ .

( $\gamma'$ )  $a = \sum_R \lambda_R a_R$ , where  $\lambda_R$  are numbers and the  $a_R$  are functions (called “elementary particles”) satisfying:

(i)  $\text{supp } a_R \subset \tilde{R} \subset \Omega$  where  $R = I \times J$ ,  $I, J$  dyadic intervals, and  $\tilde{R}$  denotes the triple of  $R$ .

(ii)

$$\left\| \frac{\partial^L a_R}{\partial x_1^L} \right\|_\infty \leq \frac{1}{\sqrt{|R|} |I|^L} \quad \text{and} \quad \left\| \frac{\partial^L a_R}{\partial x_2^L} \right\|_\infty \leq \frac{1}{\sqrt{|R|} |J|^L}$$

for all  $L \leq [2/p - 1/2]$

(iii)

$$\int a(\tilde{x}_1, x_2) x_2^k dx_2 = 0 \quad \text{and} \quad \int a(x_1, \tilde{x}_2) x_1^k dx_1 = 0$$

for all  $(\tilde{x}_1, \tilde{x}_2) \in \mathbf{R}^2$  and all  $k \leq [2/p - 3/2]$ . And

$$\left( \sum_R \lambda_R^2 \right)^{1/2} \leq |\Omega|^{1/2-1/p}.$$

If the “atoms” are Chang-Fefferman atoms, then Theorem A is true for  $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$  [2] [3].

Until now, proofs of the atomic decomposition have relied on showing that  $u^* \in L^p$  implies that some auxiliary function (such as the “grand” maximal function or the  $S$ -function) is in  $L^p$ . In this paper, we give proofs which get the atoms directly from “ $u^* \in L^p$ ”.

**REMARK.** Our argument is somewhat like that of A. P. Calderón in [1]. Calderón’s “ $u^*$ ” is the sum of two real-variable maximal functions. He writes his reproducing formula (see below) in terms of one kernel, and uses the other kernel to control the  $L^\infty$  size of his atoms. Our proof uses Green’s Theorem to get  $L^2$  bounds. This approach lets us adapt our proof to the bidisc setting, where  $L^\infty$  atoms do not seem to be the “right” ones.

**II. The case  $H^p(\mathbf{R}_+^2)$ .** Let  $\psi \in C^\infty(\mathbf{R})$  be real, radial,  $\text{supp } \psi \subset \{|x| \leq 1\}$ ,  $\psi$  has the cancellation property  $\gamma$ ), and

$$\int_0^\infty e^{-\theta} \hat{\psi}(\theta) d\theta = -1.$$

For  $y > 0$ , set  $y^{-1}\psi(t/y) = \psi_y(t)$ .

Take  $f \in L^2 \cap H^p$ ,  $f$  real-valued,  $u = P_y * f$  (the Poisson integral of  $f$ ). By Fourier transforms

$$f = \int_{\mathbf{R}_+^2} \frac{\partial u}{\partial y}(t, y) \psi_y(x - t) dt dy \quad \text{in } \mathcal{S}'.$$

(This trick is due to A. P. Calderón.) For  $k = 0, \pm 1, \pm 2, \dots$ , define

$$E^k = \{u_2^* > 2^k\} = \bigcup_{j=1}^{\infty} I_j^k$$

where the  $I_j^k$  are component intervals. For  $I$  an interval, let

$$\hat{I} = \{(t, y) \in \mathbf{R}_+^2 : (t - y, t + y) \subset I\}$$

be the “tent” region. Define  $\hat{E}^k = \bigcup \hat{I}_j^k$ ,  $T_j^k = \hat{I}_j^k \setminus \hat{E}^{k+1}$ . Then

$$f = \sum_{k,j} \int_{T_j^k} \frac{\partial u}{\partial y}(t, y) \psi_y(x - t) dt dy = \sum_{k,j} g_j^k = \sum_{k,j} \lambda_j^k a_j^k,$$

where  $\lambda_j^k = C2^k |I_j^k|^{1/p}$  and the  $a_j^k$  (we claim) are atoms. The  $a_j^k$  inherit  $\gamma$  from  $\psi$ , and obviously  $\text{supp } a_j^k \subset \tilde{I}_j^k$ . Note also that

$$\sum (\lambda_j^k)^p \leq C \int (u_2^*)^p dx \leq C \|u\|_{H^p}^p.$$

Thus, we are done if we can show

$$\|g_j^k\|_2 \leq C2^k |I_j^k|^{1/2}.$$

We do this by duality. Let  $h \in L^2(\mathbf{R})$ ,  $\|h\|_2 = 1$ . Then

$$\begin{aligned} \left| \int h(x) g_j^k(x) dx \right| &= \left| \int_{T^k} \frac{\partial u}{\partial y}(t, y) (h * \psi_y(t)) dt dy \right| \\ &\leq \left( \int_{T_j^k} y |\nabla u|^2 dt dy \right)^{1/2} \left( \int_{\mathbf{R}_+^2} |h * \psi_y(t)|^2 \frac{dt dy}{y} \right)^{1/2} \end{aligned}$$

