CLOPEN REALCOMPACTIFICATION OF A MAPPING

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In this note, we give a necessary and sufficient condition on $\varphi: X \to Y$ for $\nu \varphi$ to be an open perfect mapping of $\nu X$ onto $\nu Y$ and other related results.

Throughout this paper, by a space we mean a completely regular Hausdorff space and mappings are continuous and we assume familiarity with [1] whose notation and terminology will be used throughout. We denote by $\varphi: X \to Y$ a map of $X$ onto $Y$, by $\beta X$ ($\nu X$) the Stone-Čech compactification (Hewitt realcompactification) of $X$ and by $\beta \varphi$ ($\nu \varphi = (\beta \varphi)|\nu X$) the Stone extension (realcompactification) over $\beta X$ ($\nu X$) of $\varphi$.

Concerning clopenness of $\nu \varphi$ of a clopen map $\varphi: X \to Y$ the following results are known.

**Theorem A (Ishii [4]).** If $\varphi: X \to Y$ is an open quasi-perfect map, then $\nu \varphi$ is an open perfect map of $\nu X$ onto $\nu Y$.

**Theorem B (Morita [8]).** If $\varphi: X \to Y$ is a clopen map such that the boundary of each fiber is relatively pseudocompact, then $\nu \varphi$ is also a clopen map of $\nu X$ onto $\nu Y$.

In §2, concerning Theorem A we give a necessary and sufficient condition on $\varphi$ for $\nu \varphi$ to be an open perfect map of $\nu X$ onto $\nu Y$ without using the theory of hyper-spaces (Theorem 2.3 below) and a necessary and sufficient condition on $\varphi$ for $\nu \varphi$ to be an open RC-preserving map of $\nu X$ onto $\nu Y$ under some condition (Theorem 2.6 below).

We use the following notation and abbreviation: $C(X)$ is the set of real-valued continuous functions defined on $X$, $C(X; \varphi) = \{ f \in C(X); f$ is $\varphi$-bounded}, $\text{Bd} A$ = the boundary of $A$, usc = upper semicontinuous, lsc = lower semicontinuous and $\omega (\omega_1) = \text{the first infinite (uncountable)}$ ordinal, clopen = closed and open.

1. Definitions and Lemmas.

1.1. Definition. Let $\varphi: X \to Y. f \in C(X)$ is said to be $\varphi$-bounded if $\sup\{|f(x)|; x \in \varphi^{-1}(y)\} < \infty$ for every $y \in Y$. Whenever $f$ is $\varphi$-bounded,
we put
\[
f^s(y) = \sup\{f(x); x \in \varphi^{-1}(y)\} \quad \text{and} \quad f^i(y) = \inf\{f(x); x \in \varphi^{-1}(y)\}
\]
for each \(y \in Y\).

A subset \(A\) of \(X\) is \emph{relatively pseudocompact} if \(f|A\) is bounded for each \(f \in C(X)\). \(\varphi: X \to Y\) is said to be

1. \emph{WZ} if \(\text{cl}_{\beta X} \varphi^{-1}y = (\beta \varphi)^{-1}y\) for each \(y \in Y\) [5].
2. \emph{\(W_rN\)} if \(\text{cl}_{\beta X} \varphi^{-1}R = (\beta \varphi)^{-1}(\text{cl}_{\beta Y} R)\) for every regular closed set \(R\) of \(Y\) [3].

3. \emph{*-open (\(W^*-\)open)} if \(\text{int}(\text{cl} \varphi U) \supset \varphi U\) \((\text{int}(\text{cl} \varphi U) \neq \emptyset)\) for every open set \(U \subseteq X\) [2, 7].
4. \emph{\(\beta\)-open} if \(\varphi\) is \(\beta\)-open and \(W_rN\).
5. a \emph{\(d^*\)-map} if \(\bigcap \text{cl} \varphi Z_n = \emptyset\) for any decreasing sequence \(\{Z_n\}\) of zero sets of \(X\) with empty intersection [6].
6. \emph{\(RC\)-preserving (an RC-map)} if \(\varphi \text{R}\) is regular closed\(\text{(closed)}\) for every regular closed set \(R\) of \(X\) [2].

