ON THE WALLMAN ORDER COMPACTIFICATION

Darrell Conley Kent
ON THE WALLMAN ORDER COMPACTIFICATION

D. C. KENT

The Wallman order compactification \( w_0X \) of a topological ordered space \( X \) has been constructed by Choe and Park. This paper establishes necessary and sufficient conditions for their compactification to be \( T_2 \)-ordered, in which case it coincides with the Nachbin (or Stone-Čech order) compactification.

**Introduction.** Let \((X, \leq)\) be a poset. For \( x \in X \), let \( i(x) = \{ y \in X : x \leq y \} \) and let \( d(x) = \{ y \in X : y \leq x \} \). If \( A \subseteq X \), let \( i(A) = \bigcup \{ i(x) : x \in A \} \), and \( d(A) = \bigcup \{ d(x) : x \in A \} \). If \( A = iA \) (respectively, \( A = d(A) \)), then \( A \) is called an *increasing* (respectively, *decreasing*) set; a set which is either increasing or decreasing is said to be *monotone*.

A *topological ordered space* \((X, \leq , \tau)\) consists of a poset \((X, \leq)\) equipped with a topology \( \tau \). If \( \tau \) has an open subbase consisting of monotone sets, then the topological ordered space is said to be *convex*. Since only convex topological ordered spaces can have order compactifications which are \( T_2 \)-ordered (see below), we shall henceforth consider only spaces of this type. For brevity, a convex topological ordered space \((X, \leq , \tau)\) will be simply called a *space* and designated by "\( X \)".

Following McCartan [4], we define a space \( X \) to be *\( T_1 \)-ordered* if \( i(x) \) and \( d(x) \) are both closed for all \( x \in X \), and *\( T_2 \)-ordered* if the partial order relation is a closed subset of \( X \times X \). A \( T_1 \)-ordered space is *\( T_4 \)-ordered* (*normally ordered* in [5]) if, whenever \( A \) and \( B \) are closed disjoint subsets, the former decreasing and the latter increasing, there are disjoint open sets \( U \) and \( V \), the former decreasing and the latter increasing, such that \( A \subseteq U \) and \( B \subseteq V \). The "\( T_3 \)-ordered" property is defined in [4], and "\( T_{3,5} \)-ordered" can be taken to mean "completely regular ordered" as defined in [5], but it will not be necessary to repeat these latter definitions here.

Nachbin has constructed a Stone-Čech type order compactification \( \beta_0X \) of an arbitrary \( T_{3,5} \)-ordered space \( X \) with the property that any continuous, increasing function from \( X \) into a \( T_2 \)-ordered, compact space can be lifted to \( \beta_0X \). For details of the Nachbin compactification, see [3]. More recently, Choe and Park showed that \( X \) is *\( T_4 \)-ordered* whenever \( w_0X \) is \( T_2 \)-ordered, but were unable to prove the converse. Our main result establishes that \( w_0X \) is \( T_2 \)-ordered if and only if \( X \) is strongly \( T_4 \)-ordered.
(this term is defined below), and consequently that $w_0X$ and $\beta_0X$ are equivalent compactifications of a strongly $T_4$-ordered space $X$.

Let $X$ be a topological ordered space. If $A \subseteq X$, let $I(A)$ (respectively, $D(A)$) be the smallest increasing (respectively, decreasing) closed set containing $A$, and let $A^\sim = I(A) \cap D(A)$. Let $\mathcal{C}_X = \{ A \subseteq X : A = A^\sim \}$. Note that all members of $\mathcal{C}_X$ are closed and convex; we shall call the members of $\mathcal{C}_X$ c-sets. All monotone closed sets are c-sets, and thus $\mathcal{C}_X$ is a closed subbase for $\tau$. One can easily verify that every set of the form $A^\sim$, for $A \subseteq X$, is a c-set, and also that $\mathcal{C}_X$ is closed under finite intersections.

