THE GENERALIZED SCHWARZ LEMMA FOR THE BERGMAN METRIC

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The function-theoretic criterion for the Bergman metric to be dominated by the Kobayashi metric on the domain in \( \mathbb{C}^n \) is given. For this, we use the distinguished family of plurisubharmonic functions and \( P \)-metric of N. Sibony.

1. Introduction. Let \( D \) be a hyperbolic domain in \( \mathbb{C}^n \) (cf. Kobayashi [7]). On \( D \) we can define some intrinsic metrics: the Carathéodory metric \( C_D \), the Kobayashi metric \( K_D \), and the Bergman metric \( B_D \). It is known that \( C_D \leq K_D \) and \( C_D \leq B_D \). In this paper we investigate when the Bergman metric is dominated by the Kobayashi metric. Using N. Sibony's \( P \)-metric we give a function-theoretic criterion for the following condition (\#) to hold:

\[
\# \quad B_D \leq c K_D \quad \text{on the tangent bundle of } D, \quad c > 0 \text{ constant.}
\]

Under (\#) every holomorphic mapping \( F: U \to D \) satisfies

\[
F* B_D < \frac{1}{2} \frac{c B_U}{2^{1/2}}
\]

where \( U \) is the unit disc in \( \mathbb{C} \) with the Bergman metric \( B_U \). This theorem is called the generalized Schwarz lemma for the Bergman metric. According to N. Sibony [10], we introduce the family of functions

\[
S_p(D) = \{ u: D \to [0,1); u(p) = 0, C^2\text{-class in a neighborhood of } p \text{ and } \log u \text{ is plurisubharmonic in } D \}.
\]

Taking a Bergman kernel \( k(z, \bar{w}) \) of a domain \( D \), we construct a function \( \phi_w \) for a fixed point \( w \) in \( D \) as follows;

\[
\phi_w(z) = \phi_{w, \alpha}(z) = 1 - \left( \frac{|k(z, \bar{w})|^2}{k(z, \bar{z}) k(w, \bar{w})} \right)^\alpha = 1 - \nu^\alpha,
\]

where \( \alpha \) is a positive constant chosen for \( D \). It is clear that \( 0 \leq \phi_w \leq 1 \) and \( \phi_w(w) = 0 \). Our main results are stated as follows.

(I) Let \( D \) be a Bergman domain. If there is a constant \( \alpha > 0 \) such that \( \phi_w = \phi_{w, \alpha} \) belongs to \( S_w(D) \) for each \( w \) in \( D \), then \( B_D \leq \alpha^{1/2} K_D \); hence \( B_D \) satisfies the generalized Schwarz lemma.
(II) If the Bergman metric $B_D$ of a domain $D$ satisfies the following condition; there exists a positive constant $\alpha$ such that, for each $w$ in $D$,

$$\phi_w(z) B_D^2(z, \xi) \geq \alpha |\partial_\xi \log v(z)|^2$$

for all $\xi \in \mathbb{C}^n$ and $z \in \{ z \in D; 0 < v(z) < 1 \}$,

then $B_D \leq \alpha^{-1/2} K_D$; therefore, $B_D$ satisfies the generalized Schwarz lemma.

In §2 we give some properties of the family $S_p(D)$. In §3 we arrange the basic properties of the intrinsic metrics, especially of the $P^*$-metric. In §4 we prove the main results.

In §5, for the classical domains, we construct the function $\phi_w$ and directly verify that each $\phi_w$ belongs to $S_w(D)$. Hence, we have the generalized Schwarz lemma for the classical domains (cf. Kobayashi [7, 8]).

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2. The family $S_p(D)$. Though our argument is available on a complex manifold, we work mainly on a domain in $\mathbb{C}^n \ (n \geq 1)$.

Let $p$ be a fixed point of $D$ and

$$S_p(D) = \{ u: D \to [0, 1); u(p) = 0, C^2\text{-class in a neighborhood of } p \text{ and } \log u \text{ is plurisubharmonic in } D \},$$

$$A_p(D) = \{ u = |f|^2; f \in \text{Hol}(D, U), f(p) = 0 \},$$

where $U$ is the unit disc in $\mathbb{C}$ and $\text{Hol}(D, U)$ denotes the family of all holomorphic mappings $f: D \to U$. It is clear that $S_p(D) \supseteq A_p(D)$.

A nonnegative plurisubharmonic function $u$ such that $\log u$ is also plurisubharmonic ($\log 0 = -\infty$) is called a logarithmically plurisubharmonic function. Hereafter we abbreviate them p.s.h. and log.p.s.h. $S_p(D)$ is a special family of log.p.s.h. functions on $D$. We give some lemmas about the family $S_p(D)$.

**Lemma 2.1.** (1) If $u_1, u_2$ belong to $S_p(D)$, then $ru_1 + (1 - r)u_2 \in S_p(D)$ for any real number $r$ with $0 \leq r \leq 1$.