(Plancherel) 
$$\leq C \left( \int_{T_j^k} y |\nabla u|^2 dt dy \right)^{1/2}$$

We estimate the last integral by Green’s Theorem. It is bounded by

$$\left( \int_{\partial T_j^k} \left( |u| y \left| \frac{\partial u}{\partial \nu} \right| + \frac{1}{2} u^2 \left| \frac{\partial y}{\partial \nu} \right| \right) ds \right)$$

( $\partial/\partial \nu$  is outward normal;  $\partial T_j^k$  is just smooth enough to let us use Green’s Theorem). Because of the “2” (in  $u_2^*$ ), both  $|u|$  and  $y|\nabla u|$  are bounded by  $C2^k$  on  $\partial T_j^k$ . Since  $|\partial y/\partial \nu| \leq 1$  and  $|\partial T_j^k| \leq C|I_j^k|$ , the last term is no larger than  $C2^k |I_j^k|^{1/2}$ . □

**III. The case  $H^p(\mathbf{R}_+^{n+1})$ .** Let  $\psi$  be as in II, except now  $\psi \in C^\infty(\mathbf{R}^n)$ . Let  $f \in H^p \cap L^2$  and  $u$  be as before. Define

$$E^k = \{u_{10^n}^* > 2^k\} = \bigcup_{j=1}^\infty \Omega_j^k;$$

where the  $\Omega_j^k$  are Whitney cubes (for the definition see [9], p. 167). For  $\Omega$  a cube in  $\mathbf{R}^n$ , define

$$\hat{\Omega} = \{(t, y) : t \in \Omega, 0 < y < l(\Omega)\}$$

where  $l(\Omega)$  = sidelength of  $\Omega$ . Define

$$\hat{E}^k = \bigcup \hat{\Omega}_j^k, \quad T_j^k = \hat{\Omega}_j^k \setminus \hat{E}^{k+1}.$$

With these modifications, the preceding argument goes over practically verbatim; the details are left to the reader.

**IV. The case  $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ .** We first show that the proof in II yields a Chang-Fefferman decomposition for  $\mathbf{R}_+^2$ . For  $I \subset \mathbf{R}$  a dyadic interval, let

$$I^+ = \{(t, y) : t \in I, |I|/2 < y \leq |I|\}.$$

Define

$$\mathcal{J}_j^k = \{Q = I^+ \cap T_j^k\},$$

$$g_Q = \int_Q \frac{\partial u}{\partial y}(t, y) \psi_y(x - t) dt dy = \lambda_j^k \lambda_Q a_Q \quad \text{for } Q \in \mathcal{J}_j^k,$$

where we set

$$\lambda_Q = C(\lambda_j^k)^{-1} \left( \int_Q y |\nabla u|^2 dt dy \right)^{1/2}.$$

Then it is easily verified that the  $a_Q$  have the right cancellation, support and smoothness properties for elementary particles. And obviously

$$a_j^k = \sum_{Q \in \mathcal{J}_j^k} \lambda_Q a_Q,$$

$$\left( \sum_{Q \in \mathcal{J}_j^k} \lambda_Q^2 \right)^{1/2} \leq |\tilde{I}_j^k|^{1/2-1/p}.$$

In order to do our proof in  $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ , we need tents, and we need a way to do Green's Theorem. For these, we need some notation.

For  $(t, y) = (t_1, y_1, t_2, y_2) \in (\mathbf{R}_+^2)^2$ , let  $R_{t,y}$  be the rectangle with sides parallel to the coordinate axes, centered at  $(t_1, t_2) \in \mathbf{R}^2$ , and with dimensions  $2y_1 \times 2y_2$ .

Take  $f \in L^2 \cap H^p$ ,  $u = P_{y_1} \cdot P_{y_2} * f$  (the double Poisson integral of  $f$ ). Let  $\psi$  be as in II but with cancellation corresponding to (iii). Then

$$f = \int_{(\mathbf{R}_+^2)^2} \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y) \psi_{y_1}(x_1 - t_1) \psi_{y_2}(x_2 - t_2) dt dy \quad \text{in } \mathcal{S}'.$$

Let  $M$  be the strong maximal function. Let  $\varepsilon > 0$  be small, to be chosen later. Define

$$E^k = \{u_{100}^* > 2^k\}, \quad F^k = \{M\chi_{E^k} > \varepsilon\}.$$

It is a fact that  $|F^k| \leq C_\varepsilon |E^k|$ . Set

$$\hat{F}^k = \{(t, y) : R_{t,y} \subset F^k\},$$

$$T^k = \hat{F}^k \setminus \hat{F}^{k+1},$$

$$g^k = \int_{T^k} \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y) \psi_{y_1}(x_1 - t_1) \psi_{y_2}(x_2 - t_2) dt dy = \lambda_k a_k,$$

where we set  $\lambda_k = C2^k |E^k|^{1/p}$ .

