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JEFFREY LAWRENCE CARUSO AND STEFAN WANER

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Let V be a real orthogonal countable-dimensional representation of the Lie group G and denote by $\Omega^V \Sigma^V X$ the space of maps $S^V \rightarrow \Sigma^V X = X \wedge S^V$, where S^V denotes the one-point compactification of V and where X is an arbitrary G -space with stationary basepoint (if V is infinite-dimensional, $\Omega^V \Sigma^V X$ is taken as the natural colimit over spaces indexed on the finite-dimensional submodules of V). Since G acts on $\Omega^V \Sigma^V X$ by conjugation, the fixed-set $(\Omega^V \Sigma^V X)^G$ is the subspace of G -equivariant maps. We present here an approximation to $(\Omega^V \Sigma^V X)^G$ in the stable case (V large). This approximation will take the form of a space of “configurations” of G -orbits in V .

In the Geometry of Iterated Loop Spaces [M1], J. P. May carries out this program using an approximation

$$\alpha_n: C_n X \rightarrow \Omega^n \Sigma^n X$$

for $1 \leq n$. The map α_n is a homotopy-equivalence when X is connected, and in general is a group-completion. This means that α_n is an H -map between an H -space and a group-like H -space, and $(\alpha_n)_*: H_*(C_n X) \rightarrow H_*(\Omega^n \Sigma^n X)$ is a localization of the ring $H_*(C_n X)$ at its multiplicative submonoid $\pi_0 X$ for field coefficients. (See [M2, Ch. 15].)

We wish to carry out such a program in the equivariant case, where all spaces are acted upon by a group G . The right space to approximate in place of $\Omega^n \Sigma^n X$ is the space $\Omega^V \Sigma^V X$. Such an approximation exists in the case where G is finite ([H1], [S2]). But the case where G is a compact Lie group is much deeper, and our approximation to $(\Omega^V \Sigma^V X)^G$ is therefore not the greatest generality one could hope for. Indeed, a suitable approximation to $\Omega^V \Sigma^V X$ (even in the stable case) would suffice to prove an equivariant “recognition principle”—a simple test enabling one to determine whether a given space admits deloopings of all orders. (When G is finite, May, Hauschild and Waner have developed such a principle.) In the last section we indicate what one could hope for in this regard, and plan to address the actual development of a recognition principle in a future paper.

1. Definitions and notations. Let G be a compact Lie group, and V a finite-dimensional real representation of G . Throughout the text, \mathcal{S}_j will stand for the symmetric group on j letters.

In this section, we will define a space $C_G(V, X)$ of “configurations” of G -orbits in V which will be the finite-dimensional version of an approximation to $(\Omega^V \Sigma^V X)^G$.

If H is a closed subgroup of G and Y is any G -space we will denote by Y^H and $Y_{(H)}$ the subspaces

$$(1.1) \quad \begin{aligned} Y^H &= \{y \in Y \mid G_y \geq H\} \quad (\text{the subspace of } Y \text{ fixed by } H), \\ Y_{(H)} &= \{y \in Y \mid G_y \sim H\}, \end{aligned}$$

where (H) stands for the class of subgroups conjugate to H , and

$$G_y = \{g \in G \mid yg = y\}$$

is the isotropy group of y . In particular, if W is a representation, W^H is a vector space acted on by the normalizer NH of H , $W_{(H)}$ is a G -manifold, and $W_{(H)} = W_{(H')}$ if $H \sim H'$.

Let $L(H)$ denote the tangent space of G/H at the coset $H \cdot e$. Note that if $H' = k^{-1}Hk$ is conjugate to H , there is a G -map

$$(1.1a) \quad G/H \rightarrow G/H' \quad \text{taking} \quad Hg \mapsto Hk^{-1}gk$$

and this induces a linear isomorphism $k_*: L(H) \rightarrow L(H')$. Note that if $k \in NH$, k_* is an automorphism of $L(H)$, so that $L(H)$ is a representation of NH . Let $K(H)$ denote $L(H)^H$.

If Y is any G -space, define the space of configurations of G -orbits

$$(1.2) \quad F^G(Y, j) = \{(y_1, \dots, y_j) \in Y^j \mid y_1 \cdot G, \dots, y_j \cdot G \text{ are pairwise disjoint orbits}\}.$$

This can be thought of as a generalization of a “configuration space” of n -tuples of distinct points, which is so important in the nonequivariant case ([M1], [B2]).

Let X be a based G -space (with G -fixed basepoint), and then define

$$F^G(V_{(H)}, j) \overline{\times}_{\mathcal{S}_j} (\Sigma^{L(H)^H} X^H)^j$$

as the subspace of

$$F^G(V_{(H)}, j) \times_{\mathcal{S}_j} \left(\prod_{H' \sim H} \Sigma^{(H')^H} X^{H'} \right)^j$$

consisting of points $((v_1, \dots, v_j), (x_1, \dots, x_j))$ such that $v_i \in V^{H'}$ if $x_i \in \Sigma^{K(H')} X^{H'}$, for $i = 1, \dots, j$.

Then let

$$(1.3) \quad C_G(V, X)_{(H)} = \coprod_{j \geq 0} F^G(V_{(H)}, j) \overline{\times}_{\mathcal{S}_j} (\Sigma^{K(H)} X^H)^j / \approx$$

where \approx is the equivalence relation generated by

$$(1.4a) \quad \left((v_1, \dots, v_j), (x_1, \dots, x_j) \right) \\ \approx \left((v_1 \cdot g, v_2, \dots, v_j); (x_1 \cdot g, x_2, \dots, x_j) \right)$$

for all $v_i \in V_{(H)} \cap V^{H_i}$, $x_i \in \Sigma^{L(H_i)^{H_i}} X^{H_i}$, and

$$(1.4b) \quad \left((v_1, \dots, v_j), (x_1, \dots, x_j) \right) \approx \left((v_2, \dots, v_j), (x_2, \dots, x_j) \right)$$

if $x_1 = *$. If $K \leq G$, and U is a sub- K -space of V , let

$$C_K(U, X)_{(H)} = \coprod_{j \geq 0} F^K(U \cap V_{(H)}, j) \overline{\times}_{\Sigma_j} (\Sigma^{L(H)^H} X^H)^j / \approx .$$

Finally, let

$$(1.5) \quad C_G(V, X) = \prod_{(H)} C_G(V, X)_{(H)}$$

where (H) ranges over the set of conjugacy classes of closed subgroups of G , and the product is the “weak product” (the direct limit of the finite subproducts via basepoint inclusions). Thus a point of $C_G(V, X)$ is represented by a tuple $(v_1, \dots, v_j; x_1, \dots, x_j)$ where $v_i \in V$ and

$$x_i \in (\Sigma^{L(G_{v_i})} X)^{G_{v_i}}.$$

We denote such points with square brackets in place of parentheses: $[v_1, \dots, v_j; x_1, \dots, x_j]$.

Now that we have defined the approximating space, we need an approximating map. Ideally this would be a map $\alpha: C_G(V, X) \rightarrow (\Omega^V \Sigma^V X)^G$, which would then be shown to be a group-completion under certain hypotheses. However, the orbits in $C_G(V, X)$ are not “thick enough” to make it convenient to define such a map directly, and (analogously to [C3], [M2]) we will turn to an intermediate space $\overline{C}_G(V, X)$ which will map to both $C_G(V, X)$ and $(\Omega^V \Sigma^V X)^G$.

First we will define a space of “thick orbits” in V . Let $\overline{F}^G(V)$ denote the spaces of discs D in V such that

- (i) D is a normal slice at its center v to the orbit $v \cdot G$ in V , and
- (ii) the map $D \times_H G \rightarrow V$ taking (d, g) to dg is an embedding, where $H = G_v$.

There is an injection $\iota: \overline{F}^G(V) \rightarrow V \times \mathbf{R}$ given by sending D to $(v, \text{radius } D)$, and we topologize $\overline{F}^G(V)$ as a subspace of $V \times \mathbf{R}$ via ι .

