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Dedicated to the memory of my daughter Saraswati

This paper is a contribution toward developing a theory of (the variety H_+ of) Heyting algebras with dual pseudocomplementation.

1. Introduction. Initially, we set out to give a simple proof that the variety H_+ has CEP — a problem kindly posed to the author by Professor H. Rasiowa in 1980 while she was visiting with him at the Instituto de Matematica, Universidade Federal da Bahia, Brasil. Our proof of CEP eventually led us to the basic result of this paper (Theorem 3.3).

The theory of H_+ subsumes a significant part of the theory of double Heyting algebras and regular double p-algebras developed (separately) by Beazer, Katriňâk, Köhler, and others, thus providing a unifying framework. Furthermore, the study of subvarieties of H_+ leads us to an infinite supply of discriminator varieties, and it turns out that all finite simple members in H_+ are quasiprimal algebras. (Other results on the lattice of subvarieties of H_+ will be published elsewhere.)

More specifically, we show in §3 that the congruences on H_+-algebras are determined by normal filters, and for double Heyting algebras the H_+-congruences coincide with the double Heyting algebra-congruences. In §4 the simples, subdirectly irreducibles and directly indecomposables are characterized. Section 5 deals with characterizations of H_+-algebras with Boolean congruence lattices and Stone congruence lattices. Some of the results of §§3 and 4 were presented to the 87th Annual Meeting of the American Mathematical Society in January 1981 and announced in [**16**].

2. Preliminaries. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \rightarrow, +, 0, 1\rangle$ is a *Heyting algebra with dual pseudocomplment* if $\langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $+$ is the dual pseudocomplement ($b \geq a^+$ iff $b \vee a = 1$). H_+ denotes the class of all Heyting algebras with dual pseudocomplement, and we refer to members of H_+ as H_+-algebras. The class H_+ is an equational class defined by the following identities (see [**2**]):

(1) The identities defining lattices with 0 and 1

(2) $x \wedge (x \rightarrow y) \approx x \wedge y$

(3) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$

(4) $z \wedge [(x \wedge y) \to x] \approx z$
(5) $x \vee (x \vee y)^+ \approx x \vee y^+$
(6) $x \vee 1^+ \approx x$
(7) $1^{++} \approx 1$.

If $\mathbf{L} \in H_+$ then $a \to 0$ is the pseudocomplement of $a \in L$, denoted by a^*, and thus $\mathbf{L}$ gives rise to a distributive double p-algebra $\langle L, \vee, \wedge, *, +, 0, 1\rangle$. Hence, following [**11**], we define $a^{n(+*)}$, $n \in \omega$, in $\mathbf{L}$ recursively as follows: $a^{o(+*)} = a$, and $a^{(n+1)(+*)} = b^{+*}$, where $b = a^{n(+*)}$. $\mathbf{L}$ is *of finite range* if, for $a \in L$, there exists $n \in \omega$ such that $a^{n(+*)} = a^{(n+1)+*}$; and $B(L) = \{x \in L: x = x^{**}\}$, while $\overline{B}(L)$ is defined dually. The *center* of L, Cen L, is the Boolean sublattice of complemented elements of $\mathbf{L}$. We introduce the notion of *normal filter* in $\mathbf{L}$ which plays a crucial role for the theory of H_+-algebras. A lattice filter F [ideal I] of $\mathbf{L}$ is *normal* iff $a^{+*} \in F$ [$a^{*+} \in I$] whenever $a \in F$ [$a \in I$]. We denote the (algebraic) lattice of normal filters in L by NF(L) and the set of its compact elements by Comp(NF(L)). For $a, b \in L$ we write $a \uparrow b$ if $a^{n(+*)} \leq b$ for some $n \in \omega$.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, \leftarrow, 0, 1\rangle$ is a *double Heyting algebra* if $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\langle A, \vee, \wedge, \leftarrow, 0, 1\rangle$ is a dual Heyting algebra. If $\mathbf{A}$ is a double Heyting algebra, then $\langle A, \vee, \wedge, \to, +, 0, 1\rangle$ is an H_+-algebra with $a^+ = a \leftarrow 1$. A double p-algebra is *regular* if $a^* = b^*$ and $a^+ = b^+$ imply $a = b$ for its elements a, b. A regular double p-algebra can be regarded (see [**11**]) as a double Heyting algebra, where $a \to b$ is defined as