We note that (1) a closed map is a \(Z\)-map and a \(Z\)-map is \(WZ\) [5], (2) an open map is \(\beta\)-open and a \(\beta\)-open map is \(W^*\)-open [7], (3) a space \(Y\) is \(\text{cb}^*\) iff any \(d^*\)-map onto \(Y\) is hyper-real, i.e., \(v\varphi\) is a perfect map onto \(vY\) [6], (4) an \(RC\)-preserving map is \(RC\) and (5) an open \(WZ\)-map is \(\beta\)-open by 1.2 (1, 5) below. Thus it is easy to see that if \(\varphi\) is \(\beta\)-open, then \((\beta \varphi)|Z: Z \to (\beta \varphi)Z\) is \(\beta\)-open for each \(Z\) with \(X \subseteq Z \subseteq \beta X\). \(Y \supseteq B\) is said to be \(\varphi-d^*\) if \((\beta \varphi)^{-1}B \subseteq vX\). By 1.2(4) below, \(\varphi\) is a \(d^*\)-map iff \(Y\) is \(\varphi-d^*\).

**Lemma 1.2.** Let \(\varphi: X \to Y\).

1. If \(\varphi\) is \(WZ\), then \(\varphi\) is open iff \(\beta \varphi\) is open [5].
2. If \(\varphi\) is open \((WZ)\), then \(f^i\) is usc \((\text{usc})\) and \(f^s\) is lsc \((\text{lsc})\) for every \(f \in C(X; \varphi)\) (for example, see [5]).
3. If \(\varphi\) is open \(WZ\), then \(f^i\) and \(f^s \in C(Y)\) for every \(f \in C(X; \varphi)\) [5].
4. \(\varphi\) is a \(d^*\)-map iff \((\beta \varphi)^{-1}Y \subseteq vX\) [6].
5. \(\varphi\) is \(\beta\)-open iff \(\beta \varphi\) is open [7].
6. If \(\varphi\) is an \(RC\)-map, then \(\varphi\) is \(WZ\) [3].
7. \(\varphi\) is \(RC\)-preserving iff \(\varphi\) is a \(W^*\)-open \(RC\)-map [2].

2. **Main Theorems.**

**Lemma 2.1.** Let \(\varphi: X \to Y\). Then the following are equivalent:

1. \(\varphi\) is \(WZ\) \((\text{open})\).
2. \(f^i\) is lsc \((\text{usc})\) for every \(f \in C(X; \varphi)\)
3. \(f^s\) is usc \((\text{lsc})\) for every \(f \in C(X; \varphi)\).
Proof. (2) \(\Leftrightarrow\) (3) is evident. (1) \(\Rightarrow\) (2). From 1.2(2).

We will prove (2) \(\Rightarrow\) (1). Suppose that \(\varphi\) is not \(WZ\). Then there are \(y \in Y\) and \(p \in \beta X\) with \(p \in (\beta \varphi)^{-1}y = \text{cl}_{\beta X} \varphi^{-1}y\). Since \(p \notin \text{cl}_{\beta X} \varphi^{-1}y\), there is \(g \in C(\beta X)\) such that \(p \in \text{int}_{\beta X} Z(g)\) and \(g = 1\) on \(\text{cl}_{\beta X} \varphi^{-1}y\). Let us put \(f = g\vert X\). Then \(f \in C(X), f'(y) = 1, A = Z(f) \neq \emptyset\) and \(p \in \text{cl}_{\beta X} A\). On the other hand, \(\text{cl}_{\beta Y} \varphi A = \text{cl}_{\beta Y} (\beta \varphi) A = (\beta \varphi) \text{cl}_{\beta X} A \supseteq (\beta \varphi)p = y\). This shows \(y \in \text{cl} \varphi A\) and hence for each neighborhood \(V\) of \(y\), there is \(z \in V\) with \(f^i(z) = 0\), i.e., \(f^i\) is not lsc.