Let $\bar{F}^G(V)_{(H)}$ denote the subspace of $\bar{F}^G(V)$ of discs whose centers lie in $V_{(H)}$.

Finally, define

$$(1.6) \quad \bar{F}^G(V, j)_{(H)} = \left\{ (D_1, \dots, D_j) \in (\bar{F}^G(V)_{(H)})^j \mid D_1G, \dots, D_jG \text{ are pairwise disjoint sets} \right\}.$$

Now, in parallel with the previous definition, let

$$(1.7) \quad \bar{C}_G(V, X) = \left\{ (D_1, \dots, D_j; [x_1, t_1], \dots, [x_j, t_j]) \in \prod_{(H)} \bar{C}_G(V, X)_{(H)} \mid D_1G, \dots, D_jG \text{ are disjoint} \right\}.$$

Note that $C_G(V, X)$ and $\bar{C}_G(V, X)$ depend only on G, V and the G -homotopy type of X .

Now we are ready to define our approximating maps.

(1.8) LEMMA. *Let $\gamma_{(H)}: \bar{F}^G(V)_{(H)} \rightarrow V_{(H)}$ take a disc D to its center point. This induces a $G \wr \Sigma_j$ -equivariant homotopy equivalence*

$$\gamma'_{(H)}: \bar{F}^G(V, j)_{(H)} \rightarrow \bar{F}^G(V_{(H)}, j)$$

and so a homotopy equivalence

$$\gamma: \bar{C}_G(V, X) \rightarrow C_G(V, X).$$

(See [C5] Lemma 2.9.)

The other half of the approximation is a map

$$\alpha_v: \bar{C}_G(V, X) \rightarrow (\Omega^V \Sigma^V X)^G$$

which we define as follows.

Suppose D is a disc in $\bar{F}^G(V)_{(H)}$ with center $v \in V^H$ and $[x, t] \in (\Sigma^{L(H)} X)^H$. D spans the normal space N to $v \cdot G$ at v , and there is a homeomorphism

$$\psi = \psi_D: D \rightarrow N$$

given by

$$(1.9) \quad \psi_D(v + v') = \frac{v'}{\frac{1}{2} \text{diam}(D) - \|v'\|}.$$

The inclusion $G/H \rightarrow V$ taking Hg to $v \cdot g$ induces an identification ϕ of $L(H)$ with the tangent space T to $v \cdot G$ at v .

Define a map

$$\alpha[D, x, t]: S^V \rightarrow X \wedge S^V \approx X \wedge S^T \wedge S^N$$

by letting

$$(1.10) \quad \alpha[D, x, t](u) = \begin{cases} * & \text{if } u = \infty \text{ or } u \in V - (D \cdot G), \\ [xg, \phi(t)g, \psi(n)g] & \text{if } u = n \cdot g \text{ with } n \in D, g \in G. \end{cases}$$

This is a G -map, since

$$\alpha[D, x, t](n \cdot g\hat{g}) = [xg\hat{g}, \phi(t)g\hat{g}, \psi(n)g\hat{g}] = [xg, \phi(t)g, \psi(n)g] \cdot \hat{g}.$$

Since $D \times_H G \rightarrow D \cdot G$ is a homeomorphism, $\alpha[D, x, t] = \alpha[Dg, xg, tg]$. In fact, if $n \in D$,

$$(1.11) \quad \begin{aligned} \alpha[Dg, xg, tg](ng \cdot \hat{g}) &= [xg \cdot \hat{g}, \phi(tg)\hat{g}, \psi(ng)\hat{g}] \\ &= [x \cdot g\hat{g}, \phi(t)g\hat{g}, \psi(n)g\hat{g}] = \alpha[D, x, t](n \cdot g\hat{g}). \end{aligned}$$

Now we can define $\alpha = \alpha_v$.

Let $z = [D_1, \dots, D_j; [x_1, t_1], \dots, [x_j, t_j]] \in \bar{C}_G(V, X)$. Then we define

$$S^V \xrightarrow{\alpha(z)} X \wedge S^V$$

by

$$(1.12) \quad \alpha(z)(u) = \begin{cases} * & \text{if } \alpha[D_i, x_i, t_i](u) = * \text{ for all } i, \\ u_i & \text{if } \alpha[D_i, x_i, t_i](u) = u_i \neq * \text{ for some } i. \end{cases}$$

This is well-defined since the sets $D_i \cdot G$ are pairwise disjoint, and (1.11) holds. There is a general lemma which guarantees its continuity:

(1.13) LEMMA. *Let X be a filtered space, $\eta_j: F_j X \rightarrow (F_1 X)^j$, and let M be a partial monoid with j th multiplication $\mu_j: M_j \rightarrow M$. Let $\alpha: X \rightarrow M$ be a function and $\bar{\alpha}_j$ a function $(F_1 X)^j \rightarrow M_j$ such that for all $j \geq 1$,*

$$\begin{array}{ccc} F_j X & \xrightarrow{\alpha|_{F_j M}} & M \\ \eta_j \downarrow & & \uparrow \mu_j \\ (F_1 X)^j & \xrightarrow{\bar{\alpha}_j} & M_j \\ (\alpha|_{F_1 X})^j & \searrow \quad \swarrow & M^j \end{array}$$

commutes. Then α is continuous if $\alpha|_{F_1 X}$ is.

Proof. α is continuous if $\alpha|_{F_j X}$ is continuous for all j . □

In the present case, $(\Omega^V \Sigma^V X)^G$ is a partial monoid with $(\Omega^V \Sigma^V X)_j^G$ as the set of j -tuples $\langle f_1, \dots, f_j \rangle$ of maps from S^V to $\Sigma^V X$ such that no two of $f_1(v), \dots, f_j(v)$ are different from $*$ for any $v \in S^V$, and composition is done by combining f_1, \dots, f_j as in (1.12).

The space

$$Z = \prod_{(H)} \left(\coprod_{j \geq 0} \bar{F}^G(V_{(H)}, j) \bar{x}((\Sigma^{L(H)} X)^H)^j \right)$$

is filtered by

$$(1.14) \quad F_j Z = \left\{ z|_{z_{(H)}} \neq * \text{ for only finitely many } (H)\text{'s,} \right. \\ \left. (H_1), \dots, (H_k), z_{(H_i)} \in \bar{F}^G(V_{(H_i)}, l_i) \times ((\Sigma^{L(H_i)} X)^{H_i})^{l_i} \right. \\ \left. \text{and } l_1 + \dots + l_k \leq j \right\}.$$

The space Z projects onto $\bar{C}_G(V, X)$ via a quotient map ρ , and the composite $\alpha\rho$ satisfies the hypotheses of the lemma and hence is continuous; thus α is continuous.

We can now state the main theorems.

(1.15) **THEOREM.** *Let W be a G -vector space containing an infinite-dimensional trivial representation \mathbf{R}^∞ , such that W is the direct limit of its finite-dimensional subspaces. Define*

$$\bar{C}_G(W, X) = \varinjlim \bar{C}_G(V, X),$$

and

$$\Omega^W \Sigma^W X = \varinjlim \Omega^V \Sigma^V X,$$

where V ranges over finite-dimensional G -subspaces of W , and let $\alpha_W = \varinjlim \alpha_V$. Then if X is a countable G -CW complex,

$$\alpha_W: \bar{C}_G(W, X) \rightarrow (\Omega^W \Sigma^W X)^G \text{ is a group-completion.}$$

Let $A(G)$ denote the Burnside ring of G . In the finite case, this is the universal enveloping ring for the semi-ring of isomorphism types of finite G -sets, under the operations of disjoint union and Cartesian product. Tom Dieck defines it analogously for Lie groups. (See **D2**.) We will compute

the structure of $\pi_0(C_G(W, X))$, and (1.8) and (1.15) will allow us to deduce the additive part of tom Dieck's result in [D1]:

(1.16) COROLLARY. *Define $W = \mathbf{GR}^\infty$ to be the direct sum $\bigoplus_i V_i^\infty$, where V_1, V_2, \dots are the irreducible real representations of G , and V_i^∞ denotes the direct sum of infinitely many copies of V_i . Then*

$$A(G) \cong \varinjlim_{V < W} [S^V, S^V]$$

where V runs over all finite-dimensional G -subspaces of W and the direct limit is taken over suspension homomorphisms.