(2) If $f_j \ (j = 1, \ldots, k)$ are holomorphic in $D$ and vanish at $p$ and $u(z) = \sum |f_j(z)|^2 < 1$ on $D$, then $u \in S_p(D)$.

**Proof.** (1) It is sufficient to prove that if $\log u_j \ (j = 1, 2)$ is subharmonic in an open set $G$ in $\mathbb{C}$, then $\log(r_1 u_1 + r_2 u_2)$ is subharmonic in $G$ for $r_j \geq 0$, with $r_1 + r_2 = 1$. As in the book of Hörmander [4], we take a
disc $G^* \subset G$ and a polynomial $P(t)$ ($t \in \mathbb{C}$) such that $\log u \leq \Re P$ on $\partial G^*$, where $u = r_1 u_1 + r_2 u_2$. Then $u \leq \exp(\Re P)$ on $\partial G^*$. Since $\log u_j - \Re P$ is subharmonic in $G^*$, $r_j u_j |\exp(-P)|$ is also subharmonic in $G^*$. Hence $(r_1 u_1 + r_2 u_2)|\exp(-P)| = u|\exp(-P)|$ is subharmonic in $G^*$. From $u|\exp(-P)| \leq 1$ on $\partial G^*$ and the maximum principle for the subharmonic function, we have $u|\exp(-P)| \leq 1$ on $G^*$, that is, $\log u \leq \Re P$ in $G^*$. Since $G^*$ is an arbitrary disc in $G$, $\log u$ is subharmonic in $G$.

(2) follows directly from (1).

The following lemma will be used in §4.

**Lemma 2.2.** Let $v(z)$ be a $C^2$-function on a domain $D$ with $0 < v < 1$.

1) $1/v(z)$ is log. p.s.h. if and only if

$$\left| \sum v_i \xi_i \right|^2 - v \sum v_i \xi_i^2 j \xi_j \geq 0 \quad \text{for all } \xi \in \mathbb{C}^n,$$

where $v_i = \partial v/\partial z_i$, $v_j = \partial v/\partial \bar{z}_j$ and $v_{ij} = \partial^2 v/\partial z_i \partial \bar{z}_j$.

2) The function $\phi(z) = 1 - v(z)^\beta$ ($\beta$ is a positive constant) is log. p.s.h. if and only if

$$\phi \left[ \left| \sum v_i \xi_i \right|^2 - v \sum v_i \xi_i^2 j \xi_j \right] \geq \beta \left| \sum v_i \xi_i \right|^2 \quad \text{for all } \xi \in \mathbb{C}^n.$$

**Proof.** We show (2) only. It is clear that $(\log \phi)_{ij} = (\phi_{i} \phi - \phi_{i} \phi_{j}) \phi^{-2}$. We have

$$\phi_i = -\beta v^{\beta-1} v_i \quad \text{and} \quad \phi_{ij} = \beta v^{\beta-2} ((1 - \beta) v_i v_j - v v_{ij}).$$

Therefore,

$$\phi_{ij} \phi - \phi_i \phi_j = \beta v^{\beta-2} \left[ (\phi - \beta) v_i v_j - \phi_{ij} v \right].$$

Hence (2.3) is equivalent to the fact that the matrix $[(\log \phi)_{ij}]$ is positive semidefinite.

3. **The intrinsic metrics.** Let $M$ be a complex manifold and $TM$ its holomorphic tangent bundle. According to Kobayashi [8], we call a function $X = X_M(p, \xi)$ on $TM$ a complex Finsler metric on $M$ when it satisfies the following conditions;

(i) $X_M(p, \xi)$ is an upper semicontinuous positive function on $TM$,

(ii) $X_M(p, \lambda \xi) = |\lambda| X_M(p, \xi)$ for any $\lambda \in \mathbb{C}$.

The intrinsic metric is the biholomorphic invariant complex Finsler metric which is determined only by the complex analytic structure of the complex manifold. As examples of intrinsic metrics, there are the Carathéodory, the Kobayashi, and the Bergman metrics. In this section we
give the basic properties of them and introduce a new intrinsic metric, that is, the $P^*$-metric (cf. [10] and [14]).

When $M$ is a domain in $\mathbb{C}^n$, since $TD \cong D \times \mathbb{C}^n$, we may assume a tangent vector $\xi \in \mathbb{C}^n$. Let $(z_1, \ldots, z_n)$ be the canonical coordinates of $\mathbb{C}^n$, and let $(p, \xi)$ denote a pair of $TD$.

The Carathéodory metric (C-metric) $C_D$ of a domain $D$ is given by

$$C_D(p, \xi) = \sup \left\{ \left| \partial \xi f(p) \right| ; f \in \text{Hol}(D, U), f(p) = 0 \right\},$$

where $\partial \xi f(p) = \sum_i \frac{\partial f}{\partial z_i}(p) \xi_i$.