Now suppose that \(\varphi\) is not open. Then there are a point \(x\) and an open set \(U \ni x\) such that \(V - \varphi U \neq \emptyset\) for every open set \(V \ni y = \varphi(x)\). Let \(f \in C(X; \varphi)\) such that \(x \in \text{int} Z(f) \subset U\) and \(f = 1\) on \(X - U\). Obviously \(f^i(y) = 0\) and \(f^i = 1\) on \(V - \varphi U\). This shows that \(f^i\) is not usc.

Using 2.1, it is easy to see the following:

**Theorem 2.2.** \(\varphi: X \to Y\) is open \(WZ\) iff \(f^i\) and \(f^s \in C(Y)\) for every \(f \in C(X; \varphi)\) equivalently,

\[
C(Y) = \{ f^i; f \in C(X; \varphi) \} = \{ f^s; f \in C(X; \varphi) \}.
\]

**Theorem 2.3.** \(\varphi: X \to Y\) is a \(\beta\)-open \(d^*\)-map iff \(\nu \varphi\) is an open perfect map of \(\nu X\) onto \(\nu Y\).

**Proof.** \(\Leftarrow\) From 1.2(1, 4, 5) and \((\beta \varphi)^{-1}Y \subset (\beta \varphi)^{-1}\nu Y = \nu X\). \(\Rightarrow\) By 1.2(5), \(\beta \varphi\) is open. We will prove that \(\nu \varphi\) is a perfect map on \(\nu Y\). To do this, it suffices to show that \((\beta \varphi)p = q \in \beta Y - \nu Y\) for every \(p \in \beta X - \nu X\). Let \(p \in \beta X - \nu X\). Then there is \(f \in C(\beta X)\) with \(p \in Z(f) \subset \beta X - \nu X\). \(\beta \varphi\) being open \(WZ\) by 1.2(5), it follows from 2.2 that \(f^i \in C(\beta Y), f^i(q) = 0\) and \(f^i > 0\) on \(Y\). This shows \(q \in \beta Y - \nu Y\), so \(\nu \varphi\) is a perfect map on \(\nu Y\). Since \(\beta(\nu \varphi) = \beta \varphi\) and \(\beta \varphi\) is open, \(\nu \varphi\) is open by 1.2(1). Thus \(\nu \varphi\) is an open perfect map of \(\nu X\) onto \(\nu Y\).

2.4. **Example.** Let \(X = [0, \omega_1]^2 - \{ (\omega_1, \alpha); \omega \leq \alpha \leq \omega_1 \}, Y = [0, \omega_1]\) and \(\varphi\) the projection of \(X\) onto \(Y\). It is obvious that \(\varphi\) is not \(WZ\) and hence not closed and \(\varphi^{-1}(\omega_1)\) is not compact. On the other hand \(\beta \varphi: \beta X = \nu X = [0, \omega_1]^2 \to Y = \nu Y = \beta Y\) is open perfect (compare with the assumption of Theorem A).

2.5. **Lemma.** If \(\varphi: X \to Y\) is a \(*\)-open RC-map, then \(\varphi\) is open.

**Proof.** Let \(U\) be open in \(X\) and \(x \in U\). Take a regular closed set \(R\) with \(x \in \text{int} R \subset R \subset U\). Since \(\varphi\) is a \(*\)-open RC-map, we have \(y = \varphi(x) \in \text{int}(\text{cl} \varphi(\text{int} R)) \subset \varphi R \subset \varphi U\), so \(y \in \text{int} \varphi U\). Thus \(\varphi\) is open.
In the following we put
\[ Y_d = \{ y \in Y; \varphi^{-1}y \text{ is open but not relatively pseudocompact} \}, \]
\[ Y_e = X - Y_d. \]

**Theorem 2.6.** \( \varphi: X \to Y \) is a \( \beta \)-open map such that \( Y_e \) is \( \varphi \)-d* iff \( \nu \varphi \) is an open RC-preserving map of \( \nu X \) onto \( \nu Y \) such that \( \text{cl}_{\nu Y} Y_e \) is \( (\varphi \nu) \)-d*.

