WEIGHTED REVERSE WEAK TYPE INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

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Kenneth Andersen and Wo-Sang Young recently obtained a suffi-
cient condition and a different necessary condition on a pair of weight
functions for which a reverse weak type norm inequality holds for the
Hardy-Littlewood maximal function. It is shown here that their necessary
condition is also sufficient. The consequences of the sufficiency theorem
that they proved are strengthened by use of this result.

1. Introduction. Define the oriented cubes in \( \mathbb{R}^n \) to be those with
sides parallel to the coordinate axes, and for a set \( E \) in \( \mathbb{R}^n \) let \( |E| \) denote
the Lebesgue measure of \( E \). For \( f \) locally integrable on \( \mathbb{R}^n \), define
the Hardy-Littlewood maximal function \( Mf \) by

\[
(Mf)(x) = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(t)| dt,
\]
where the supremum is taken over all oriented cubes containing \( x \). The
local sufficiency result proved in [1] for a reverse weak type inequality for
the Hardy-Littlewood maximal function is as follows.

**Theorem (1.1).** If \( U(x) \) and \( V(x) \) are nonnegative functions defined on
an oriented cube \( Q_0 \) such that

\[
\frac{1}{|Q|} \int_{Q} U(x) \, dx \geq A \text{ ess sup}_{x \in Q} V(x)
\]
for all oriented cubes \( Q \subset Q_0 \), then

\[
\int_{Q_0 \cap \{x: (Mf)(x) > \lambda\}} U(x) \, dx \geq \frac{A^{2^{-n}}}{\lambda} \int_{\{x: |f(x)| > \lambda\}} |f(x)| V(x) \, dx
\]
for \( f \) supported on \( Q_0 \) and \( \lambda \geq (1/|Q_0|) \int_{Q_0} |f(x)| \, dx \).

The local necessity result proved in [1] is the following; note that for a
cube \( Q \) in \( \mathbb{R}^n \) the notation \( SQ \) denotes the cube concentric with \( Q \) with
sides \( S \) times as long.
THEOREM (1.3). If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube $Q_0$ such that

$$\int_{Q_0 \cap \{ x: \lambda < U(x) \}} U(x) \, dx \geq \frac{A}{\lambda} \int_{\{ x: \lambda < f(x) \}} |f(x)| V(x) \, dx$$

for all $f$ which are characteristic functions of measurable sets $E \subset Q_0$ and $\lambda \geq (1/|Q_0|) \int_{Q_0} |f(x)| \, dx$, then for oriented cubes $Q \subset Q_0$,

$$\frac{1}{|Q|} \int_{2Q \cap Q_0} U(x) \, dx \geq A 2^{-n} \text{ess sup}_{x \in Q} V(x).$$

If $U(x)$ satisfies the doubling condition $\int_{2Q \cap Q_0} U(x) \, dx \leq C \int_{Q} U(x) \, dx$ for oriented $Q \subset Q_0$, conditions (1.2) and (1.5) are equivalent. It is easy to see, however, that (1.2) and (1.5) are not equivalent in general by the example $U(x) = V(x) = \chi_{\{1/2\}Q_0}(x)$ which satisfies (1.5) for $A = 1$ but does not satisfy (1.2) for any finite $A$.

Condition (1.2) resembles the well-known weight condition for $A_1$, $(1/|Q|) \int_{Q} U(x) \, dx \leq A \text{ ess inf}_{x \in Q} U(x)$, introduced on page 214 of [4], and (1.2) looks more “natural” than (1.5). It is, therefore, surprising that (1.5) is in fact a necessary and sufficient condition for the reverse weak type inequality (1.4). The main result of this paper is the following.

THEOREM (1.6). If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube $Q_0$ such that

$$\int_{Q_0 \cap \{ x: (Mf)(x) > \lambda \}} U(x) \, dx \geq A 300n^{-n} \frac{A}{\lambda} \int_{\{ x: \lambda < f(x) \}} |f(x)| |V(x)| \, dx$$

for $f$ supported on $Q_0$ and $\lambda \geq \lambda_0 = (1/|Q_0|) \int_{Q_0} |f(x)| \, dx$.