$$a \to b = (a^* \vee b^{**})^{**} \wedge \left[(a \vee a^*)^+ \vee a^* \vee b \vee b^*\right],$$

and $a \leftarrow b$ is defined dually — this remark will be used frequently without mention in the sequel.

Unless otherwise stated, L denotes an arbitrary H_+-algebra throughout. If $A \subseteq L$ and $A \neq \varnothing$, then $N(A)$ denotes the normal filter of L generated by A (where $N(a)$ denotes $N(\{a\})$, and A^l denotes the set of lower bounds of A in L, while $D(L) = \{x \in L: x^* = 0\}$).

LEMMA 2.1. *If $A \subseteq L$ and $A \neq \varnothing$, then*

$$N(A) = \left\{x \in L: x \geq (a_1 \wedge \cdots \wedge a_n)^{m(+*)} \right.$$
$$\left. \text{for some } a_i \in A \text{ and } m \in \omega\right\};$$

and if $a \in L$ then $N(a) = \bigvee_{m \in \omega} \left[a^{m(+*)}\right)$.

COROLLARY 2.2. *If J is a filter of* Cen L *then*

$$N(J) = \{x \in L;\ x \geq a \text{ for some } a \in J\}.$$

LEMMA 2.3. *Let* $a \in \operatorname{Cen} L$. *Then*
(1) $(a] \in H_+$,
(2) *The function* $x \mapsto x \wedge a$ *is an* H_+*-homomorphism of* L *onto* $(a]$.

Proof. It is well known (see [2]) that $(a]$ is a Heyting algebra. For $x \in (a]$ we let $x^{+a} = x^+ \wedge a$. Then it is easy to see that x^{+a} is the dual pseudocomplement of x in $(a]$, since $a \in \operatorname{Cen} L$.

COROLLARY 2.4 *If* $a \in \operatorname{Cen} L$, *then* $L \cong (a] \times (a']$.

3. Congruence lattices as normal filter lattices. In this section we prove our basic results namely that the congruences on an H_+-algebra are determined by normal filters, and H_+-congruences (preserving $\wedge$, $\vee$, $\rightarrow$, $+$) coincide with the congruences (preserving $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$) on a double Heyting algebra, and we give several applications of these results, which include some recent results of Katriňák, Köhler and Beazer on double Heyting algebras and regular double p-algebras.

LEMMA 3.1. *Let* $\theta \in \operatorname{Con} L$. *Then* $1/\theta$ *is a normal filter in* **L**.

Proof. Observe $1^{+}{}^{*} = 1$.

LEMMA 3.2. *Let* $F \in \operatorname{NF}(L)$ *and define a relation* $\theta(F)$ *by* $x \equiv y\theta(F)$ *iff* $x \wedge f = y \wedge f$ *for some* $f \in F$. *Then* $\theta(F)$ *is a congruence on* L *such that* $1/\theta(F) = F$.

Proof. It is well known that $\theta(F)$ is a Heyting algebra-congruence (e.g., see [1]) such that $1/\theta(F) = F$. Now let $\langle a, b\rangle \in \theta(F)$; then $a \wedge f = b \wedge f$ for some $f \in F$ and so $a^+ \vee f^+ = b^+ \vee f^+$ using the identity $(x \wedge y)^+ \approx x^+ \vee y^+$. Hence, $(a^+ \vee f^+) \wedge f^{+}{}^{*} = (b^+ \vee f^+) \wedge f^{+}{}^{*}$, from which it follows that $a^+ \wedge f^{+}{}^{*} = b^+ \wedge f^{+}{}^{*}$. Since F is normal, $f^{+}{}^{*} \in F$ so that $\langle a^+, b^+\rangle \in F$, proving the lemma.