The Kobayashi metric (K-metric) $K_D$ of $D$ is defined by

$$K_D(p, \xi) = \inf \left\{ 1/r ; F \in \text{Hol}(U, D), F(0) = p, F'(0) = r\xi, r > 0 \right\}.$$}

Though they are not always positive, we mainly work on hyperbolic domains. The following theorem is well known.

**Theorem 3.1** [7, 8]. Let $D$ be a hyperbolic domain in $\mathbb{C}^n$.

1. The $C$-metric is continuous on $TD$ and the $K$-metric is upper semicontinuous on $TD$.

2. The $C$ and $K$-metrics are decreasing for any holomorphic mappings: for $F \in \text{Hol}(D, E)$ ($D, E$ are domains),

$$C_E(F(p), F'(p)\xi) \leq C_D(p, \xi),$$

$$K_E(F(p), F'(p)\xi) \leq K_D(p, \xi).$$

N. Sibony [10] introduced the $P$-metric as follows:

$$P_D(p, \xi) = \sup \left\{ L(u; p, \xi)^{1/2} ; u \in S_p(D) \right\},$$

where

$$L(u; p, \xi) = \sum \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(p) \xi_i \bar{\xi}_j$$

is the Levi form of $u$. $P_D$ is locally integrable, but its upper semicontinuity is unknown yet. We define

$$P_D^*(p, \xi) = \limsup P_D(q, \xi) \text{ as } (q, \xi) \to (p, \xi).$$

Then $P_D^*$ is upper semicontinuous and $P_D \leq P_D^*$ and $P^*_U(0, 1) = 1$ for the unit disc $U$ in $\mathbb{C}$. $P_D^*$ is called the $P^*$-metric of $D$.

**Lemma 3.2** (cf. [10], [14]). The $P^*$-metric is an intrinsic metric having the decreasing property for holomorphic mappings, and $C_D \leq P_D^* \leq K_D$ on $TD$. 


Proof. The first half follows from the result of Sibony [10] (see also [14]). We give a simple proof for the last half. Noting that \(S_p(D) \supset A_p(D)\) and \(L(|f|^2; p, \xi) = |\partial_\xi f(p)|^2\), we have \(C_D \leq P_D^*\). Taking a mapping \(F\) in \(\text{Hol}(U, D)\) with \(F(0) = p, F'(0) = r\xi (r > 0)\), from the decreasing property for \(F\), we have

\[
P_D^*(p, r\xi) = P_D^*(F(0), F'(0)) \leq P_U^*(0, 1) = 1.
\]

Therefore, \(P_D^*(p, \xi) \leq 1/r\). Since the K-metric is the infimum of such \(1/r\), we obtain the inequality \(P_D^* \leq K_D\).

Let \(X_D(p, \xi)\) be one of the intrinsic metrics on a domain \(D\). A point \(p\) in \(D\) is said to be an \(X\)-hyperbolic point if there is a neighborhood \(V\) of \(p\) and a constant \(c > 0\) such that \(X_D(q, \xi) \geq c||\xi||\) for all \(\xi \in \mathbb{C}^n\) at every point \(q\) in \(V\), where \(||\cdot||\) is the Euclidean norm of \(\mathbb{C}^n\). If each point of \(D\) is \(X\)-hyperbolic point, then \(D\) is said to be an \(X\)-hyperbolic domain.

4. The generalized Schwarz lemma. Let \(D\) be a domain in \(\mathbb{C}^n\). The reproducing kernel \(k(z, \bar{w})\) of the Hilbert space \(L^2H(D)\) of \(L^2\)-holomorphic functions on \(D\) is called the Bergman kernel of \(D\). It is holomorphic in \(D \times \bar{D}\) (where \(\bar{D}\) is the complex conjugate of \(D\).) Defining

\[
B_D^2(z, \xi) = \sum \left( \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log k(z, \bar{z}) \right) \xi_i \bar{\xi}_j, \quad \xi \in \mathbb{C}^n,
\]

we have the Bergman metric \(B_D\) of \(D\) provided that the right side of the above is positive for all \(\xi \neq 0\). The Bergman metric is a Kähler metric and an intrinsic metric, but does not always have the decreasing property for holomorphic mappings.

We call a domain \(D\) with \(B_D \neq 0\) the Bergman domain.

For the upper semicontinuous Finsler metric, the holomorphic curvature is defined (cf. [12]) and coincides with the holomorphic sectional curvature if the metric is \(C^2\)-hermitian. Let \(U\) be the unit disc in \(\mathbb{C}\) with the canonical metric \((1 - |t|^2)^{-2}|dt|^2\). The following lemma is a generalization of the Schwarz lemma (cf. [7, III, Theorem 2.1]) to the upper semicontinuous case.