The following global version of Theorem (1.3) is proved in [1] as a corollary of Theorem (1.3).

THEOREM (1.9). If $U(x)$ and $V(x)$ are nonnegative functions defined on $\mathbb{R}^n$ and

$$\int_{\{ x: (Mf)(x) > \lambda \}} U(x) \, dx \geq A 300n^{-n} \frac{A}{\lambda} \int_{\{ x: \lambda < f(x) \}} |f(x)| |V(x)| \, dx$$

for all $f$ which are characteristic functions of measurable sets $E \subset \mathbb{R}^n$ and $\lambda \geq (1/|Q_0|) \int_{Q_0} |f(x)| \, dx$. 
for all $f$ which are characteristic functions of measurable sets $E \subset \mathbb{R}^n$ and all $\lambda > 0$, then
\[
\frac{1}{|Q|} \int_{2Q} U(x) \, dx \geq A 2^{-n} \text{ess sup} \, V(x)_{x \in Q}
\]
for all oriented cubes $Q \subset \mathbb{R}^n$.

The sufficiency of this condition will be proved here as the following corollary of Theorem (1.6). This and Theorem (1.13) generalize results by Stein in [5].

**Theorem (1.10).** If $U(x)$ and $V(x)$ are nonnegative functions defined on $\mathbb{R}^n$ and
\[
\frac{1}{|Q|} \int_{2Q} U(x) \, dx \geq A \text{ess sup} \, V(x)_{x \in Q}
\]
for all oriented cubes $Q \subset \mathbb{R}^n$, then
\[
\int_{\{x: (Mf)(x) > \lambda\}} U(x) \, dx \geq \frac{A(300n)^{-n}}{\lambda} \int_{\{x: |f(x)| > \lambda\}} |f(x)|V(x) \, dx
\]
for all measurable $f$ and $\lambda > 0$.

The other results in [1] can be proved with weaker hypotheses by using Theorem (1.6) or Theorem (1.10) in their proofs. These strengthened versions are as follows.

**Theorem (1.13).** If $U(x)$ and $V(x)$ are nonnegative functions defined on an oriented cube $Q_0$ such that (1.7) holds for all oriented cubes $Q \subset Q_0$ with $A > 0$, $f$ is supported on $Q_0$ and $\int_{Q_0} (Mf)(x)U(x) \, dx < \infty$, then
\[
\int_{Q_0} |f(x)| \left[ \log^+ |f(x)| \right] V(x) \, dx < \infty.
\]

For the next theorem define $e_i$ as the vector in $\mathbb{R}^n$ with $i$th entry 1 and all other entries 0. Define $M_if$, the Hardy-Littlewood maximal function in the $i$th variable by $(M_if)(x) = \sup_{h \neq 0} \left( \frac{1}{h} \int_0^h |f(x + te_i)| \, dt \right)$. The following theorem generalizes a result in [2].

**Theorem (1.14).** If $U(x)$ is a nonnegative function on $\mathbb{R}^n$, $K$ is an integer, $1 \leq K \leq n$, for $h > 0$, $1 \leq i \leq K$ and almost every $x$ in $\mathbb{R}^n$
\[
\frac{1}{2h} \int_{-2h}^{2h} U(x + te_i) \, dt \geq A \text{ess sup} \, U(x + te_i)_{|t| \leq h}
\]
with \( A \) independent of \( i, x \) and \( h \), and \( T_f(x) = (M_K \cdots M_1 f)(x) \), then for all \( \lambda > 0 \),

\[
(1.16) \quad \int_{\{x : T_f(x) > \lambda\}} U(x) \, dx \\
\geq \frac{300 - K^A}{(K - 1)!\lambda} \int_{\{x : |f(x)| > \lambda\}} |f(x)| \left[ \log^+ \left( \frac{|f(x)|}{\lambda} \right) \right]^{K-1} U(x) \, dx.
\]

If it is also assumed that \( \lim_{|x| \to \infty} T_f(x) = 0 \) and \( \int_{|x| < r} T_f(x) U(x) \, dx < \infty \) for every finite \( r \), then

\[
\int_{\mathbb{R}^n} |f(x)| \left[ \log^+ |f(x)| \right]^K U(x) \, dx < \infty.
\]

2. Proofs. The proof of Theorem (1.6) is based on the following version of the Whitney covering lemma. As usual, \( E^\circ \) will denote the interior of the set \( E \), and \( \chi_E \) is the characteristic function of the set \( E \).