The following theorem is basic to what follows and will be frequently used without explicit reference to it.

THEOREM 3.3. $\operatorname{Con} L \cong \operatorname{NF}(L)$.

Proof. We claim that the function θ: $\operatorname{NF}(L) \rightarrow \operatorname{Con} L$, $F \mapsto \theta(F)$, is an isomorphism. If $\alpha \in \operatorname{Con} L$ then, in view of Lemma 3.2 it suffices to prove $\alpha = \theta(1/\alpha)$. If $\langle a, b\rangle \in \alpha$, using $y \rightarrow y \approx 1$ we have

$$\langle (a \rightarrow b) \wedge (b \rightarrow a), 1\rangle \in \alpha$$

so that

$$f = (a \to b) \wedge (b \to a) \in 1/\alpha.$$

Use the identity $x \wedge (x \to y) \approx x \wedge y$ to conclude that $a \wedge f = b \wedge f$.

COROLLARY 3.4. *The variety H_+ has CEP.*

DEFINITION 3.5. Let H_{+n} denote the subvariety of H_+ defined by the identity ($n \in \omega$)

$$x^{n(+*)} \approx x^{(n+1)(+*)}.$$

Note that H_{+0} is the (sub)variety of Boolean algebras.

COROLLARY 3.6. *For each $n \in \omega$ the variety H_{+n} has equationally definable principal congruences.*

The following (surprising) result will enable us to show that a significant part of the theories of double Heyting algebras and regular double p-algebras can be subsumed in the theory of H_+-algebras.

THEOREM 3.7. *Let A be a double Heyting algebra. Then every H_+-congruence on A is a congruence on A.*

Proof. Let θ be an H_+-congruence on A and $\langle x, y\rangle \in \theta$ and $\langle u, w\rangle \in \theta$. Then $x \wedge f = y \wedge f$ and $u \wedge g = w \wedge g$ for some $f, g \in 1/\theta$; hence, $x \vee f^+ = y \vee f^+$ and $u \vee g^+ = w \vee g^+$. Now

$$(x \leftarrow u) \vee f^+ \vee g^+ = f^+ \vee g^+ \vee [(f^+ \vee g^+ \vee x) \leftarrow (f^+ \vee g^+ \vee u)]$$

by the dual of (3)

$$= f^+ \vee g^+ \vee [(f^+ \vee g^+ \vee y) \leftarrow (f^+ \vee g^+ \vee w)]$$

$$= f^+ \vee g^+ \vee (y \leftarrow w).$$

It is now immediate that $\langle x \leftarrow u, y \leftarrow w\rangle \in \theta$, since $\langle f^+ \vee g^+, 0\rangle \in \theta$.

COROLLARY 3.8. (*Köhler* [**13**].) *If A is a double Heyting algebra then* $\mathrm{Con}\, A \cong \mathrm{NF}(L)$.

We remarked in §2 that a regular double p-algebra is a double Heyting algebra, so we have the following corollary.

COROLLARY 3.9. (*Katriňák* [**1**], *Beazer* [**3**].) *If A is a regular double p-algebra then* $\mathrm{Con}\, A \cong \mathrm{NF}(A)$.

Thus, on a regular double p-algebra double p-algebracongruences, H_+-congruences and double Heyting algebracongruences all coincide.

We conclude this section by noting that there are easy examples of H_+algebras (e.g., 4-element chain) such that not every lattice-congruence compatible with $+$ and $*$ is an H_+-congruence.

4. Simples, (finitely) subdirectly irreducibles and indecomposables. In this section we characterize directly indecomposables (see [7] for definition), finitely subdirectly irreducibles, subdirectly irreducibles and simples in H_+, influenced by the works of Beazer, Katriňák and Köhler.