**Lemma 4.1** (cf. [14].) Let \(X_M(p, \xi)\) be a complex Finsler metric on a K-hyperbolic manifold \(M\). If its holomorphic curvature is bounded from above by a negative constant \(-c^2\) \((c > 0)\), then \(X_M \leq 2c^{-1}K_M\), and for any holomorphic mapping \(F: U \to M, X_M(F(t), F'(t)) \leq 2c^{-1}K_U(t, 1)\).

Since the holomorphic curvature coincides with the holomorphic sectional curvature for the Bergman metric, we have the following lemma.
**Lemma 4.2.** Let \( D \) be a Bergman domain. If the holomorphic sectional curvature of the Bergman metric \( B_D \) of \( D \) is bounded from above by a negative constant \(-k\), then \( B_D \leq 2k^{-1/2}K_D \), and for every holomorphic mapping \( F: U \to D \),

\[
(*) \quad B_D(F(t), F'(t)) \leq \left(\frac{2}{k}\right)^{1/2} B_U(t, 1), \quad t \in U.
\]

**Proof.** The first half follows from Lemma 4.1. For any holomorphic mapping \( F: U \to D \), we have

\[
B_D(F(t), F'(t)) \leq 2k^{-1/2}K_D(F(t), F'(t)) \leq 2k^{-1/2}K_U(t, 1) = \left(\frac{2}{k}\right)^{1/2} B_U(t, 1),
\]

since \( \sqrt{2}K_U = B_U \). \( \square \)

From the proof of this lemma, we see that if the Bergman metric is dominated by the Kobayashi metric, then any \( F \in \text{Hol}(U, D) \) is decreasing with respect to the Bergman metric (i.e. (*) holds).

For a domain \( D \) in \( \mathbb{C}^n \), we consider the following condition:

(\#) \( B_D \leq cK_D \) on \( TD \) for some constant \( c > 0 \).

(A) If \( D \) is a bounded homogeneous domain, then \( C_D, K_D, P_D^* \) and \( B_D \) are all equivalent metrics because of their biholomorphic invariantness and the homogeneity of \( D \). Especially, (\#) is satisfied.

(B) If the Bergman metric of \( D \) has strictly negative holomorphic sectional curvature, then by Lemma 4.2, (\#) is satisfied.

Here we shall give a function-theoretic criterion for (\#). Let \( D \) denote the Bergman domain in \( \mathbb{C}^n \) with the Bergman kernel \( k(z, \bar{w}) \). For a fixed point \( w \) in \( D \), we construct a function on \( D \) by

\[
\phi_w(z) = \phi_{w, \alpha}(z) = 1 - \left( \frac{|k(z, \bar{w})|^2}{k(z, \bar{z})k(w, \bar{w})} \right)^\alpha,
\]

where \( \alpha \) is a positive constant properly chosen for \( D \). When \( \alpha = 1/2 \), this is a square of the invariant distance \( \rho_D(z, w) \) of Skwarczynski (cf. [11]).

**Lemma 4.3.** With the above notations, the following hold.

(1) \( 0 \leq \phi_w \leq 1 \) and \( \phi_w(w) = 0 \).

(2) If \( k(z, \bar{w}) \neq 0 \), then \( L(\phi_w; w, \xi) = \alpha B_D^2(w, \xi) \).

(3) \( \phi_w \) is biholomorphically invariant, i.e. for any biholomorphic automorphism \( g \) of \( D \), \( \phi_{g(w)}(g(z)) = \phi_w(z) \).

**Proof.** (1) is clear. (3) follows from the fact that

\[
k(g(z), \bar{g(w)})J_g(z)J_g'(w) = k(z, \bar{w}),
\]
where \( J_g \) is the Jacobian determinant of \( g \). It remains to show (2). By simple calculations we have

\[
\partial^2\phi_w/\partial z_i\partial \bar{z}_j(w) = \alpha k(w, \bar{w})^{-2}\left\{ k_{ij}(w, \bar{w})k(w, \bar{w}) - k_i(w, \bar{w})k_j(w, \bar{w}) \right\}
\]

\[
= \alpha \left( \partial^2 \log k/\partial z_i\partial \bar{z}_j \right)(w).
\]

Hence, \( L(\phi_w; w, \xi) = \alpha B_D^2(w, \xi) \).

A domain \( D \) in \( \mathbb{C}^n \) is called a Lu Qi-keng domain if \( k(z, \bar{w}) \neq 0 \) for all \( z, w \) in \( D \).