We also have a splitting theorem:

(1.17) COROLLARY. *$(\Omega^W \Sigma^W X)$ is equivalent to a product*

$$\prod_{(H)} \varinjlim_{V < W} \text{Map}_0(V^H / (V^H - V_{(H)}), \Sigma^{V^H} X^H)^{NH}.$$

Finally it is worth noting how nicely this result restrict to the finite case:

(1.18) THEOREM. *Let*

$$C(V, X) = \left(\coprod_{j < 0} F(V, j) \times_{\Sigma_j} X^j \right) / \sim$$

be the usual configuration space. If $W \geq \mathbf{R}^\infty$, and X is a countable G -CW complex, then $C(W, X)$ is a based G -space, and there is a map

$$\alpha: C(W, X) \rightarrow \Omega^W \Sigma^W X$$

such that $\alpha^H: C(W, X)^H \rightarrow (\Omega^W \Sigma^W X)^H$ is a group-completion for all $H \leq G$.

Proof. Since G is finite, $L(H) = 0$ for all H , and G/H is a finite set. Define a homeomorphism

$$h: C_G(W, X) \xrightarrow{\cong} C(W, X)^G$$

by

$$h[\langle v_1, \dots, v_j \rangle; x_1, \dots, x_j] = [\langle v_i \cdot g \mid i = 1, \dots, j, g \in G \rangle, \langle x_i \cdot g \rangle].$$

The approximation

$$C_G(W, X) \xleftarrow[\cong]{\gamma} K(W, X) \xrightarrow{\alpha} \Omega^W \Sigma^W X$$

using little convex bodies K restricts to

$$\alpha^G: C(W, X)^G \xleftarrow[\approx]{h} C_G(W, X) \xleftarrow[\approx]{\gamma} \overline{C}_G(W, X) \xrightarrow{\alpha_w} (\Omega^W \Sigma^W X)^G;$$

restricting to a subgroup H lets us conclude that α^H is a group-completion for all $H < G$. \square

The nice point here is that we needn't approximate the various fixed-point sets of $\Omega^W \Sigma^W X$ separately when G is finite.

2. Fiberings of equivariant function spaces. Let V be a real representation of G .

Here we will produce fiberings involving various subspaces of $(\Omega^V \Sigma^V X)^G$, and we will use these later to reduce the proof of the main theorem to consideration of spaces of functions which are nontrivial only on maximal orbits.

(2.1) an *orbit-type family* \mathcal{F} for G is a collection of closed subgroups such that if $H \in \mathcal{F}$ and K is subconjugate to H , then $K \in \mathcal{F}$.

An *orbit of class* \mathcal{F} is any G -orbit isomorphic to G/H for some $H \in \mathcal{F}$. If 1 denotes the trivial subgroup, the *maximal orbits* are those of class $\{1\}$.

We will examine subspaces of $(\Omega^V \Sigma^V X)^G$. Let K be a subgroup of G , let U be a sub- K -space of V , and let Y be any based K -space. Then define

$$(2.2) \quad (\Omega^U Y)_{\mathcal{F}}^K = \{f \in (\Omega^U Y)^K \mid f(U \cap V^H) = * \text{ for all } H \notin \mathcal{F}\}.$$

That is, $(\Omega^U Y)_{\mathcal{F}}^K$ consists of K -maps f which are non-trivial only on orbits which are of class \mathcal{F} in V .

We will construct fibrations in the situation where \mathcal{F}_1 and \mathcal{F}_2 are *successive* orbit families, that is, $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{F}_2 - \mathcal{F}_1$ consists of just one conjugacy class (H), so that \mathcal{F}_1 contains all proper subgroups of H . In this case we may form the following sequence

$$(2.3) \quad (\Omega^V Y)_{\mathcal{F}_1}^G \xrightarrow{i} (\Omega^V Y)_{\mathcal{F}_2}^G \xrightarrow{p} (\Omega^{V^H} Y^H)_{\mathcal{F}_2}^{NH}$$

where i is inclusion of subspaces and p is obtained by restricting each element of its domain to S^{V^H} .

(2.4) **THEOREM.** *The sequence (2.3) is a fiber sequence onto the image of p , which is a union of components of $(\Omega^{V^H} Y^H)_{\mathcal{F}_2}^{NH}$.*

Proof. Suppose we have an NH -map

$$h: Z \wedge I^+ \wedge S^{V^H} \rightarrow Y^H$$

such that $h(Z \wedge I^+ \wedge S^{V^K}) = *$ for all $K \leq H$ such that $K \notin \mathcal{F}_2$ that is, for all K such that $H \not\leq K \leq NH$. Suppose also that we have a G -map

$$H_0: Z \wedge \{0\}^+ \wedge S^V \rightarrow Y$$

such that $pH_0 = h_0$.

Since h is NH -equivariant and nontrivial only on $(Z \wedge I^+ \wedge (S^{V^H})_{(H)})$, it extends uniquely to a G -map

$$\bar{h}: Z \wedge I^+ \wedge (S^{V^H} \cdot G) \rightarrow Y^H \cdot G \rightarrow Y$$

where $S^{V^H} \cdot G \subseteq S^V$. In fact, this inclusion is a cofibration, and hence the homotopy extension property for $(S^V, S^{V^H} \cdot G)$ implies that H_0 extends to a lift H of h . Also it is clear that $p^{-1}(*) = (\Omega^V Y)_{\mathcal{F}_1}^G$.

This establishes the result, including the fact that the image of p is a union of components. \square

3. Decomposition of configuration spaces. It is convenient to develop a decomposition-theory for the spaces $\bar{C}_G(V, X)$ parallel to that developed in §2 for mapping spaces. It turns out that the theory and the arguments are quite a bit simpler.

Recall the notations of the previous section. Define

$$\bar{C}_K(U, Y)_{\mathcal{F}}$$

to be subspace of $\bar{C}_K(U, Y)$ consisting of configurations $[D_1, \dots, D_j; [x_1, t_1], \dots, [x_j, t_j]]$, where for each i the center point v_i of D_i gives rise to an orbit $v_i \cdot G$ in V of type \mathcal{F} ; that is, $v_i \in V_{(H)}$ for some $H \in \mathcal{F}$. Then

$$(3.1) \quad \bar{C}_K(U, Y) = \prod_{(H) \subset \mathcal{F}} \bar{C}_K(U, Y)_{(H)}.$$

If H is a subgroup of G , maximal in \mathcal{F} , we can define a homeomorphism

$$(3.2) \quad k: \bar{C}_G(V, X)_{(H)} \rightarrow \bar{C}_{NH}(V^H, X^H)_{(H)} = \bar{C}_{NH}(V^H, X^H)_{\mathcal{F}}$$

as follows. Let

$$k[D_1, \dots, D_j; [x_1, t_1], \dots, [x_j, t_j]] = [D'_1, \dots, D'_j; [x'_1, t'_1], \dots, [x'_j, t'_j]],$$

where D'_i is related to D_i as follows: if $v_i \in V^{gHg^{-1}}$, let $v'_i = v_i g$, and let D'_i be the disc normal to the orbit $v'_i \cdot NH$ at v'_i in V^H , with the same radius

as D_i . Let $t'_i = g_*(t_i)$, where

$$g_*: L(gHg^{-1})^{gHg^{-1}} \rightarrow L(H)^H \cong (\tau(NH/H)_{eH})^H$$

is the map in (1.12), and let $x'_i = x_i \cdot g$.