**Lemma (2.1).** Given an open set \( G \) in \( \mathbb{R}^n \), its complement \( F \) and a closed cube \( Q \) such that \( G \subset 3Q \), there exists a sequence \( \{Q_j\} \) of closed cubes with sides parallel to those of \( Q \) such that

(a) \( Q_j \subset Q \),
(b) \( \partial Q_j \supset G \cap Q^\circ \),
(c) \( Q_i^\circ \cap Q_j^\circ = \emptyset \) if \( i \neq j \),
(d) \( \text{diam} \, Q_j \leq \text{dist}(Q_j, F) \leq 4 \text{diam} \, Q_j \),
(e) \( \sum_j \chi_{2Q_j}(x) \leq (33\sqrt{n})^n \chi_G(x) \).

The definition of the \( Q_j \)'s and the proof of (a)–(d) are essentially the proof of Theorem 1, p. 167 of [6]. The principal change is that the initial mesh \( \mathcal{M}_0 \) should be based on \( Q \), i.e. \( \mathcal{M}_0 \) consists of \( Q \) and translations of \( Q \) parallel to its edges by integral multiplies of \( S = |Q|^{1/n} \). The cubes in the mesh \( \mathcal{M}_K \) then have side length \( S2^{-k} \) and the layers \( \Omega_K \) are defined by \( \Omega_K = \{ x : 2^{1-k}S\sqrt{n} < \text{dist}(x, F) \leq 2^{2-K}S\sqrt{n} \} \). Let \( A \) be the set of all cubes \( q \) for which there is an integer \( K \) such that \( q \in \mathcal{M}_K \) and \( q \cap \Omega_K \neq \emptyset \). As shown in [6], we have

\[
(2.2) \quad \text{diam}(q) \leq \text{dist}(q, F) \leq 4 \text{diam}(q)
\]

for \( q \) in \( A \) and

\[
(2.3) \quad G = \bigcup_{q \in A} q.
\]

Since \( G \subset 3Q \), we have \( \text{dist}(q, F) \leq \frac{1}{2} \text{diam}(G) \leq \frac{3}{2} \text{diam}(Q) \) for any cube \( q \); this and (2.2) show that \( \text{diam}(q) \leq \frac{3}{2} \text{diam}(Q) \) for \( q \in A \). Because
of the choice of $M_0$, we conclude that every $q$ in $A$ is a subset of $Q$ or has interior disjoint from $Q$. Let the $Q_j$ be the members of $A$ that are subsets of $Q$ and which are not proper subsets of other members of $A$. Then properties (a) and (c) are immediate, property (d) follows from (2.2) and property (b) follows from (2.3).

The proof of property (e) is an adaptation of the proof of property (4) of the decomposition lemma on pages 15–16 of [3]. For this proof, fix an $x$ in $\bigcup_{j \geq 1} 2Q_j$. If $2Q_j$ contains $x$, then $\text{dist}(x, F) \leq \text{dist}(x, Q_j) + \text{diam}(Q_j) + \text{dist}(Q_j, F)$. Since $x \in 2Q_j$, it follows that $\text{dist}(x, Q_j) \leq (\frac{1}{2}) \text{diam}(Q_j)$, and by (2.2) we have $\text{dist}(Q_j, F) \leq 4 \text{diam}(Q_j)$. Therefore,

\begin{equation}
\text{dist}(x, F) \leq \left(\frac{11}{2}\right) \text{diam}(Q_j).
\end{equation}

Similarly, $\text{dist}(x, F) \geq \text{dist}(Q_j, F) - \text{dist}(x, Q_j)$ implies

\begin{equation}
\text{dist}(x, F) \geq \left(\frac{1}{2}\right) \text{diam}(Q_j).
\end{equation}