**Theorem 4.4.** Let \( D \) be a Bergman domain in \( \mathbb{C}^n \). If there is a positive constant \( \alpha \) such that \( \phi_w = \phi_{w, \alpha} \) belongs to \( S_w(D) \) for each \( w \) in \( D \), then \( D \) is a Lu Qi-keng domain and

\[
B_D(w, \xi) \leq \alpha^{-1/2} P_D^*(w, \xi) \leq \alpha^{-1/2} K_D(w, \xi) \quad \text{on} \quad TD.
\]

**Proof.** When \( \phi_w \) belongs to \( S_w(D) \) for each \( w \) in \( D \), \( \phi_w \) is a p.s.h. function. Hence, by the maximum principle, \( 0 \leq \phi_w < 1 \) in \( D \), that is, \( k(z, w) \neq 0 \) for all \( z, w \) in \( D \). So \( D \) is a Lu Qi-keng domain. From the definition of \( P^* \)-metric and Lemma 4.3(2), we have

\[
\alpha^{1/2} B_D \leq P_D^* \leq K_D \quad \text{on} \quad TD.
\]

Using Lemma 2.2 we can rewrite the condition that \( \phi_w \) belongs to \( S_w(D) \) for each \( w \) in \( D \). For a fixed point \( w \) in \( D \), let \( v(z) = |k(z, \bar{w})|^2/k(z, \bar{z})k(w, \bar{w}) \). We remark that \( k(z, \bar{w}) \) is holomorphic in \( z \in D \). We set

\[
A_w = \{ z \in D; v(z) = 0 \} = \{ z \in D; k(z, \bar{w}) = 0 \},
\]

\[
E_w = \{ z \in D; v(z) = 1 \} = \{ z \in D; \phi_w(z) = 0 \},
\]

\[
D_w = D \setminus (A_w \cup E_w).
\]

Then \( A_w \) is an analytic set in \( D \), and \( E_w \) contains at least \( w \).

**Proposition 4.5.** For fixed \( w \) in \( D \), \( \phi_w = 1 - v^\alpha \) is log.p.s.h. if and only if the following inequality holds:

\[
(4.6) \quad \phi_w(z) B_D^2(z, \xi) \geq \alpha |\partial_\xi \log v(z)|^2
\]

for all \( \xi \in \mathbb{C}^n - \{0\} \) at all \( z \in D_w \).

**Proof.** Since \( 0 < v < 1 \) on \( D_w \), we may verify that (2.3) in Lemma 2.2 is equivalent to (4.6) (we set \( \beta = \alpha \)). Assume that this was proved. Then
log $\phi_w$ is p.s.h. in $D_w$. For $z$ in $E_w$, we set $\log \phi_w(z) = -\infty$. Hence log $\phi_w$ is p.s.h. in $D \setminus A_w$. Remarking that log $\phi_w$ is negative in $D$ and $A_w$ is the analytic set in $D$, we can extend log $\phi_w$ plurisubharmonically to $D$ (by the p.s.h. extension theorem of Grauert and Remmert). Thus it remains to show that (2.3) is equivalent to (4.6) for each $w$ in $D$. For a fixed $w$ in $D$, we write $\phi = \phi_w, k = k(z, \bar{z}), h = k(z, w), a = k(w, \bar{w})$. Then $\nu = \frac{h\bar{h}}{ak}$, $\phi = 1 - \left(\frac{h\bar{h}}{ak}\right)^\alpha$. By partial differentiation we have the following:

$$v_i = \frac{\bar{h}(h_i k - h k_i)}{ak^2},$$

$$v_{ij} = \left\{ h_i \bar{h}_j k^2 \left( - \frac{\bar{h}_h k_j + h h_j + h k_{i,j}}{k} \right) \right\} / ak^3.$$ 

Substituting these in

$$A = \phi \left| \sum v_i \xi_i \right|^2 - \phi \nu \sum v_{ij} \xi_i \bar{\xi}_j - \alpha \left| \sum v_i \xi_i \right|^2,$$

we get

$$A = \nu^2 \left\{ k \sum k_{i,j} \xi_i \bar{\xi}_j - \left| \sum k_i \xi_i \right|^2 \right\} / k^2 - \alpha \nu^2 \left| \sum v_i \xi_i \right|^2$$

$$= \nu^2 \left\{ \phi B_D^2(z, \xi) - \alpha |\partial_\xi \log \nu|^2 \right\}.$$ 

Therefore (2.3) holds if and only if (4.6) holds because of $0 < \nu < 1$ on $D_w$. 

**Theorem 4.7.** If the Bergman metric $B_D$ of a Bergman domain $D$ satisfies the following condition: for each $w$ in $D$ there is a positive constant $\alpha$ such that (4.6) holds, then $B_D \leq \alpha^{-1/2} K_D$. Hence every holomorphic mapping $F: U \to D$ satisfies $F^* B_D \leq (2\alpha)^{-1/2} B_U$. 

**Proof.** By Proposition 4.5 we have $\phi_w \in S_w(D)$ for each $w$ in $d$. From Theorem 4.4 the conclusion is obtained.