The equivalence relation \approx in (1.4) and (1.6) shows that k is well-defined, and a well-defined inverse can be given by sending $[D'_i, [x'_i, t'_i]]$ to $[D''_i, [x'_i, t'_i]]$, where D''_i is the disc with the same diameter as D'_i , normal in V to $v'_i \cdot G$.

Thus we may define a product bundle

$$(3.3) \quad \bar{C}_G(V, X)_{\mathcal{F}_1} \xrightarrow{i} \bar{C}_G(V, X)_{\mathcal{F}_2} \xrightarrow{p} \bar{C}_{NH}(V^H, X^H)_{\mathcal{F}_2}$$

where i is inclusion and p is the projection

$$\prod_{(K) \subset \mathcal{F}_2} \bar{C}_G(V, X)_{(K)} \rightarrow \bar{C}_G(V, X)_{(H)}$$

composed with k .

4. Duality in equivariant function spaces. The previous two sections reduce the main work of proving (1.15) to showing that α restricts to an equivalence of $\bar{C}_{NH}(W^H, X^H)_{\mathcal{F}}$ and $(\Omega^{W^H} \Sigma^{W^H} X^H)_{\mathcal{F}}^{NH}$ when H is maximal in \mathcal{F} . This “maximal orbit type” case can be attacked by something similar to methods used by Becker and Schultz in [B1].

In this section, we will obtain an equivalence

$$(4.1) \quad \varepsilon: (\Omega^{W^H} \Sigma^{W^H} X^H)_{\mathcal{F}}^{NH} \rightarrow \Omega^\infty \Sigma^\infty (EJ^+ \wedge_J \Sigma^L X^H)$$

where $\mathbf{R}^\infty \leq W$, H is maximal in \mathcal{F} , $J = NH/H$ and $L = L(J/1)$ is the Lie algebra of J . The space $\bar{C}_{NH}(W^H, X^H)_{\mathcal{F}}$ is analyzed in the following section.

Recall the following definitions and notation from [B1]. A *sectioned bundle* is a bundle $\xi: E \rightarrow B$ equipped with a section $\Delta_\xi: B \rightarrow E$ (so that $\xi \circ \Delta_\xi = \text{id}$). If B is fixed, sectioned bundles over B form a topological category, and if ξ, η are sectioned bundles over B , define

$$\text{Bund}_0(\xi, \eta)$$

to be the space of morphisms from ξ to η . These are bundle maps f over B such that $f \circ \Delta_\xi = \Delta_\eta$.

The fiber $\xi^{-1}(b)$ of ξ over b may be thought of as a based space with basepoint $\Delta_\xi(b)$, and we may construct several functors on the category of sectioned bundles over B from standard functors on the category of based spaces. For example, if ξ and η are sectioned bundles over B , let $\xi \wedge \eta$

denote the “fiberwise smash product” of ξ and η ; the fiber of $\xi \wedge \eta$ over b is

$$(\xi \wedge \eta)^{-1}(b) = \xi^{-1}(b) \wedge \eta^{-1}(b),$$

with basepoint $\Delta_\xi(b) \wedge \Delta_\eta(b)$.

If $A \subseteq B$, $\xi|_A$ is defined to be the bundle $\xi^{-1}(A) \rightarrow A$ whose projection and section are the restrictions of ξ and Δ_ξ .

If X is a based space, let \dot{X} denote the product bundle $p: X \times B \rightarrow B$ with $p(x, b) = b$ and $\Delta(b) = (*, b)$.

If $\alpha: E \rightarrow B$ is a vector bundle, let $\bar{\alpha}$ denote the based sphere bundle obtained by taking the fiberwise one-point compactification of E and letting Δ be the cross-section at infinity. Note that $\overline{\alpha \oplus \beta}$ is canonically isomorphic to $\bar{\alpha} \wedge \bar{\beta}$.

There is a functor T from sectioned bundles to based spaces defined by

$$T(\xi) = E/\Delta_\xi(B).$$

If α is a vector bundle, $T(\bar{\alpha})$ is just the usual Thom space of α , also denoted $T\alpha$ or B^α . If $A \subseteq B$, define

$$(B, A)^\xi = E/(\Delta(B) \cup \xi^{-1}(A)).$$

We may also define a category of pairs (ξ, ξ') where ξ is a sectioned bundle of B and ξ' is a subbundle of $\xi|_A$; if (η, η') is another such pair let the morphism space

$$\text{Bund}_0(\xi, \xi'; \eta, \eta')$$

be the subspace of $\text{Bund}_0(\xi, \eta)$ of maps sending ξ' into η' .

(4.2) LEMMA. (i) *There is a natural isomorphism $T(\dot{X} \wedge \xi) \approx X \wedge T(\xi)$.*

(ii) *T is a continuous functor, and induces a homeomorphisms*

$$\text{Bund}_0(\xi, \xi|_A; \dot{X}, *) \xrightarrow{\cong} \text{Map}_0((B, A)^\xi, X).$$

This is a simple check.

If H is maximal in \mathcal{F} , Y is an NH/H -space, and V is a sub- NH -space of W^H , then $(\Omega^V \Sigma^V Y)_{\mathcal{F}}^{NH}$ is homeomorphic to the mapping space

$$\text{Map}(S^V, S^V - (V \cap W_{(H)}); \Sigma^V Y, *)^J$$

where $J = NH/H$. We will thus need to make some remarks about mapping spaces of G -manifolds.

(4.3) REMARKS. Let $(M, \partial M)$ be a compact smooth manifold with boundary, on which NH acts such that $M = M_{(H)}$. Then J is a finite group acting on M , so M/J is a manifold, and

$$\mu: M \rightarrow M/J$$

is a principal G -bundle. Suppose that M is contained in a J -vector space V such that $\dim M = \dim V$. Then the natural projection

$$M \times_J V \rightarrow M/J$$

is equivalent to the Whitney sum of the tangent bundle τ of M/J and the bundle

$$\pi: M \times_J L \rightarrow M/J$$

where L is the Lie algebra of J .

Consider the mapping space $\text{Map}_J(M, \partial M; \Sigma^V Y, *)$ of J -maps from M to $\Sigma^V X$ taking ∂M to $*$, where Y is a based J -space. This can be identified with the space of sections of the bundle

$$\eta: M \times_J \Sigma^V Y \rightarrow M/J$$

taking the value $*$ on $\partial M/J$. This is the space of bundle maps

$$\text{Bund}_0(\dot{S}^0, \dot{S}^0 | (\partial M/J); \eta, *).$$

From the note on $M \times_J V$, we see that η may be decomposed as

$$(4.4) \quad \eta = \bar{\tau} \wedge \bar{\pi} \wedge \xi$$

where $\xi: M \times_J Y \rightarrow M/J$ is the natural projection with section $\Delta[m] = [m, *]$.

Choose some embedding $i: M/J \rightarrow \mathbf{R}^s$ and let v be its normal bundle. Then we have a Pontryagin-Thom map

$$C: S^s \rightarrow (M/J, \partial M/J)^v,$$

and a natural isomorphism of $v \oplus \tau$ to the trivial vector bundle with fiber \mathbf{R}^s . Define a map $\varepsilon(M)$ as the composite

$$(4.5) \quad \begin{array}{c} \text{Map}(M, \partial M; \Sigma^V Y, *)^J \\ \downarrow \mathcal{R} \\ \text{Bund}_0(\dot{S}^0, \dot{S}^0 | (\partial M/J); \bar{\tau} \wedge \bar{\pi} \wedge \xi, *) \\ \downarrow \sigma \\ \text{Bund}_0(\bar{v}, \bar{v} | (\partial M/J); \overline{v \oplus \tau} \wedge \bar{\pi} \wedge \xi, *) \\ \parallel \\ \text{Bund}_0(\bar{v}, \bar{v} | (\partial M/J); \dot{S}^s \wedge \bar{\pi} \wedge \xi, *) \\ \downarrow T \\ \text{Map}_0((M/J, \partial M/J)^v, \Sigma^s(M/J)^{\bar{\pi} \wedge \xi}) \\ \downarrow C^* \\ \text{Map}_0(S^s, \Sigma^s(M/J)^{\bar{\pi} \wedge \xi}) \\ \parallel \\ \Omega^s \Sigma^s(M^+ \wedge_J \Sigma^L Y), \end{array}$$

where σ is suspension by \bar{v} .