THEOREM 4.1. *Let* $|L| > 1$. *Then TFAE*:
(1) L *is directly indecomposable*
(2) Cen $L = \{0, 1\}$.

Proof. We may assume $|L| > 2$. If $a \in \text{Cen } L - \{0,1\}$ then $L \cong (a] \times (a']$, showing that (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (1) let $\theta, \bar{\theta}$ be a pair of factor congruences different from Δ and ∇ (see [7]). Then $\theta \wedge \bar{\theta} = \Delta$ and $\theta \circ \bar{\theta} = \nabla$. Let $F_1 = 1/\theta$ and $F_2 = 1/\bar{\theta}$; then $F_1 \wedge F_2 = \{1\}$ and there exists $a \in L - \{0,1\}$ such that $0\ \theta\ a\ \bar{\theta}\ 1$. Hence, $a \in F_2$ and $a^* \in F_1$ so that $a \vee a^* \in F_1 \wedge F_2 = \{1\}$; thus $a \in \text{Cen } L$, which shows that (2) $\Rightarrow$ (1).

Using Theorem 3.7 we now have

COROLLARY 4.2. *The equivalence of* (1) *and* (2) *in Theorem* 4.1 *holds for double Heyting algebras and regular double p-algebras.*

THEOREM 4.3. *Let* $|L| > 2$. *Then TFAE*:
(1) L *is simple*
(2) Cen $L = \{0,1\}$, *and for every* $x \in D(L) - \{1\}$ *there exists* $n \in \omega$ *such that* $x^{n(+*)} = 0$.

Proof. Suppose L is simple. Then it follows from Theorem 4.1 that Cen $L = \{0,1\}$. Let $x \in D(L) - \{1\}$, and since L is simple, $N(x) = L$ so that $0 \in N(x)$. Thus there exists $n \in \omega$ such that $x^{n(+*)} = 0$. Conversely, if $a \in L - \{1\}$ then it suffices to show that $N(a) = L$. Now $x = a \vee a^* \in D(L)$. Since Cen$(L) = \{0,1\}$ we see that $x \neq 1$. Then from the hypothesis we conclude that $0 \in N(x) \leq N(a)$, so that $N(a) = L$.

Theorems 4.3 and 3.7 together yield the following.

COROLLARY 4.4. (*Köhler* [13], *Beazer* [5].) *The equivalence of* (1) *and* (2) *is also true of double Heyting algebras and of regular double p-algebras.*

COROLLARY 4.5. *Let L be of finite range and $|L| > 1$. Then TFAE*:
(1) *L is simple*
(2) *L is subdirectly irreducible*
(3) *L is finitely subdirectly irreducible*
(4) *L is directly indecomposable*
(5) Cen $L = \{0, 1\}$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are well known and (4) $\Rightarrow$ (5) follows from Theorem 4.1, while (5) $\Rightarrow$ (1) by Theorem 4.3.

The above corollary is an extension, and an improvement, of corresponding results for double Heyting algebras and regular double p-algebras due to Beazer, Katriňák and Köhler. The corollary also implies that every simple algebra is hereditarily simple.

LEMMA 4.6. *Let $a \in L$. Then the pseudocomplement of $N(a)$ in* NF(L) *is given by*

$$N(a)^* = \{x \in L : x^+ \leq a^{n(+*)} \text{ for every } n \in \omega\}.$$

THEOREM 4.7. *Let $|L| > 1$. Then TFAE*:
(1) *L is finitely subdirectly irrecducible* (*see* [7] *for definition*)
(2) $\bigwedge_{n\in\omega} a^{n(+*)} = 0$ *for every* $a \in L - \{1\}$
(3) $\bigwedge_{n\in\omega} a^{n(+*)} = 0$ *for every* $a \in \bar{B}(L) - \{1\}$
(4) $N(a)$ *is dense in* NF(L) *for every* $a \in L - \{1\}$.