Also, for any $F \in \text{Hol}(U, D)$,

$$B_D(F(t), F'(t)) \leq \alpha^{-1/2} K_D(F(t), F'(t))$$

$$\leq \alpha^{-1/2} K_U(t, 1) = (2\alpha)^{-1/2} B_U(t, 1),$$

since $B_U = 2^{1/2} K_U$. 

**Remarks.** (1) $E_w$ is the polar set of log $\phi_w$. If the coordinate functions $z_1, \ldots, z_n$ are in $L^2 H(D)$ and the volume of $D$ is finite, then $E_w = \{ w \}$.

(2) There exist the bounded homogeneous domains in $\mathbb{C}^n (n \geq 7)$ of which the Bergman metrics have positive holomorphic sectional curvatures (cf. D’Atri [2]).
(3) There is a bounded pseudoconvex domain in $\mathbb{C}^3$ with $C^\infty$-boundary, which does not satisfy condition (#) (cf. Diederich-Fornaess [3]).

(4) The annulus $A = \{ t \in \mathbb{C}; \ r < |t| < 1 \}$ is not the Lu Qi-keng domain. Therefore $\log \phi_w$ is not p.s.h. for some $w \in A$ (cf. Skwarczynski [11]).

5. The classical domains. In this section we construct the function $\phi_w$ for the classical domains (the bounded symmetric domains of four main type) $R$ and directly verify that they belong to $S_w(R)$ for some properly chosen $\alpha$.

We begin with the unit ball.

Example 5.1. Let $D$ be the unit ball $B_n = \{ z \in \mathbb{C}^n; \| z \| < 1 \}$. Then its Bergman kernel is $k(z, \bar{w}) = c_n(1 - z \cdot \bar{w})^{-n-1}$, where $c_n = n!\pi^{-n}$. Let 0 be the origin of $\mathbb{C}^n$. Taking $\alpha = 1/(n + 1)$ we have $\phi_0(z) = \| z \|^2$ for $w = 0$. It is clear that $\phi_0 \in S_0(B_n)$. For any other point $w$ in $B_n$, taking an automorphism $g$ of $B_n$ with $g(0) = w$, we have $\phi_w(z) = \phi_0(g(z)) = \| g(z) \|^2$, which belongs to $S_w(B_n)$, and

$$1 - (1 - \| z \|^2)(1 - \| w \|^2)/|1 - z \cdot \bar{w}| = \| g(z) \|^2.$$ 

The last formula is well known (cf. Rudin [9], p. 26).

The four classical domains $R_1, R_{II}, R_{III},$ and $R_{IV}$, are given as follows ($M(m, n)$ denotes the set of all $m \times n$ matrices):

$R_1 = \{ Z \in M(m, n); \ I_n - Z^*Z > 0 \},$

$R_{II} = \{ Z \in M(n, n); \ Z' = Z, I_n - Z^*Z > 0 \},$

$R_{III} = \{ Z \in M(n, n); \ Z' = -Z, I_n - Z^*Z > 0 \},$

$R_{IV} = \{ z \in \mathbb{C}^n; \ |\sum z_i^2|^2 + 1 - 2\| z \|^2 > 0, \ |\sum z_i^2| < 1 \},$

where $I_n$ is the $n \times n$ unit matrix and $Z'$ is the transpose of $Z$ and $Z^* = \overline{Z'}$.

Let $R$ or $R_j$ denote one of these domains, and 0 be the zero matrix or the origin of $\mathbb{C}^n$.

Proposition 5.2. (1) For $R_1$, choosing $\alpha = 1/(m + n)n$, we set

$$\phi_0(Z) = 1 - (\det(I_n - Z^*Z))^{1/n},$$

(2) For $R_{II}$, choosing $\alpha = 1/(n + 1)n$, we set

$$\phi_0(Z) = 1 - (\det(I_n - Z^*Z))^{1/n},$$
(3) For $R_{III}$, choosing $\alpha = 1/(n - 1)\lceil n/2 \rceil$, we set
\[
\phi_0(Z) = 1 - (\det(I_n - Z^*Z))^{\lceil n/2 \rceil},
\]
(4) For $R_{IV}$, choosing $\alpha = 1/2n$, we set
\[
\phi_0(z) = 1 - \left(1 + \left|\sum z_i^2\right|^2 - 2\|z\|^2\right)^{1/2}.
\]
Then each $\phi_0$ is log.p.s.h., hence it belongs to $S_0(R)$; also $\phi_w(Z) = \phi_0(g(Z))$ belongs to $S_w(R)$ for any point $w$ in $R$, where $g$ is an automorphism of $R$ with $g(w) = 0$.

We quote two lemmas to prove this proposition.