The manifold we wish to study is a manifold $M = M(V)$ constructed by letting $\delta > 0$ be small enough that

$$M = \left[V - B_\delta(V - V \cap W_{(H)}) \right] \cap \overline{B_{1/\delta}(0)}$$

is a deformation retract of $V \cap W_{(H)}$, where

$$B_\delta(A) = \text{the } \delta\text{-neighborhood of } A.$$

Then

$$(\Omega^V \Sigma^V Y)_{\mathcal{F}}^{NH} \approx \text{Map}_0(M/\partial M, \Sigma^V Y)^J,$$

because $M/\partial M$ is homeomorphic to $S^V/(S^V - V \cap W_{(H)})$.

We cannot prove that $\varepsilon(M(V))$ is an equivalence if V is finite-dimensional. However, let W be a representation of G containing an orbit isomorphic to G/H and a copy of \mathbf{R}^∞ . Then there is a sequence of finite-dimensional sub- NH -spaces

$$(4.6) \quad V_1 < V_2 < \dots < W^H$$

such that $W^H = \bigcup_n V_n$. Define $M_n = M(V_n)$ for $n = 1, 2, 3, \dots$, and choose δ_n so that

$$(M_1, \partial M_1) \subset (M_2, \partial M_2) \subset \dots$$

The union $\bigcup_n M_n$ is a free J -space which can be shown to be contractible and hence may be thought of as the total space EJ of the universal bundle of J .

We may also choose the embeddings $i_n: M_n/J \rightarrow \mathbf{R}^{s_n}$ so that $s_n < s_{n+1}$, and so that

$$(4.7) \quad \begin{array}{ccc} M_n/J & \xrightarrow{i_n} & \mathbf{R}^{s_n} \\ \downarrow & & \downarrow i' \\ M_{n+1}/J & \xrightarrow{i_{n+1}} & \mathbf{R}^{s_{n+1}} \end{array}$$

commutes, where $i'(\underline{t}) = (\underline{t}, 0)$ is the standard inclusion. It follows that

$$(4.8) \quad \begin{array}{ccc} \text{Map}(M_n, \partial M_n; \Sigma^{V_n} Y, *)^J & \xrightarrow{\varepsilon(M_n)} & \Omega^{s_n} \Sigma^{s_n} (M_n^+ \wedge_J \Sigma^{L} Y) \\ j \downarrow & & \downarrow \sigma \\ \text{Map}(M_{n+1}, \partial M_{n+1}; \Sigma^{V_{n+1}} Y, *)^J & \xrightarrow{\varepsilon(M_{n+1})} & \Omega^{s_{n+1}} \Sigma^{s_{n+1}} (M_{n+1}^+ \wedge_J \Sigma^{L} Y) \end{array}$$

commutes, where σ is suspension composed with the inclusion induced from $M_n \hookrightarrow M_{n+1}$, and j is defined as making the following diagram commute:

$$\begin{array}{ccc} \text{Map}(M_n, \partial M_n; \Sigma^{V_n} Y, *)^J & \approx & (\Omega^{V_n} \Sigma^{V_n} Y)^J_{\mathcal{F}} \\ j \downarrow & & \downarrow \sigma \\ \text{Map}(M_{n+1}, \partial M_{n+1}; \Sigma^{V_{n+1}} Y, *)^J & \approx & (\Omega^{V_{n+1}} \Sigma^{V_{n+1}} Y)^J_{\mathcal{F}} \end{array}$$

(4.10) PROPOSITION. *Let $U = W^H$ and $Y = X^H$. Taking the direct limit over the sequence (4.6) via diagram (4.8), define ε to be the composite*

$$\begin{aligned} (\Omega^U \Sigma^U Y)^{NH}_{\mathcal{F}} &\xrightarrow{h} \varinjlim \text{Map}(M_n, \partial M_n; \Sigma^{V_n} Y, *) \\ &\xrightarrow{\lim} \lim_{\rightarrow} \Omega^{s_n} \Sigma^{s_n} (M_n^+ \wedge_j \Sigma^L Y) = \Omega^\infty \Sigma^\infty (EJ^+ \wedge_j \Sigma^L Y). \end{aligned}$$

Then ε is an equivalence of H -spaces.

Proof. The H -structure on $(\Omega^U \Sigma^U Y)^{NH}_{\mathcal{F}}$ comes from the loop multiplication, since $\mathbf{R}^\infty \leq W^H$, and this is carried over to the mapping spaces since if $V_n \oplus \mathbf{R} \subseteq V_{n+1}$, then $M_n \times \mathbf{R} \subseteq M_{n+1}$. The map ε is an H -map since we may choose the V_n 's to have the form $V'_n \oplus \mathbf{R}$, where $U = (\bigcup_n V'_n) \oplus \mathbf{R}$, and we can choose $i_n: M_n/J \rightarrow \mathbf{R}^{s_n}$ to have the form

$$(4.11) \quad M_n/J \approx M(V'_n)/J \times I \xrightarrow{i_n^{\times 1}} \mathbf{R}^{s_n-1} \times I \rightarrow \mathbf{R}^{s_n}$$

where $I = [0, 1] \subseteq \mathbf{R}$.

The proof that ε is an equivalence occupies the remainder of this section.

Some of the maps we will use are duality maps from a fiberwise duality involving $\bar{\pi} \wedge \xi$. We recall some facts of equivariant topology.

(4.12) LEMMA. *Let (X, A) be a pair of finite G -CW complexes. Then X embeds in the unit ball of a representation V such that*

- (i) X is a G -Euclidean neighborhood retract (G -ENR) of some invariant neighborhood U ,
- (ii) there is a G -deformation $\bar{U} - X \rightarrow \partial \bar{U}$, and
- (iii) A embeds in a hemisphere E^+ of the unit sphere with (i) and (ii) restricting appropriately.

In this case, an argument similar to that of Atiyah in [A1] shows that $\bar{U}/\partial \bar{U}$ is S -dual to X/A , and $\bar{U}/(\partial \bar{U} \cup \bar{U} \cap E^+)$ is S -dual to X .

A fiberwise duality is a map $\gamma \wedge \hat{\gamma} \rightarrow \dot{S}^t$ of bundles which restricts to a duality on fibers. The above lemma proves the existence of a fiberwise dual for a bundle γ whose fibers are finite G -CW complexes. Hence assume from now on that Y is finite, and let $\hat{\gamma}$ be the dual to the bundle $\gamma = \bar{\pi} \wedge \xi$, and $i': M \times_J \Sigma^L Y \rightarrow M/J \times \mathbf{R}^t$ the inclusion as a fiberwise ENR.