**Proof.** (\( \Leftarrow \)) Since \( \nu \varphi \) is open \( WZ \) by 1.2(6), \( \beta \varphi \) is open by 1.2(1) and \( \varphi \) is a \( \beta \)-open map by 1.2(5). The fact that \( \text{cl}_{\nu Y} Y_e \) is \( (\varphi \nu) \)-d* implies that \( Y_e \) is \( \varphi \)-d*.

(\( \Rightarrow \)) (1) We will first prove that if \( p \in \beta X - \nu X \) and \( (\beta \varphi) p = q \in \nu Y \), then there is a clopen subset \( D \) of \( Y \) such that \( q \in \text{cl}_{\nu Y} D, D \subset Y_d \) and \( \text{cl}_{\nu Y} D \cap \text{cl}_{\nu Y} Y_e = \emptyset \). There is \( f \in C(\beta X) \) with \( p \in Z(f) \subset \beta X - \nu X \). By 1.2(5), \( \beta \varphi \) is open. Thus \( f' \in C(\beta Y) \). Since \( Y_e \) is \( \varphi \)-d*, \( f' > 0 \) on \( Y_e \) and hence \( Z(f') \cap Y_e = \emptyset \). Since \( f'(q) = 0, q \in \nu Y \) and \( Z(f') \) is closed. \( D = Z(f') \cap Y_d = Z(f') \cap Y \) is a non-empty clopen discrete subset of \( Y \) contained in \( Y_d \). \( \text{cl}_{\nu Y} D = Z(f') \cap \nu Y \) implies \( q \in \text{cl}_{\nu Y} D \) and \( \text{cl}_{\nu Y} D \cap \text{cl}_{\nu Y} Y_e = \emptyset \).

(2) Let us put \( \mathcal{D} = \{ D \subset Y_d; D \text{ is a clopen subset of } Y \} \) and \( \text{cl}_{\nu Y} \mathcal{D} = \bigcup \{ \text{cl}_{\nu Y} D; D \in \mathcal{D} \} \). Then it is easy to see the following
\[ \nu Y = \text{cl}_{\nu Y} \mathcal{D} \cup \text{cl}_{\nu Y} Y_e, \quad \text{cl}_{\nu Y} \mathcal{D} \cap \text{cl}_{\nu Y} Y_e = \emptyset \]
and
\[ (\beta \varphi)^{-1} \text{cl}_{\nu Y} Y_e \subset \nu X. \]

(3) \( \nu \varphi \) is onto \( \nu Y \). Let \( q \in \text{cl}_{\nu Y} D, D \in \mathcal{D} \). For each \( y \in D \), let us pick a point \( p(y) \) from \( \varphi^{-1}y \) and put \( A = \{ p(y); y \in D \} \). Then \( A \) is a discrete closed C-embedded subset of \( X \). Thus \( \nu A = \text{cl}_{\nu X} A \) is homeomorphic to \( \text{cl}_{\nu Y} D \) under the map \( \nu \varphi \). Thus we have \( \nu \varphi(\nu X) = \nu Y \).

(4) \( \nu \varphi \) is an RC-map. Let \( F \) be regular closed in \( \nu X \) and \( E = (\nu \varphi) F \). Suppose that there is \( q \in \text{cl}_{\nu Y} E - E \). By (2) and the clopeness of \( \varphi^{-1}y, y \in Y_d, \) we have \( q \notin Y_d \cup \text{cl}_{\nu Y} E_e \). Thus there is \( D \in \mathcal{D} \) with \( q \in \text{cl}_{\nu Y} D \) and \( \text{cl}_{\nu Y} D \cap \text{cl}_{\nu Y} Y_e = \emptyset \) by (2). Since \( \beta \varphi \) is open by 1.2(5), \( \nu \varphi \) is also \( * \)-open and we have that \( E \supset (\nu \varphi) \text{int}_{\nu X} F \) is dense in \( \text{cl}_{\nu Y} E \) because \( F \) is regular closed. Let \( M = E \cap D \cap Y_d \). Then \( q \in \text{cl}_{\nu Y} M \). Let us pick a point \( p(y) \) from \( \varphi^{-1}(y) \cap F, y \in M \). \( A = \{ p(y); y \in M \} \) is a discrete closed C-embedded subset of \( X \) and hence \( \nu A = \text{cl}_{\nu X} A \subset F \) and \( \nu A \) is homeomorphic to \( \nu M = \text{cl}_{\nu Y} M \), so \( q \in E \) a contradiction.