**Lemma 5.3** ([7], p. 34). Let $B_j$ be the Bergman metric of $R_j$ ($j = I, II, III, IV$). Then

\[
\begin{align*}
B_{I_1}^2(Z, \xi) &= (m + n)\text{Trace}\left[ (I_n - Z^*Z)^{-1}\xi(I_m - ZZ^*)^{-1}\xi^* \right], \\
B_{I_2}^2(Z, \xi) &= (n + 1)\text{Trace}\left[ (I_n - Z^*Z)^{-1}\xi(I_n - ZZ^*)^{-1}\xi^* \right], \\
B_{III}^2(Z, \xi) &= (n - 1)\text{Trace}\left[ (I_n - Z^*Z)^{-1}\xi(I_n - ZZ^*)^{-1}\xi^* \right], \\
B_{IV}^2(z, \xi) &= 2nA^{-2}\xi\left[ A(I_n - 2z^*z) + 2(I_n - z^*z)z^*z(I_n - z^*z) \right]\xi^*,
\end{align*}
\]

where
\[
A = \left|\sum z_i^2\right|^2 + 1 - 2\|z\|^2.
\]

**Lemma 5.4** ([5]). (1) For $Z \in R_I$, there exists an element $g$ of $G_0$ (the isotropy subgroup of automorphism group of $R_I$) such that
\[
gZ = \begin{pmatrix}
\lambda_1 \\
& \ddots \\
& & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
& & & \lambda_n \\
0 \\
\end{pmatrix}, \quad 1 > \lambda_1 \geq \cdots \geq \lambda_n \geq 0, m \geq n.
\]
(2) For $Z \in R_{II}$, there is a $g \in G_0$ such that
\[
gZ = \text{diagonal}[\lambda_1, \ldots, \lambda_n], \quad 1 > \lambda_1 \geq \cdots \geq \lambda_n \geq 0.
\]
(3) For $Z \in R_{III}$, there is a $g \in G_0$ such that
\[
gZ = \text{diagonal}\left[\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}, (0)\right],
\]
where $k = \lceil n/2 \rceil$, and the last term is 0 if $n$ is odd.
For \( z \in R_{IV} \), there is a \( g \in G_0 \) such that 
\[
g_0 = (\lambda, i\mu, 0, \ldots, 0), \quad \lambda, \mu \text{ real numbers.}
\]

**Proof of Proposition 5.2.** Since 
\[
\phi_0(Z) = 1 - \left( |k(Z, 0)|^2 / (k(Z, \overline{Z}) k(0, 0)) \right) \alpha
\]
and \( k(Z, 0) \) is a constant, each \( \phi_0 \) is as above. (For the Bergman kernel of the classical domains, see, for example, [7, p. 34].) We directly verify (4.6) for the domain \( R_1 \) (\( m \geq n \)). Note that \( R_1 \) is a Lu Qi-keng domain; thus \( A_0 \) is the empty set and \( E_0 \) is point 0 only. From the biholomorphic invariantness of \( \phi_0 \) and Lemma 5.4, it is sufficient to show (4.6) at the point 
\[
Z_0 = \begin{pmatrix}
\lambda_1 \\
0 \\
\vdots \\
0 \\
\lambda_n
\end{pmatrix}
\]
\((1 > \lambda_1 \geq 0)\).

Thus 
\[
V(Z_0) = \prod_{i=1}^{n} (1 - \lambda_i^2)^{1/n};
\]

\[
\phi_0(Z_0) = 1 - \prod_{i=1}^{n} (1 - \lambda_i^2)^{1/n}.
\]

From Lemma 5.3 we have 
\[
B_0^2(Z_0, \xi) = (m + n) \left[ \sum_{i,j}^n \frac{|\xi_{ij}|^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} + \sum_{i=1}^{n} \sum_{k > n}^m \frac{|\xi_{ik}|^2}{1 - \lambda_i^2} \right].
\]

By partial differentiation 
\[
\frac{\partial}{\partial Z_{\mu \nu}} \det(I_n - Z^*Z) = (-1)^{\mu+1} \delta_{\mu \nu} \lambda_\mu \prod_{i \neq \mu} (1 - \lambda_i^2) \quad \text{at } Z = Z_0.
\]