In this case, we may define an embedding

$$(4.13) \quad j: M \times_J \Sigma^L Y \xrightarrow{i'} M/J \times \mathbf{R}^{t \times 1} \rightarrow \mathbf{R}^s \times \mathbf{R}^t = \mathbf{R}^{s+t}.$$

Then we can prove

(4.14) PROPOSITION. *The Thom space $(M/J, \partial M/J)^{\hat{\gamma} \wedge \bar{v}}$ is S -dual to $M^+ \wedge_J \Sigma^L Y$, where v is normal bundle of i .*

Proof. Let $B = M/J$, $E = M \times_J \Sigma^L Y$; thus $\partial B = (\partial M)/G$. Let $\gamma: E \rightarrow B$ be the bundle, $i: B \rightarrow \mathbf{R}^s$, and $i': E \rightarrow B \times \mathbf{R}^t$. Finally let $\gamma': U \rightarrow B$ be the projection to B of the neighborhood U of $\text{im}(i')$ in $B \times \mathbf{R}^t$, and define $\hat{\gamma}: \bar{U}/\partial \bar{U} \rightarrow B$ to be the fiberwise collapse of $\partial \bar{U}$. Now $B^\gamma = E/\Delta_\gamma(B)$. Define i and i' so that the embedding $(i \times 1) \circ i'$ sends E into $I^{s+t} \subset \mathbf{R}^{s+t}$ with $\Delta(B)$ embedded in $I^s \times 0$. By Atiyah's argument, $E/\Delta(B)$ is dual to $I^{s+t} - E$, which is equivalent to $I^{s+t}/I^{s+t} - U'$, U' being a regular neighborhood of E in I^{s+t} . This is the same as collapsing out $\gamma^{-1}(\partial B)$, and collapsing the boundary of $(\gamma')^{-1}(b) \times v^{-1}(b)$ to a point for all $b \in B$.

But this is just the Thom space of $(\gamma'/\partial\gamma') \wedge (v/\partial v) = \hat{\gamma} \wedge \bar{v}$ over $(B, \partial B)$. Hence

$$B^\gamma \text{ is } S\text{-dual to } (B, \partial B)^{\hat{\gamma} \wedge \bar{v}}. \quad \square$$

In the above proposition, the duality map

$$(M^+ \wedge_J \Sigma^L Y) \wedge (M/J, \partial M/J)^{\bar{v} \wedge \hat{\gamma}} \rightarrow S^{s+t}$$

comes from the embedding j in (4.13). This and the fiberwise duality $\gamma \wedge \hat{\gamma} \rightarrow \dot{S}^t$ induce maps

$$(4.15) \quad \text{Bund}_0(\mu, \mu' \wedge \gamma) \rightarrow \text{Bund}_0(\mu \wedge \hat{\gamma}, \mu' \wedge \dot{S}^t)$$

and

$$\text{Map}_0(Y, M^+ \wedge_J \Sigma^L Y) \rightarrow \text{Map}_0(Y \wedge (M/J, \partial M/J)^{\bar{v} \wedge \hat{\gamma}}, S^{s+t})$$

which will be generically denoted by D . We now use these maps to complete the proof that ε is an equivalence.

By construction $\varepsilon = \lim_{\rightarrow} C_n^* \circ T_n \circ \sigma(\bar{v}_n)$. The connectivity of the fiber of \bar{v}_n tends to infinity with n , and a suspension theorem [J1] applies to prove that $\lim_{\rightarrow} \sigma(\bar{v}_n)$ is an equivalence

To show that $\lim_{\rightarrow} C_n^* \circ T_n$ is a equivalence, we note that the following diagram commutes for $M = M_n$:

(4.16)

$$\begin{array}{ccc}
 \text{Bund}_0(\bar{v}, \bar{v}|(\partial M/J); \overline{\tau \oplus v \wedge \gamma}, *) P & \xrightarrow{D} & \text{Bund}_0(\bar{v} \wedge \hat{\gamma}, \bar{v} \wedge \hat{\gamma}|(\partial M/J); \dot{S}^{s+t}, *) \\
 T \downarrow & & \downarrow T \\
 \text{Map}_0((M/J, \partial M/J)^v, \Sigma^s(M^+ \wedge_J \Sigma^L Y)) & & \text{Map}_0((M/J, \partial M/J)^{\bar{v} \wedge \hat{\gamma}}, \Sigma^{s+t}(M/J^+)) \\
 C^* \downarrow & & \downarrow p \\
 \text{Map}_0(S^s, \Sigma^s(M^+ \wedge_J \Sigma^L Y)) & & \text{Map}_0((M/J, \partial M/J)^{\bar{v} \wedge \hat{\gamma}}, S^{s+t}) \\
 \sigma \downarrow & \swarrow \sigma & \searrow D \nearrow \\
 & M^+ \wedge_J \Sigma^L Y & \\
 \sigma \downarrow & & \downarrow \sigma \\
 \text{Map}_0(S^N, \Sigma^N(M^+ \wedge_J \Sigma^L Y)) & \xrightarrow{D} & \text{Map}_0(\Sigma^N(M/J, \partial M/J)^{\bar{v} \wedge \hat{\gamma}}, S^{s+t+N})
 \end{array}$$

where the σ 's are suspensions, N is any number $\geq s$, and $p: \Sigma^{s+t}(M/J^+) \rightarrow S^{s+t}$ collapses M/J to a point. By (4.2) (ii), the composite pT is a homeomorphism since $\tau \oplus v = \dot{S}^{s+t}$ is trivial.

Passing to the limit over M_n , the suspensions and duality maps become equivalences, and hence so does $\lim_{\rightarrow} C_n^* T_n$.

This was all done assuming X finite, but now a simple colimit argument allows us to deduce the same result when X is a countable G -CW complex as in the hypotheses to (1.15). \square

5. Duality and configuration spaces. This is parallel to §4; we will exhibit homotopy-equivalences

$$(5.1) \quad \mu: C_{NH}(U, Y)_{\mathcal{F}} \rightarrow C(\mathbf{R}^\infty, EJ^+ \wedge_J \Delta^L Y)$$

where U, Y , and L are as in (4.10).

Let $V_1 < V_2 < \dots < U$ be as in (4.6), and $M_n = M(V_n)$ for $n = 1, 2, \dots$. Now define $C_{NH}(M|\partial M, Y)$ as the space of configurations in $C_{NH}(V, Y)_{\mathcal{F}}$ whose V -coordinates all lie in $M - \partial M$. This suggestive notation is to indicate that they may be thought of as approximations to $\text{Map}_{NH}(M/\partial M, \Sigma^V Y)$, though this will not be developed here. Note that the natural homeomorphism $M - \partial M \approx V \cap W_{(H)}$ induces a homeomorphism

$$(5.2) \quad C_{NH}(V, Y)_{\mathcal{F}} \approx C_{NH}(M|\partial M, Y).$$

Recall the embedding $i: M/G \rightarrow \mathbf{R}^s$ and define

$$(5.3) \quad \phi: C_{NH}(M|\partial M, Y) \rightarrow C(\mathbf{R}^s, M^+ \wedge_J \Sigma^L Y)$$

by sending

$$\begin{aligned} & (m_1, \dots, m_k; [x_1, l_1], \dots, [x_k, l_k]) \quad \text{to} \\ & (i[m_1], \dots, i[m_k]; [m_1, x_1, l_1], \dots, [m_k, x_k, l_k]) \end{aligned}$$

and

$$\begin{aligned} \psi: C(\mathbf{R}^s, M^+ \wedge_J \Sigma^L Y) & \rightarrow C_{NH}(\mathbf{R}^s \times (M - \partial M), Y)_{\mathcal{F}} \\ & = C_{NH}(M(\mathbf{R}^s \times V)|\partial M(\mathbf{R}^s \times V), Y) \end{aligned}$$

by

$$(p, [m, x, l]) \mapsto (p, j(m); [x, l])$$

where $j: M \rightarrow M$ is a self-embedding with $j(M) \subseteq M - \partial M$.

We may choose V_1, V_2, \dots so that

$$V_n \oplus \mathbf{R}^{s_n} < V_{n+1},$$

and so that

$$(5.4) \quad \begin{array}{ccc} (M_n \times \mathbf{R}^{s_n})/J & \xrightarrow{i} & M_{n+1}/J \\ \cong & & \downarrow l_{n+1} \\ M_n/J \times \mathbf{R}^{s_n} & & \downarrow \\ i_n \times 1 \downarrow & & \downarrow \\ \mathbf{R}^{s_n} \times \mathbf{R}^{s_n} & \xrightarrow{i'} & \mathbf{R}^{s_{n+1}} \end{array}$$

commutes.