(5) \( \nu \varphi \) is open RC-preserving. Since \( \nu \varphi \) is an RC-map, \( \nu \varphi \) is WZ by 1.2(6). Thus the openness of \( \beta \varphi \) implies that \( \nu \varphi \) is open by 1.2(1) and RC-preserving by 1.2(7).
As a direct consequence of the above theorem, we have the following corollary which is a generalization of the result obtained in [5] if $X$ is realcompact and $\varphi: X \to Y$ is an open $WZ$ map with $\text{Bd } \varphi^{-1}y = \text{compact}$ for each $y \in Y$, then $Y$ is also realcompact.

**COROLLARY 2.7.** If $X$ is realcompact and $\varphi: X \to Y$ is a $\beta$-open map such that $Y_e$ is $\varphi$-$d^*$, then $Y$ is also realcompact.

**THEOREM 2.8.** Let $\varphi: X \to Y$ and $Z = (\beta \varphi)^{-1}Y_d \cup vX$. Then the following are equivalent:

1. $Z$ is a realcompact and $\varphi$ is a $\beta$-open map such that $Y_e$ is $\varphi$-$d^*$.
2. $\varphi' = (\beta \varphi)|Z$ is an open perfect map of $Z$ onto $vY$.
3. $v \varphi$ is a clopen map of $vX$ onto $vY$ such that $\text{Bd}(v \varphi)^{-1}q$ is compact for every $q \in vY$.
4. $v \varphi$ is a clopen map of $vX$ onto $vY$ such that $(vY)_e$ is $(v \varphi)$-$d^*$.

**Proof.** (1) $\Rightarrow$ (2) If $Z = \beta X$, then $\varphi' = \beta \varphi$ and $\varphi'$ is an open perfect map onto $vY$. Let $p \in \beta X - Z$ and $q = (\beta \varphi)p$. Then $Z = vZ, \beta Z = \beta X$ and there is $f \in C(\beta X)$ such that $p \in Z(f) \subset \beta X - Z$ and $0 \leq f \leq 1$. Since $\beta \varphi$ is open $WZ$ and $Y_e$ is $\varphi$-$d^*$, it is easy to see that $f^i(q) = 0$ and $f^i > 0$ on $Y$. Thus $q \in \beta Y - vY$, so $\varphi'$ is a perfect map onto $vY$. The openness of $\varphi'$ follows from 1.2(1, 5).

(2) $\Rightarrow$ (3) We shall show that $v \varphi$ is closed. Let $F$ be closed in $vX$ and $q \in \text{cl}_{vY}(v \varphi)F - (v \varphi)F$. Since $\varphi'$ is perfect and every point of $Y_d$ is isolated, we have $q \notin Y_d$, so $(\beta \varphi)^{-1}q = (v \varphi)^{-1}q$ is disjoint from $\text{cl}_ZF$, and hence $q \notin v \varphi'(\text{cl}_ZF)$, a contradiction. Thus $v \varphi$ is closed. The verifications of other parts are easy. (3) $\Rightarrow$ (4) Evident.

(4) $\Rightarrow$ (1) Since $v \varphi$ is clopen, $\beta(v \varphi) = \beta \varphi$ is open by 1.2(1) and hence $\varphi$ is $\beta$-open by 1.2(5). Since $vY = (vY)_e \cup Y_d$, the $(v \varphi)$-$d^*$-ness of $(vY)_e = vY - Y_d$ implies the $\varphi$-$d^*$-ness of $Y_e$.

Since $Y_d = (vY)_d$ and $(vY)_e$ is $(v \varphi)$-$d^*$, we have $Z = (\beta \varphi)^{-1}vY$, and hence $\varphi': Z \to vY$ is an open perfect map which shows that $Z$ is realcompact.

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Received August 18, 1983 and in revised form October 28, 1983.

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