Hence, 
\[
\alpha |\partial_\xi \log v(Z_0)|^2 = \frac{m + n}{n} \left| \partial_\xi \log \det(I_n - Z_0^*Z_0) \right|^2
\]
\[
= \frac{m + n}{n} \left| \sum_{\mu, \nu} \frac{\partial}{\partial Z_{\mu \nu}} (\det(I_n - Z_0^*Z_0) \xi_{\mu \nu}) \right|^2 \left| \det(I_n - Z_0^*Z_0) \right|^{-2}
\]
\[
= \frac{m + n}{n} \left| \sum_{\mu} \frac{\lambda_\mu \xi_{\mu \mu}}{1 - \lambda_\mu^2} \right|^2.
\]
Let $T(Z_0)$ be the difference of the left and right sides of (4.6) at $Z_0$. Then

$$T(Z_0) = (m + n) \left(1 - \prod_j (1 - \lambda_j^2)^{1/n}\right) \cdot \left(\sum_{i,j} \frac{|\xi_{ij}|^2}{(1 - \lambda_i^2)(1 - \lambda_j^2)} + \sum_i \sum_{k > n} \frac{|\xi_{ki}|^2}{1 - \lambda_i^2}\right) - \frac{m + n}{n} \left|\sum_{\mu} \frac{\lambda_{\mu}}{1 - \lambda_{\mu}^2} \xi_{\mu\mu}\right|^2.$$

Noting that $\prod(1 - \lambda_j^2)^{1/n} \leq 1 - (1/n)\sum \lambda_j^2$, we have

$$T(Z_0) \geq \frac{m + n}{n} \left(\sum \lambda_j^2\right) \left(\sum |\xi_{ij}|^2 (1 - \lambda_i^2)^{-1}(1 - \lambda_j^2)^{-1}\right) - \frac{m + n}{n} \left|\sum_{\mu} \lambda_{\mu} \xi_{\mu\mu} (1 - \lambda_{\mu}^2)^{-1}\right|^2 \geq 0$$

by Schwarz' inequality.

It is the same for (2) and (3) as (1). For $R_{IV}$ we may verify (2.3) directly at $(\lambda, i\mu, 0, \ldots, 0)$. The proof is reduced to the following inequality;

$$(\phi - 1/2) \left[\lambda^2(\lambda^2 - \mu^2 - 1)^2|\xi_1|^2 + \mu^2(\lambda^2 - \mu^2 + 1)|\xi_2|^2 + i\lambda\mu(\lambda^2 - \mu^2 - 1)(\lambda^2 - \mu^2 + 1)(\xi_1\xi_2 - \xi_2\xi_1)\right] - 2\phi v \left[(2\lambda^2 - 1)|\xi_1|^2 + (2\mu^2 - 1)|\xi_2|^2 + 2i\lambda\mu(\xi_1\xi_2 - \xi_2\xi_1)\right] + 2\phi v \left(\sum_3 |\xi_i|^2\right) \geq 0 \quad \text{for all } \xi \in \mathbb{C}^n,$$

where $v = |\lambda^2 - \mu^2|^2 + 1 - 2(\lambda^2 + \mu^2)$ (0 < $v$ < 1), $\phi = 1 - v^{1/2}$, and $|\lambda^2 - \mu^2| < 1$. Simple but long calculations show that the above inequality holds.

\textbf{Corollary 5.5.} $B_1^2 \leq (m + n)nK_1^2$, $B_2^2 \leq (n + 1)nK_2$, $B_3^2 \leq (n - 1)[n/2]K_3^2$, $B_4^2 \leq 2nK_4^2$. 

Proof. From Theorem 4.4 and Proposition 5.2, we have, for example,

\[ B^*(0, \xi) \leq (m + n)nK^2_f(0, \xi) \quad \text{for all } \xi \in \mathbb{C}^n. \]

Using the homogeneity of \( R_j \) and invariantness of the Bergman and Kobayashi metrics, we have the conclusion. \( \square \)

REMARKS. For the classical domains \( R_j, C, P^*, \) and \( K \)-metrics all coincide. The \( K \)-metric \( K^1_f(0, \xi) \) is given by

\[ K^1_f(0, \xi) = \max \{ \text{eigenvalues of } \xi^*\xi \} \]

(cf. [13]). Hence,

\[ K^2_f(0, \xi) \geq n^{-1} \text{Trace } \xi^*\xi = (m + n)^{-1}n^{-1}B^2_f(0, \xi) = \alpha B^2_f(0, \xi). \]

On the other hand, the holomorphic sectional curvature \( \kappa \) of \( B^2_f \) satisfies \(-4/(m + n) \leq \kappa \leq -4/(m + n)n\), hence the constant \( k \) in Lemma 4.2 is equal to \( 4 \alpha \) and we have \( \alpha B^2_f \leq K^2_f \). For \( R_{II} \) and \( R_{III} \), we can do similarly. With respect to \( R_{IV} \),

\[ K^2_{IV}(0, \xi) = \|\xi\|^2 + (\|\xi\|^4 - |\xi^*\xi|^2)^{1/2} \geq \|\xi\|^2 = 1/2nB^2_{IV}(0, \xi), \]

and the holomorphic sectional curvature \( \kappa \) of \( B^2_{IV} \) satisfies \(-4/n \leq \kappa \leq -2/n\). Hence \( \alpha B^2_{IV} \leq K^2_{IV} (\alpha = 1/2n) \) (cf. [13]).

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