Let $F(X)$ be the set of all filters on $X$; the fixed ultrafilter generated by $\{ x \}$ will be denoted by $\hat{x}$ for $x \in X$. If $\mathcal{F}, \mathcal{G} \in F(X)$, then $\mathcal{F} \cup \mathcal{G}$ will designate the filter generated by $\{ F \cap G : F \in \mathcal{F}, G \in \mathcal{G} \}$ (assuming that the latter collection does not include $\emptyset$).

For $\mathcal{F} \in F(X)$, we denote by $i(\mathcal{F})$ the filter generated by $\{ i(F) : F \in \mathcal{F} \}$; the filters $d(\mathcal{F})$, $I(\mathcal{F})$, and $D(\mathcal{F})$ are defined analogously. A filter $\mathcal{F}$ is a c-filter (respectively, a convex filter) if it has a filter base of c-sets (respectively, convex sets). Note that $\mathcal{F}$ is a c-filter (respectively, a convex filter) iff $\mathcal{F} = I(\mathcal{F}) \cup D(\mathcal{F})$ (respectively, $\mathcal{F} = i(\mathcal{F}) \cup d(\mathcal{F})$). A c-filter which is not properly contained in any other c-filter will be called a maximal c-filter. A standard Zorn's Lemma argument establishes that every c-filter is contained in a maximal c-filter.

We can assume that $X$ is a $T_1$-ordered space and define $w_0(X)$ to be the set of all maximal c-filters on $X$. Note that the only convergent maximal c-filters are the fixed ultrafilters. It will be convenient to write $w_0X = \{ \hat{x} : x \in X \} \cup X'$, where $X'$ is the set of all non-convergent maximal c-filters. An order relation "$\leq$" for $w_0X$ is defined as follows: $\mathcal{F} \leq \mathcal{G}$ iff $I(\mathcal{F}) \subseteq \mathcal{G}$ and $D(\mathcal{G}) \subseteq \mathcal{F}$. It is a simple matter to verify that $(w_0X, \leq)$ is a poset and that the canonical map $\varphi : (X, \leq) \to (w_0X, \leq)$, defined by $\varphi(x) = \hat{x}$, is increasing.

We next introduce a topology on $w_0X$. For $A \subseteq X$, define $A^* = \{ \mathcal{F} \in w_0X : A \in \mathcal{F} \}$. Then $\mathcal{C}^* = \{ A^* : A \in \mathcal{C}_X \}$ is a closed subbase for a topology on $w_0X$ which we shall denote by $w_0\tau$. Clearly, $(A \cap B)^* = A^* \cap B^*$ for all subsets $A, B$ of $X$; from this one easily deduces that $w_0X$ is a topological ordered space. It is obvious that $A = \varphi^{-1}(A^*)$ for any $A \subseteq X$; therefore $\varphi : X \to w_0X$ is a topological embedding, and both $\varphi$ and $\varphi^{-1}|(\varphi(x))$ are increasing functions.

Before proceeding further, it is desirable to compare our construction of $w_0X$ with that of Choe and Park. They define a bifilter $(\mathcal{G}, \mathcal{H})$ on $X$ to be a pair of filters such that $\mathcal{G}$ has a base of decreasing closed sets, $\mathcal{H}$ has a base of increasing closed sets, and $\mathcal{G} \cup \mathcal{H}$ exists; the set of all maximal
bifilters forms the underlying set for their compactification, which is also denoted by $w_0X$. It is easy to see that, for any bifilter $(\mathcal{G}, \mathcal{H})$ on $X$, the filter $\mathcal{F} = \mathcal{G} \lor \mathcal{H}$ is a $c$-filter, and that, for any $c$-filter $\mathcal{F}$, $(D(\mathcal{F}), I(\mathcal{F}))$ is a corresponding bifilter. If $(\mathcal{G}, \mathcal{H})$ is a maximal bifilter, then $\mathcal{F} = \mathcal{G} \lor \mathcal{H}$ is a maximal $c$-filter, and $(D(\mathcal{F}), I(\mathcal{F})) = (\mathcal{G}, \mathcal{H})$; thus a bijection exists between the set of maximal bifilters on $X$ and the set of maximal $c$-filters on $X$. A comparison of the order relation and topology defined for $w_0X$ in [2] with our definitions given above reveals the equivalence of these spaces both as posets and as topological spaces. Thus the results obtained concerning $w_0X$ in [2] are applicable here, albeit with appropriate terminological alterations. The next two results are obtained in this way.