Proof. (1) $\Leftrightarrow$ (4) is clear from Theorem 3.3 and the fact that NF(L) is a pseudocomplemented lattice.

(4) $\Rightarrow$ (2). Let $a \in L - \{1\}$ and let $m \leq a^{n(+*)}$ for every $n \in \omega$. Then $m^{*+} \leq a^{n(+*)}$ for every $n \in \omega$ by using the implication $x \leq y^{+*} \Rightarrow x^{*+} \leq y$; hence, by Lemma 4.6, $m^* \in N(a)^* = \{1\}$; thus $m = 0$, implying (2).

(2) $\Rightarrow$ (4). Suppose there is an $a \in L - \{1\}$ such that $N(a)$ is not dense. Then there exists $m \in L - \{1\}$ such that $m \in N(a)^*$ so that $0 < m^+ \leq a^{n(+*)}$ for every $n \in \omega$, hence, $\bigwedge_{n\in\omega} a^{n(+*)} \geq m^+ > 0$.

Finally, for (3) $\Rightarrow$ (2) it suffices to note that if $a \in L - \{1\}$, then $a^{n(+*)} = a^{++n(+*)}$ and $a^{++} \in \bar{B}(L) - \{1\}$.

In view of 3.7 we obtain

COROLLARY 4.8. (*Beazer* [6].) *The equivalences in the above theorem also hold for double Heyting algebras* (*and for regular double p-algebras*).

Beazer's proof of the above corollary, however, made essential use of the existence of the dual Heyting implication $\leftarrow$.

LEMMA 4.9. *If* $\bigwedge_{n\in\omega} a^{n(+*)} = 0$ *for all* $a \in \overline{B}(L) - \{1\}$, *then* Cen $L = \{0,1\}$.

Proof. The hypothesis implies by 4.7 that $\bigwedge_{n\in\omega} a^{n(+*)} = 0$ for all $a \in L - \{1\}$, and the rest of the proof is the same as that of Theorem 2 in [**1**].

THEOREM 4.10. *Let* $|L| > 2$. *Then TFAE*

(1) L *is subdirectly irreducible but not simple.*

(2) (i) $\bigwedge_{n\in\omega} a^{n(+*)} = 0$ *for all* $a \in \overline{B}(L) - \{1\}$, *and*

(ii) *there exists* $d \in D(L) - \{1\}$ *such that* $x \uparrow d \not\uparrow 0$ *for all* $x \in D(L) - \{1\}$.

(3) (i) Cen $L = \{0,1\}$ *and*

(ii) (2)(ii) *holds.*

Proof. (1) $\Rightarrow$ (2). Now (2)(i) follows from (1) by Theorem 4.7, and (1) implies that there exists a principal normal filter $N(a)$ such that $\{1\} \subset N(a) \subset L$ and $N(a)$ is contained in every normal filter $N(x)$ of L for $x \neq 1$, implying that $x^{n(+*)} \leq a$. Now let $d = a \vee a^* \in D(L)$, and $d \neq 1$ since, otherwise, $a \in$ Cen $L = \{0,1\}$, whence $a = 0$ which is a contradiction. If $d \uparrow 0$ then for some $n \in \omega$, $d^{n(+*)} = 0$, so that $0 \in N(a)$, which is impossible since $N(a) \neq L$. Finally, if $x \in D(L) - \{1\}$, then $N(a) \subseteq N(x)$, so $x^{n(+*)} \leq a \leq d$, proving (1) $\Rightarrow$ (2). Next, (2) $\Rightarrow$ (3) follows immediately from Lemma 4.9.