Furthermore, if $y \in Q_j$, $\text{dist}(x, y) \leq \text{dist}(x, Q_j) + \text{diam}(Q_j)$. This, the fact that $\text{dist}(x, Q_j) \leq (\frac{1}{2}) \text{diam}(Q_j)$ and (2.5) show that

\begin{equation}
\text{dist}(x, y) \leq 3 \text{dist}(x, F).
\end{equation}

From (2.4), (2.5) and (2.6) it follows that all $Q_j$ with $x$ in $2Q_j$ lie in a sphere of radius $3\text{dist}(x, F)$ about $x$, are disjoint and have diameter greater than or equal to $(\frac{7}{11})\text{dist}(x, F)$. Since the sphere of radius $3\text{dist}(x, F)$ has volume less than $[6\text{dist}(x, F)]^n$, the number of $Q_j$ with $x \in 2Q_j$ is bounded by $[6\text{dist}(x, F)]^n/[(2/11\sqrt{n}) \text{dist}(x, F)]^n = (33\sqrt{n})^n$. This proves (e) for $x$ in $G$. For $x$ not in $G$, property (d) shows that the left side of (e) is 0. This completes the proof of Lemma (2.1).

To prove Theorem (1.6), fix a $\lambda \geq \lambda_0$ and let $G$ be the set where $Mf(x) > \lambda$. If $x$ is not in $3Q_0$, then any oriented cube that contains $x$ and intersects $Q_0$ has sides with length greater than the sides of $Q_0$ and $Mf(x) < (1/|Q_0|) \int_{Q_0} |f(x)| (x) dx = \lambda_0$. Therefore, $G \subset 3Q_0$ and we can apply Lemma (2.1) with this $G$ and with $Q$ taken to be $Q_0$. By property (e)

\begin{equation}
\int_{G \cap Q_0} U(x) \, dx = \int_{Q_0} \chi_G(x) U(x) \, dx \\
\geq (33\sqrt{n})^{-n} \sum_{j \geq 1} \int_{Q_0} \chi_{2Q_j}(x) U(x) \, dx;
\end{equation}

combining this and (1.7) shows that

\begin{equation}
\int_{G \cap Q_0} U(x) \, dx \geq A(33\sqrt{n})^{-n} \sum_{j \geq 1} |Q_j| \text{ ess sup } V(x).
\end{equation}
Now given \( j \geq 1 \), we have \( \text{dist}(Q_j, F) \leq 4 \text{diam}(Q_j) \). Therefore, there is an \( x \) in \( F \) such that \( |x - c_j| \leq (\frac{3}{2}) \text{diam}(Q_j) \), where \( c_j \) is the center of \( Q_j \). Since \( 9\sqrt{n} Q_j \) contains the sphere of radius \( (\frac{9}{2}) \text{diam}(Q_j) \) about \( c_j \), it follows that \( x \) is in \( 9\sqrt{n} Q_j \), and since \( Mf(x) \leq \lambda \), we have

\[
\frac{1}{|9\sqrt{n} Q_j|} \int_{Q_j} |f(x)|dx \leq \lambda.
\]

This shows that the right side of (2.7) is bounded below by

\[
A(33\sqrt{n})^{-n} \sum_{j \geq 1} |Q_j| \text{ess sup}_{Q_j} V(x) \frac{1}{\lambda |9\sqrt{n} Q_j|} \int_{Q_j} |f(x)|dx.
\]

This is bounded below by

\[
\frac{A(300n)^{-n}}{\lambda} \sum_{j \geq 1} \int_{Q_j} |f(x)|V(x) dx.
\]

By property (b) of Lemma (2.1), this is bounded below by

\[
\frac{A(300n)^{-n}}{\lambda} \int_{G \cap Q_0} |f(x)|V(x) dx.
\]

Finally since \( Mf(x) \geq |f(x)| \) almost everywhere, this is bounded below by the right side of (1.8). This completes the proof of Theorem (1.6).