**Proposition 1.1.** For any $T_1$-ordered space $X$, $(w_0X, \varphi)$ is an order compactification of $X$, and $w_0X$ is a $T_1$ topological space. If $w_0X$ is $T_2$-ordered, then $X$ is $T_4$-ordered.

**Proposition 1.2.** Let $X$ be a $T_1$-ordered space, $Y$ a $T_2$-ordered compact space, and $f: X \rightarrow Y$ a continuous, increasing function. Then there is a unique, continuous, increasing function $\tilde{f}: w_0X \rightarrow Y$ such that $\tilde{f} \cdot \varphi = f$.

We define a $T_4$-ordered space $X$ to be strongly $T_4$-ordered if, whenever $A$ and $B$ are $c$-sets:

\[
I(A) \cap B = \emptyset \quad \text{implies} \quad I(A) \cap D(B) = \emptyset \\
D(A) \cap B = \emptyset \quad \text{implies} \quad D(A) \cap I(B) = \emptyset
\]

Note that a $T_4$-ordered space $X$ is strongly $T_4$-ordered iff, for a $c$-set $A$ and a decreasing open set $U$ with $A \subseteq U$, $D(A) \subseteq U$ and dually.

Priestly [6] defines a $C$-space to be a topological ordered space $X$ such that, for each closed subset $A$, $i(A)$ and $d(A)$ are also closed. The class of strongly $T_4$-ordered spaces includes the $T_4 C$-spaces, among which are the $T_2$-ordered compact spaces.

**Proposition 1.3.** A $T_1$-ordered space $X$ is strongly $T_4$-ordered if and only $w_0X$ is $T_2$-ordered

**Proof.** In Proposition 1, page 26, [5], Nachbin shows that a space is $T_2$-ordered if, whenever $a \nleq b$, there is an increasing neighborhood $V$ of $a$ and a decreasing $W$ of $b$ such that $V \cap W = \emptyset$.

Assume that $\mathcal{F}, \mathcal{G}$ are elements of $w_0X$ such that $\mathcal{F} \leq \mathcal{G}$ is false. Then either $I(\mathcal{F}) \subseteq \mathcal{F}$ or $D(\mathcal{G}) \subseteq \mathcal{F}$ is false. In the former case, since $\mathcal{G}$ is a
maximal c-filter, there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $I(F) \cap G = \emptyset$. By the assumption that $X$ is strongly $T_4$-ordered, $I(F) \cap D(G) = \emptyset$, and so there are disjoint open neighborhoods $U$ and $V$ of $I(F)$ and $D(G)$, respectively, such that $U$ is increasing and $V$ decreasing. Then $U^*$ and $V^*$ are disjoint, open neighborhoods of $\mathcal{F}$ and $\mathcal{G}$, respectively, in $w_0X$, the former increasing and the latter decreasing. This $w_0X$ is $T_2$-ordered.

Conversely, assume that $w_0X$ is $T_2$-ordered. Let $A, B$ be c-sets and suppose $I(A) \cap B = \emptyset$. Then $I(A)^* \cap B^* = \emptyset$. $I(A)^*$ is a closed, increasing subset of $w_0X$ and $B^* = D(B)^* \cap I(B)^*$ is a closed subset of $w_0X$. Let $d_w(B^*) = \{ \mathcal{F} \in w_0X: \mathcal{F} \leq \mathcal{G} \text{ for some } \mathcal{G} \in B^* \}$. By Proposition 4, page 44, $[5]$, $d_w(B^*)$ is a closed subset of $w_0X$, and it follows that $I(A)^* \cap d_w(B^*) = \emptyset$. Then $\varphi^{-1}(I(A)^* \cap d_w(B^*)) = \varphi^{-1}(I(A)^*) \cap \varphi^{-1}(d_w(B^*)) = \emptyset$. Since $\varphi^{-1}(I(A)^*) = I(A)$ and $D(B) \subseteq \varphi^{-1}(d(B^*))$, it follows that $I(A) \cap D(B) = \emptyset$. A similar argument shows that if $D(A) \cap B = \emptyset$, then $D(A) \cap I(B) = \emptyset$. This conclusion that $X$ strongly $T_4$-ordered now follows with the help of Proposition 1.1.