(3) $\Rightarrow$ (1). Observe that $N(d) \neq \{1\}$ and $N(d) \neq L$, since $d \not\uparrow 0$. Let $F \in \mathrm{NF}(L)$ be such that $\{1\} \subset F \subset L$, and let $x = f \vee f^*$ for some $f \in F \setminus \{1\}$. Then, $x \in F$ and $x \in D(L)$. Since Cen $L = \{0,1\}$, we have $x \neq 1$. Thus, by hypothesis, $x^{n(+*)} \leq d$ for some $n \in \omega$, implying that $d \in F$, so $N(d) \subseteq F$. This shows that L is subdirectly irreducible but not simple.

Theorems 4.10 and 3.7 yield the following

COROLLARY 4.11. (*Katriňák* [**12**], *Beazer* [**5**].) *The equivalence of* (1), (2), *and* (3) *holds for double Heyting algebras and regular double p-algebras.*

It is shown in [**1**] that the converse of Lemma 4.9 is not true. We conclude this section by pointing out another proof of this fact: Suppose

the converse of Lemma 4.9 holds. In view of Theorems 4.1 and 4.7 it follows that every directly indecomposable algebra in H_+ is finitely subdirectly irreducible. Then by the proof of Theorem 12.5 in [7], H_+ is semisimple, which is impossible, since there are known to exist non-simple subdirectly irreducible doubly Heyting algebras, and such algebras are non-simple subdirectly irreducibles in H_+, as well in view of 3.7.

5. H_+-algebras with Boolean congruence lattices. There are many results in the literature characterizing algebras with Boolean congruence lattices in a given variety. For example, pseudocomplemented semilattices with Boolean congruence lattices are characterized in [**15**]. (See references in [**15**] for more such results.)

In this section we characterize those members of H_+ whose join-semilattice of compact congruences is actually a Boolean lattice as well as those $L \in H_+$ for which Con L is a Boolean lattice.

LEMMA 5.1. *If* $a, b \in L$ *then* $N(a) \vee N(b) = N(a \wedge b)$.

LEMMA 5.2. $F \in \mathrm{NF}(L)$ *is compact iff* $F = N(a)$ *for some* $a \in L$.

THEOREM 5.3. *TFAE*:
(1) Comp(NF(L)) *is a Boolean sublattice of* NF(L)
(2) Comp(NF(L)) *is complemented*
(3) L *is of finite range*
(4) $\mathrm{NF}(L) \cong F(\mathrm{Cen}\ L)$.

Proof. It suffices to prove the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1), since (3) $\Rightarrow$ (4) holds for any distributive double p-algebra (see [**3**]).

(2) $\Rightarrow$ (3): Let $a \in L$; then by (2) $N(a)$ has a complement in NF(L). Thus there is an $F \in \mathrm{NF}(L)$ such that $N(a) \wedge F = \{1\}$ and $N(a) \vee F = L$. Hence, by Grätzer [**10**], Lemma 5, p. 60, F is principal, say $F = N(b)$ for $b \in L$. Since $0 \in N(a) \vee N(b)$, there exists $n \in \omega$ such that $(a \wedge b)^{n(+*)} = 0$, that is, $a^{n(+*)} \wedge b^{n(+*)} = 0$. Also $a^{n(+*)} \vee b^{n(+*)} \in N(a) \wedge N(b) = \{1\}$, so $b^{n(+*)}$ is a complement of $a^{n(+*)}$, and hence $a^{n(+*)} \in \mathrm{Cen}\ L$, $a^{n(+*)} = a^{(n+1)(+*)}$ — thus proving that L is of finite range.

(3) $\Rightarrow$ (1). In view of 5.2 it suffices to show that the principal normal filters form a Boolean sublattice of NF(L). Let $a, b \in L$; then $a^{n(+*)} = a^{(n+1)(+*)} = x$ (say) and $b^{m(+*)} = b^{(m+1)(+*)} = y$ (say). Now $N(a) = [x)$ and $N(b) = [y)$, so $N(a) \wedge N(b) = [x \vee y)$, and $[x \vee y)$ is normal since $x \vee y \in \mathrm{Cen}\ L$; thus Comp(NF($L$)) is a sublattice of NF(L). Finally, since $N(a) = [x)$ and $x \in \mathrm{Cen}\ L$, it follows that the complement of $N(a)$ is the principal normal filter $[x')$.