The proof of Theorem (1.10) is a minor modification of the sufficiency proof of Corollary 1 of [1]. Let \( Q_1 \) be a fixed oriented closed cube. For \( t > 0 \) let \( f_t(x) = f(x) \) if \( |f(x)| \leq t \) and \( x \in tQ_1 \), \( f_t(x) = t \text{sgn } f(x) \) if \( |f(x)| > t \) and \( x \in tQ_1 \), and \( f_t(x) = 0 \) if \( x \notin tQ_1 \). Given \( \lambda > 0 \) and \( R \) such that \( R > t \) and

\[
(2.8) \quad \frac{1}{|R - t Q_1|} \int_{R^n} |f_t(x)|dx \leq \lambda,
\]

define \( U_R(x) = U(x) \) if \( x \in RQ_1 \) and \( U(x) = \infty \) if \( x \notin RQ_1 \). We will apply Theorem (1.6) to \( f_t \) with \( Q_0 \), \( U \) and \( V \) taken respectively as \( 2RQ_1 \), \( U_R \) and \( V \). Hypothesis (1.7) is satisfied if \( 2Q \subset RQ_1 \) by (1.11) while if \( 2Q \subsetneq RQ_1 \), hypothesis (1.7) is satisfied because the left side is infinite. Inequality (2.8) implies that \( \lambda \) is greater than the required \( \lambda_0 \). Theorem (1.6) then implies that

\[
(2.9) \quad \int_{2RQ_1 \cap \{ x : (Mf_t)(x) > \lambda \}} U_R(x) dx
\]

\[
\geq \frac{A(300n)^{-n}}{\lambda} \int_{\{ x : |f_t(x)| > \lambda \}} |f_t(x)|V(x) dx.
\]
Now (2.8) implies that \((Mf_t)(x) \leq \lambda\) for \(x \notin RQ_1\); therefore, (2.9) is equivalent to

\[\int_{\{x: (Mf_t)(x) > \lambda\}} U(x) \, dx \geq \frac{A(300n)^{-n}}{\lambda} \int_{\{x: |f_t(x)| > \lambda\}} |f_t(x)| V(x) \, dx.\]

Now let \(t\) approach \(\infty\) and use the monotone convergence theorem to complete the proof of Theorem (1.10).

Theorem (1.13) is proved from Theorem (1.6) in the same way that Theorem 2 of [1] is proved from Theorem 1 of [1]. Theorem (1.14) is proved from Theorem (1.10) in the same way that Theorem 3 and Corollary 2 of [1] are proved from Theorem 2 of [1].

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Bruce Alan Barnes, Algebraic elements of a Banach algebra modulo an ideal ................................................................. 219
Howard D. Fegan and Peter Gilkey, Invariants of the heat equation .......... 233
Erica Flapan, Necessary and sufficient conditions for certain homology 3-spheres to have smooth $Z_p$-actions .................................. 255
Philip R. Heath, Product formulae for Nielsen numbers of fibre maps .... 267
Derbiau Frank Hsu and A. Donald Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs. II ........ 291
Taqdir Husain, Orthogonal primitive idempotents and Banach algebras isomorphic with $l_2$ ......................................................... 313
Gary M. Lieberman, Regularized distance and its applications ............. 329
William W. Menasco, Determining incompressibility of surfaces in alternating knot and link complements ............................... 353
Benjamin Muckenhoupt, Weighted reverse weak type inequalities for the Hardy-Littlewood maximal function ................................. 371
John Dacey O’Neill, Direct summands of direct products of slender modules ............................................................................... 379
Andreas Rapp, Elimination of Malitz quantifiers in stable theories .......... 387
Francisco José Ruiz, A unified approach to Carleson measures and $A_p$ weights ......................................................................... 397
Hanamatagouda Pandappa Sankappanavar, Heyting algebras with dual pseudocomplementation ................................................ 405
Annette Sinclair, Strong Carleman and strong uniform approximation .... 417
Masaaki Suzuki, The generalized Schwarz lemma for the Bergman metric .............................................................................. 429
Brian Thorpe and Ludwig Tomm, Universal approximation by regular weighted means .......................................................... 443