**Corollary 1.4.** A $T_4$-ordered space $X$ is strongly $T_4$-ordered if and only if, for any c-set $A$, $d(A)$ and $i(A)$ are both closed.

**Proof.** The condition is obviously sufficient. Suppose that $X$ is strongly $T_4$-ordered and $x \notin d(A)$. Then $i(x)^* \cap A^* = \emptyset$, and consequently $i(x)^* \cap d_w(A^*) = \emptyset$. It follows that $i(x) \cap \varphi^{-1}(d_w(A^*)) = \emptyset$. Since the closure of $d(A)$ in $X$ is a subset of $\varphi^{-1}(d_w(A^*))$, $x$ is not in the closure of $d(A)$. Thus $d(A)$ is closed.

**Corollary 1.5.** Let $X$ be $T_{3.5}$-ordered. Then the compactifications $w_0X$ and $\beta_0X$ are equivalent if and only if $X$ is strongly $T_4$-ordered.

If the order relation of $X$ is trivial, then the c-sets are simply the closed sets, and the compactification $w_0X$ is identical with the ordinary Wallman compactification. In this case, Corollary 1.5 yields the well-known equivalence of the Wallman and Stone–Čech compactifications for $T_4$ topological space.

We conclude by considering the Wallman order compactification for a simple and familiar class of spaces. We define a **totally ordered space** to be a totally ordered set with its order topology. If $X$ is a totally ordered space, then one can show that $w_0X$ (and hence $\beta_0X$) is a totally ordered space and a complete lattice. If $X = R$ is the totally ordered space of real numbers, then $w_0X$ can be identified with the extended real line $[-\infty, \infty]$. 
If \( X = \mathbb{Q} \) is the space of rationals, then \( w_0X \) can also be regarded as the extended real line, but with each irrational “occurring twice”; by identifying these “irrational pairs”, one obtains \( w_0\mathbb{R} \) as a quotient space of \( w_0\mathbb{Q} \).

**References**


Received October 14, 1983 and in revised form January 12, 1984.

WASHINGTON STATE UNIVERSITY
PULLMAN, WA 99164–2930
Dan Amir, On Jung’s constant and related constants in normed linear spaces . . . 1
Abdul Aziz, On the location of the zeros of certain composite polynomials . . . 17
Joseph Barback, On hereditarily odd-even isols and a comparability of summands property .......................................................... 27
Matthew G. Brin, Klaus Johannson and Peter Scott, Totally peripheral 3-manifolds .......................................................... 37
Robert F. Brown, A topological bound on the number of distinct zeros of an analytic function .......................................................... 53
K. C. Chattopadhyay, Not every Lodato proximity is covered .................. 59
Beverly Diamond, Some properties of almost rimcompact spaces ............ 63
Manfred Dugas and Rüdiger Göbel, On radicals and products ................. 79
Abdelouahab El Kohen, A hyperbolic problem .................................... 105
Harry Gonshor, Remarks on the Dedekind completion of a nonstandard model of the reals .......................................................... 117
William H. Kazez, On equivalences of branched coverings and their action on homology .......................................................... 133
Darrell Conley Kent, On the Wallman order compactification .................. 159
Martin Andrew Magid, Lorentzian isoparametric hypersurfaces ............... 165
Milan Miklavčič, Stability for semilinear parabolic equations with noninvertible linear operator .................................................. 199
Richard Dean Neidinger and Haskell Paul Rosenthal, Norm-attainment of linear functionals on subspaces and characterizations of Tauberian operators .................................................. 215
Johannes Vermeer, Closed subspaces of $H$-closed spaces ...................... 229