For  $R = I \times J$ ,  $I, J$  dyadic intervals, let  $R^+ = I^+ \times J^+ \subset \mathbf{R}_+^2 \times \mathbf{R}_+^2$ . Set

$$\mathcal{J}_k = \{Q = R^+ \cap T^k\},$$

$$\begin{aligned} g_Q &= \int_Q \frac{\partial^2 u}{\partial y_1 \partial y_2}(t, y) \psi_{y_1}(x_1 - t_1) \psi_{y_2}(x_2 - t_2) dt dy \\ &= \lambda_k \lambda_Q a_Q \quad (Q \in \mathcal{J}^k), \end{aligned}$$

where we set

$$\lambda_Q = C(\lambda_k^{-1}) \left( \int_Q y_1 y_2 |\nabla_1 \nabla_2 u|^2 dt dy \right)^{1/2}$$

with

$$|\nabla_1 \nabla_2 u|^2 = \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 + \left| \frac{\partial^2 u}{\partial x_1 \partial y_2} \right|^2 + \left| \frac{\partial^2 u}{\partial y_1 \partial x_2} \right|^2 + \left| \frac{\partial^2 u}{\partial y_1 \partial y_2} \right|^2.$$

Then, in exact analogy to case II, everything will be done once we show

$$(2) \quad \int_{T^k} y_1 y_2 |\nabla_1 \nabla_2 u|^2 dt dy \leq C2^{2k} |E^k|.$$

For this we need a lemma of Merryfield. The lemma requires a little more notation.

Let  $\eta \in C^\infty(\mathbf{R})$ ,  $\eta \geq 0$ ,  $\text{supp } \eta \subset [-1, 1]$ ,  $\eta \geq \frac{1}{2}$  on  $[-\frac{1}{2}, \frac{1}{2}]$  and  $\int \eta = 1$ . Define

$$\Phi_{y_1 y_2}(t_1, t_2) = \eta_{y_1}(t_1) \cdot \eta_{y_2}(t_2).$$

For  $E \subset \mathbf{R}^2$ , set

$$V_E(t, y) = \Phi_y * \chi_E(t), \quad (t, y) \in (\mathbf{R}_+^2)^2.$$

Now,  $V_E(t, y)$  is essentially the density of  $E$  in  $R_{t,y}$ . In particular, if this density is greater than  $1 - \epsilon$ ,  $\epsilon$  small, then  $V_E(t, y) > 10^{-6}$ .

Merryfield's lemma is [8]:

LEMMA. Let  $u \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ ,  $p < 2$ , and let  $u_{100}^* \leq \lambda$  on  $E \subset \mathbf{R}^2$ . Then

$$\int_{(\mathbf{R}_+^2)^2} y_1 y_2 |\nabla_1 \nabla_2 u|^2 V_E^2(t, y) dt dy \leq C \lambda^2 |E|.$$

(Note: Merryfield states this for  $E$  open, but openness, as his proof shows, is not required.)

Let us set  $G^k = F^k \setminus E^{k+1}$ . Merryfield's lemma says that

$$\int_{\mathbf{R}_+^2} y_1 y_2 |\nabla_1 \nabla_2 u|^2 V_{G^k}^2(t, y) dt dy \leq C 2^{2k} |G^k| \leq C 2^{2k} |E^k|.$$

Therefore, we will have (2) (and be done) if we can show

$$V_{G^k} > 10^{-6} \quad \text{on } T^k.$$

Take  $(t, y) \in T^k$ . Then  $R_{t,y} \subset F^k$  but  $R_{t,y} \not\subset F^{k+1}$ . So there is an  $x \in R_{t,y} \cap (F^k \setminus F^{k+1})$ . Since  $x \notin F^{k+1}$ ,  $M \chi_{E^{k+1}}(x) \leq \epsilon$ . From the definition of  $M$ , this implies

$$|R_{t,y} \cap E^{k+1}| / |R_{t,y}| \leq \epsilon.$$

Since  $R_{t,y} \subset F^k$ ,

$$|R_{t,y} \cap (F^k \setminus E^{k+1})| / |R_{t,y}| \geq 1 - \epsilon.$$

But  $F^k \setminus E^{k+1} = G^k$ , and this implies that  $V_{G^k}(t, y) > 10^{-6}$ , for  $\epsilon$  small.  $\square$

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