Then the following diagram commutes:

$$(5.5) \quad \begin{array}{ccccc} C_{NH}(V_n, Y)_{\mathcal{F}} & \cong & C_{NH}(M_n|\partial M_n, Y) & & \\ \downarrow & & \downarrow C(i_0) & \searrow \phi_n & C(\mathbf{R}^{s_n}, M_n^+ \wedge_J \Sigma^L Y) \\ & & C_{NH}(M_n \times \mathbf{R}^{s_n}|\partial M_n \times \mathbf{R}^{s_n}, Y) & \swarrow \psi_n & \downarrow \\ & & \downarrow & & C(\mathbf{R}^{s_{n+1}}, M_n^+ \wedge_J \Sigma^L Y) \\ C_{NH}(V_{n+1}, Y)_{\mathcal{F}} & \cong & C_{NH}(M_{n+1}|\partial M_{n+1}, Y) & \xrightarrow{\phi_{n+1}} & \downarrow (C1, i_n) \\ & & & & C(\mathbf{R}^{s_{n+1}}, M_{n+1}^+ \wedge_J \Sigma^L Y). \end{array}$$

We pass to limits and obtain the composite

$$(5.6) \quad \begin{array}{c} \left[\begin{array}{c} C_{NH}(U, Y)_{\mathcal{F}} \\ \parallel \\ \varinjlim C_{NH}(V_n, Y)_{\mathcal{F}} \\ \parallel \wr \\ \varinjlim C_{NH}(M_n | \partial M_n, Y) \\ \lim_{\rightarrow} \phi_n \downarrow \quad \uparrow \lim_{\rightarrow} \psi_n \\ \varinjlim C(\mathbf{R}^{s_n}, M_n^+ \wedge_J \Sigma^L Y) \\ \parallel \\ C(\mathbf{R}^\infty, EJ^+ \wedge_J \Sigma^L Y) \end{array} \right. \\ \mu \end{array}$$

which is an equivalence of H -spaces.

A similar theorem is provable for $\bar{C}_{NH}(U, Y)$; in fact there is an equivalence $\bar{\mu}$ such that

$$\begin{array}{ccc} C_{NH}(U, Y) & \xleftarrow{\gamma} & \bar{C}_{NH}(U, Y) \\ \mu \downarrow & & \downarrow \bar{\mu} \\ C(\mathbf{R}^\infty, EJ^+ \wedge_J \Sigma^L Y) & \xleftarrow{\gamma'} & C_\infty(EJ^+ \wedge_J \Sigma^L Y) \end{array}$$

commutes, where γ' is the map replacing each little cube with its center-point.

6. Proof of the main theorem. This section contains the proof of (1.15); in fact, we will prove the following general statement:

(6.1) If $\mathbf{R}^\infty \leq W$, then the restriction of α ,

$$\alpha_{\mathcal{F}}: C_G(W, X)_{\mathcal{F}} \rightarrow (\Omega^W \Sigma^W X)_{\mathcal{F}}^G,$$

is a group completion for any orbit-type family \mathcal{F} of closed subgroups of G .

The proof of (6.1) consists of a series of lemmas.

(6.2) LEMMA. *If H is maximal in \mathcal{F} , then the induced map*

$$\alpha^{NH}: C_{NH}(W^H, X^H)_{\mathcal{F}} \rightarrow (\Omega^{W^H} \Sigma^{W^H} X^H)_{\mathcal{F}}^{NH}$$

is a group-completion.

Proof. Assembling the results of §§4–5, we see that there is a diagram of H -spaces and H -maps:

$$\begin{array}{ccccc} C_{NH}(W^H, X^H)_{\mathcal{F}} & \xleftarrow{\gamma} & \bar{C}_{NH}(W^H, X^H)_{\mathcal{F}} & \xrightarrow{\alpha} & (\Omega^{W^H} \Sigma^{W^H} X^H)_{\mathcal{F}}^{NH} \\ \downarrow \mu & & \downarrow \bar{\mu} & & \downarrow \epsilon \\ C(\mathbf{R}^\infty, EJ^+ \wedge_J \Sigma^L X^H) & \xleftarrow{\gamma'} & C_\infty(EJ^+ \wedge_J \Sigma^L X^H) & \xrightarrow{\alpha'} & \Omega^\infty \Sigma^\infty(EJ^+ \wedge_J \Sigma^L X^H) \end{array}$$

which commutes up to homotopy, where α' is the nonequivariant approximation, and $\gamma, \gamma', \mu, \bar{\mu}$, and ε are equivalences. Thus (6.2) follows from the fact that α' is a group-completion ([C1], [C4], [S1]). \square

(6.3) LEMMA. *Let A, C be H -spaces. Let F, E, B be grouplike H -spaces (i.e., application of the functor $\pi_0(-)$ yields a group), and $F \xrightarrow{i} E \xrightarrow{p} B$ a fiber sequence such that the following diagram is homotopy-commutative:*

$$\begin{array}{ccc}
 E & \xrightarrow{\cong} & (\pi_0 E) \times E_0 \\
 p \downarrow & & \downarrow \pi_0(p) \times (p|_{E_0}) \\
 B & \xrightarrow{\cong} & (\pi_0 B) \times B_0
 \end{array}$$

Then if

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & F \\
 i_1 \downarrow & & \downarrow i \\
 A \times C & \xrightarrow{\gamma} & E \\
 pr_2 \downarrow & & \downarrow p \\
 C & \xrightarrow{\beta} & B
 \end{array}$$

is a commutative diagram of H -spaces and H -maps, and α, β are group-completions, then γ is a group-completion and $E \simeq F \times B$.

Proof. Clearly it suffices to show this for the case where B is connected ($B = B_0$) or discrete ($B \cong \pi_0 B$). The discrete case is easy algebra, and we consider only the connected case.

The product of group-completions is a group-completion [C2], [M4], so that there is an H -map $\bar{\gamma}: F \times B \rightarrow E$ such that

$$\begin{array}{ccccc}
 & & A & & \\
 & \alpha \swarrow & \downarrow & \searrow \alpha & \\
 F & & & & F \\
 \downarrow & \xrightarrow{=} & & & \downarrow \\
 F \times B & \xrightarrow{\bar{\gamma}} & A \times C & \xrightarrow{\gamma} & E \\
 \downarrow & \xrightarrow{\cong} & \downarrow & & \downarrow \\
 B & \xrightarrow{=} & C & \xrightarrow{\beta} & B
 \end{array}$$

homotopy-commutes. Hence $\bar{\gamma}$ is an equivalence, and γ is a group-completion. \square

Now partially order the conjugacy classes of subgroups of G by defining

$$(H) < (K)$$

if H is subconjugate to K . Then an orbit-type family is precisely the union of the classes in an initial segment of this partial order.

By letting \mathcal{F} run through some cofinal sequence of initial segments, one obtains the following by induction (see, for example, McClure [M5]):

(6.4) LEMMA. *The statement (6.1) is true for all \mathcal{F} if*

- (i) (6.1) is true for $\mathcal{F} = \{1\}$.
- (ii) whenever \mathcal{F}_1 and $\mathcal{F}_2 = \mathcal{F}_1 \cup (H)$ are a successive pair of families, and (6.1) is true for \mathcal{F}_1 , then it is true for \mathcal{F}_2 , and
- (iii) whenever $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n \subset \dots$ is a chain of families and $\mathcal{F} = \bigcup_n \mathcal{F}_n$, then if (6.1) is true for all the \mathcal{F}_n , it is true for \mathcal{F} .