COROLLARY 5.4. *If* $L \in H_{+n}$ *then the compact congruences on* L *form a Boolean sublattice of* $\mathrm{NF}(L)$.

This corollary and a result of S. Burris together imply that the variety H_{+n} is a discriminator variety for $n \in \omega$. However, one can also write a dicriminator term for H_{+n}. For example, if we let $d(x) = x^{n(+*)}$, then the following term is the ternary discriminator on each of the simple algebras in H_{+n}:

$$t(x, y, z) = [z \wedge d((x \vee y) \to (x \wedge y))] \vee [x \wedge (d((x \vee y) \to (x \wedge y)))^*].$$

COROLLARY 5.5. (*Epstein and Horn* [**9**].) *If* A *is a P-algebra, then* $\mathrm{NF}(L) \cong F(\mathrm{Cen}\, L)$, *and also the compact congruences form a Boolean sublattice. Hence P-algebras form a discriminator variety.*

THEOREM 5.6. *TFAE*:
(1) $\mathrm{Con}\, L$ *is a Boolean lattice*
(2) L *is of finite range and* $\mathrm{Cen}\, L$ *is finite*
(3) $\mathrm{Con}\, L$ *is a finite Boolean lattice.*

Proof. (1) $\Rightarrow$ (2). Let $x \in L$, then $N(x)$ has a complement $F \in \mathrm{NF}(L)$ by (1). Since $0 \in N(x) \vee F$, there exists an $f \in F$ and $n \in \omega$ such that

$$0 = (x \wedge f)^{n(+*)} = x^{n(+*)} \wedge f^{n(+*)}.$$

From $x^{n(+*)} \in N(x)$ and $f^{n(+*)} \in F$ it follows that $x^{n(+*)} \vee f^{n(+*)} \in N(x) \wedge F = \{1\}$; thus $x^{n(+*)} \in \mathrm{Cen}\, L$, and hence L is of finite range. Then it follows from 5.3 that $\mathrm{Cen}\, L$ is finite. For (2) $\Rightarrow$ (3), we again use 5.3 to get $\mathrm{NF}(L) \cong F(\mathrm{Cen}\, L)$, and since $\mathrm{Cen}\, L$ is finite, it follows that $\mathrm{NF}(L)$, and hence $\mathrm{Con}\, L$, is a finite Boolean lattice.

COROLLARY 5.7. *Theorem* 5.6 *also holds for double Heyting algebras.*

Theorem 5.6 generalizes a result of Beazer [**3**].

The following theorem and its corollary were presented by the author to the 57th Ontario Mathematical Meeting held at London, Ontario, Canada, on February 7, 1981. The referee of this paper pointed out to the author that these results also follows from (Theorem 3.3 and) a recent result proved by R. Beazer, *Lattices whose ideal lattice is Stone*, Proc. Edinburgh Math. Soc. (1983), 107–212. We therefore state these results without proofs.

THEOREM 5.8. *Let L be an H_+-algebra or a double Heyting algebra. Then TFAE*:

(1) $\mathrm{NF}(L)$ *is a Stone lattice*

(2) $\bigwedge F$ *exists in L for all $F \in \mathrm{NF}(L)$*

(3) $\bigwedge S$ *exists in L for all $S \subseteq \mathrm{Cen}\, L$ and $\bigwedge_{n\in\omega} a^{n(+*)}$ exists in L for all $a \in L$*

(4) $\mathrm{Con}\, L$ *is a Stone lattice.*

COROLLARY 5.9. *Let L be an H_+-algebra or a double Heyting algebra of finite range. Then* $\mathrm{Con}\, L$ *is a Stone lattice iff* $\mathrm{Cen}\, L$ *is a complete (Boolean) lattice.*

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