We use this last lemma to prove (6.1). Hypothesis (i) is just the special case $\mathcal{F} = \{1\}$, $H = 1$, $NH = G$ of Lemma (6.2). Hypothesis (iii) is easily verified by looking at the homology of

$$C_G(W, X)_{\mathcal{F}} = \varinjlim C_G(W, X)_{\mathcal{F}_n} \quad \text{and} \quad (\Omega^W \Sigma^W X)_{\mathcal{F}}^G = \varinjlim (\Omega^W \Sigma^W X)_{\mathcal{F}_n}^G.$$

Finally, to verify hypothesis (ii), we take together the results of §§2–3 and note that the following diagram commutes:

$$\begin{array}{ccc} C_G(W, X)_{\mathcal{F}_1} & \xrightarrow{\alpha_{\mathcal{F}_1}^G} & (\Omega^W \Sigma^W X)_{\mathcal{F}_1}^G \\ i \downarrow & & \downarrow i' \\ C_G(W, X)_{\mathcal{F}_2} & \xrightarrow{\alpha_{\mathcal{F}_2}^G} & (\Omega^W \Sigma^W X)_{\mathcal{F}_2}^G \\ p \downarrow & & \downarrow p' \\ C_{NH}(W^H, X^H)_{\mathcal{F}_2} & \xrightarrow{\alpha_{\mathcal{F}_2}^{NH}} & (\Omega^{W^H} \Sigma^{W^H} X^H)_{\mathcal{F}_2}^{NH} \end{array}$$

By hypothesis $\alpha_{\mathcal{F}_1}^G$ is a group-completion, and by (6.2), $\alpha_{\mathcal{F}_2}^{NH}$ is a group-completion. Hence by (6.3), $\alpha_{\mathcal{F}_2}^G$ is a group completion. □

Hence (6.1) is true for all \mathcal{F} and (1.15) follows.

7. Other results. We offer a proof of Corollary (1.16). Recall that $A(G)$ is additively the free abelian group generated by elements $[G/H]$ for which $|NH:H| < \infty$. If $A^+(G)$ is the submonoid of elements with non-negative coefficients [D1], then $A(G)$ is the universal enveloping group of $A^+(G)$.

A consequence of (1.15) is that

$$\pi_0(\alpha\gamma^{-1}): \pi_0(C_G(W, S^0)) \rightarrow \pi_0((\Omega^W S^W)^G) = [S^W, S^W]_G$$

is the inclusion of $\pi_0(C_G(W, S^0))$ into its universal enveloping group. Hence (1.16) can be shown by constructing an isomorphism

$$\Phi: A^+(G) \rightarrow \pi_0(C_G(W, S^0))$$

of monoids.

To keep our notation clear, let $S^0 = \{*, a\}$. Then any point in $C_G(W, S^0)$ may be written in the form

$$z = [v_1, \dots, v_j; a, a, \dots, a; t_1, \dots, t_n]$$

where $v_i \in W$ and $t_i \in L(G_{v_i})^{G_{v_i}}$. If some v_i has isotropy group G_{v_i} with $|NG_{v_i}: G_{v_i}| = \infty$, then $L(G_{v_i})^{G_{v_i}} \neq 0$, and so there is a path from z to the point

$$\begin{aligned} z' &= [v_1, \dots, v_j; a, \dots, a; t_1, \dots, t'_i = \infty, \dots, t_j] \\ &= [v_1, \dots, \hat{v}_i, \dots, v_j; a, \dots, a; t_1, \dots, \hat{t}_i, \dots, t_j]. \end{aligned}$$

It follows that any element of $\pi_0(C_G(W, S^0))$ may be represented by a point of the form

$$[v_1, \dots, v_j; a, \dots, a; 0, \dots, 0]$$

where G_{v_i} has finite index in its normalizer.

Now let $(H_1), \dots, (H_n), \dots$ be the conjugacy classes of subgroups of G , and choose w_1, \dots, w_n, \dots such that

$$w_n \in W_{(H_n)}.$$

Then define Φ as above by letting

$$\Phi([G/H_i]) = \{[w_i; a; 0]\}$$

and extending to $A^+(G)$ by additivity.

An inverse $\Psi: \pi_0(C_G(W, S^0)) \rightarrow A^+(G)$ to Φ may be defined by letting

$$\Psi\{[v_1, \dots, v_j; a, \dots, a; 0, \dots, 0]\} = \sum_{n=1}^j [G/G_{v_n}],$$

where G_{v_1}, \dots, G_{v_j} are of finite index in their normalizers. Then it is easily verified that $\psi\Phi = \text{id}$ and that

$$\Phi\psi\{z\} = \{z'\}$$

where z' is in the same path-component as z .

8. Other questions. The ambition highlighted in the introduction, of finding a model for $\Omega^V \Sigma^V X$ which would serve as a basis for a recognition principle, is still unsatisfied. Three basic and natural questions spring up and need to be answered:

(i) To what do the natural inclusion maps

$$(\Omega^W \Sigma^W X)^G \hookrightarrow (\Omega^W \Sigma^W X)^H$$

correspond on the configuration-space level, for $H \leq G$?

(ii) Can we construct a manageable global model $C(W, X)$ so that

$$(C(W, X))^H = C_H(W, X) \text{ for all } H \leq G$$

(as for the case where G is finite)?

(iii) What can be said for the “unstable” case where $\mathbf{R}^n \leq W$ but $\mathbf{R}^\infty \not\leq W$?

Related to these questions is that of the multiplicative structure in the Burnside ring:

(iv) Is there a natural ring space structure $C_G(W, X) \times C_G(W, X) \rightarrow C_G(W, X)$ corresponding to multiplication in $A(G)$ via (1.16)?

Finally, we are examining the following question along with (i)–(iv), which are all work in progress.

(v) Recall the homotopical model $\tilde{C}_n X$ for $\Omega^n \Sigma^n X$ [C3]. Is there a similar model for $(\Omega^W \Sigma^W X)^G$, and how does it relate to $C_G(W, X)$?

This last question may need to be answered before we can approach any of the others.

REFERENCES

- [A1] M. F. Atiyah, *Thom complexes*, Prof. London Math. Soc. (3), **11** (1961), 291–310.
- [B1] J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory I*, Comm. Math. Helv., **49** (1974), 1–34.
- [B2] J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc., **74** (1968), 1117–1122.
- [C1] J. L. Caruso, Ph.D. Thesis, Univ. of Chicago (1979).
- [C2] J. L. Caruso, F. R. Cohen, J. P. May, and L. R. Taylor, *James maps, Segal maps, and the Kahn-Priddy theorem*, to appear.
- [C3] J. L. Caruso and S. Waner, *An approximation to $\Omega^n \Sigma^n X$* , Trans. Amer. Math. Soc., **265** (1981), 147–162.
- [C4] F. R. Cohen, T. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. **533** (1976). Springer-Verlag.
- [C5] F. R. Cohen, J. P. May, and L. R. Taylor, *Splittings of certain spaces CX* , Math. Proc. Cam. Phil. Soc., **84** (1978), 465–496.
- [D1] T. tom Dieck, *The Burnside ring and equivariant stable homotopy*, Chicago Lecture Notes, 1975.
- [D2] _____, *Transformation Groups and Representation Theory*, Springer volume 766 (1979).

- [H1] H. Hauschild, *Zerspaltung äquivariante homotopiemengen*, Math. Ann., **230** (1977), 279–292.
- [J1] I. M. James, *Bundles with special structure*, J. Ann. Math., **89** (1969), 359–390.
- [M1] J. P. May, *The Geometry of Iterated LLoop Spaces*, Springer vol. 271, 1972.
- [M2] _____, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc., no. 155, 1972.
- [M3] _____, *The Homotopical Foundations of Algebraic Topology*, Acad. Press, to appear.
- [M4] J. P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology, **17** (1978), 205–224.
- [M5] J. E. McClure, *Localization and Splitting of Equivariant Homology and Cohomology Theories*, §5, preprint, 1980.
- [S1] G. B. Segal, *Configuration spaces and iterated loop spaces*, Invent. Math., **21** (1973), 213–221.
- [S2] _____, *Some results in equivariant homotopy theory*, preprint, 1979.

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