THE DIOPHANTINE EQUATION $ax + by = c$ IN $\mathbb{Q}(\sqrt{5})$ AND OTHER NUMBER FIELDS

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Solving in rational integers the linear diophantine equation

\[ ax + by = c, \quad (a, b)|c, \quad a, b, c, \in \mathbb{Z} \]

is very well known. Let \( d = (a, b) \), and put \( A = a/d, B = b/d, C = c/d \), then equation (1) becomes

\[ Ax + By + C, \quad (A, B) = 1, \quad A, B, C, \in \mathbb{Z}. \]

The purpose of this note is to discuss the solutions of this equation when \( A, B, C \) are integers in \( Q(\sqrt{5}) \) and the solutions are integers in \( Q(\sqrt{5}) \).

What makes the discussion interesting is that an algorithm which mimics the continued fraction algorithm that solves the rational integer case can be implemented.

A brief summary of the continued fraction algorithm for the rational case is as follows: To solve (1'): find the regular simple continued fraction for \( A/B \); i.e.

\[
\frac{A}{B} = r_0 + \frac{1}{r_1 + \frac{1}{\ddots + \frac{1}{r_n}}}.
\]

which we write as \( A/B = (r_0; r_1, \ldots, r_n) \). Since \( A/B \) is rational, the continued fraction is finite. The \((m + 1)\)th convergent of a continued fraction is denoted by \( P_m/Q_m = (r_0; r_1 \cdots r_m) \). If \( A/B = P_n/Q_n \) then the penultimate convergent \( P_{n-1}/Q_{n-1} \) provides a solution to \( Ax + By = 1 \) because of the well-known relation.

\[ P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n+1}. \]

It suffices therefore to take \( x = (-1)^{n+1} Q_{n-1}, y = (-1)^n P_{n-1} \). To solve (1) we take \( x = (-1)^{n+1} dCQ_{n-1} \) and \( y = (-1)^{n+1} dCP_{n-1} \).

It is well known that the integers in \( Q(\sqrt{5}) \) have the form \( s + t\lambda \), where \( s, t \in \mathbb{Z} \) and \( \lambda = (1 + \sqrt{5})/2 \). (See Hardy and Wright [1] or Niven and Zuckerman [3] for a complete discussion of this algebraic number field.) The elements in \( Q(\sqrt{5}) \) are of course the quotients of integers in the
field. In order to mimic the solution procedure above we would require a continued fraction development that essentially parallels the ordinary continued fraction representation of real numbers, that is the elements of $Q(\sqrt{5})$ should have a unique finite continued fraction representation and every other real number has a unique infinite continued fraction representation. Such a representation exists and the continued fractions will be referred to as $\lambda_5$-fractions [4].

These continued fractions were presented by the author in connection with studies on the Hecke groups [4], and are one example of the more general $\lambda_q$-fractions where $\lambda = 2 \cos(\pi/q)$. It was shown in [4], that every finite $\lambda_q$-fraction is an element in the algebraic number field $Q(\lambda_q)$, and Leutbecher [2] showed that only in the case $q = 5$, every element in $Q(\sqrt{5}) = Q(\lambda_5)$ has a finite $\lambda_5$-fraction. Hence a real number is an element of $Q(\sqrt{5})$ if and only if it has a finite $\lambda_5$-continued fraction representation and every real number has a unique $\lambda_5$-fraction representation. Thus we will show that the algorithm that solves the rational integer case (which is the case $q = 3$) will work in the $Q(\sqrt{5})$ case.

What are the $\lambda_q$-fractions? These are continued fractions of the form

$$r_0 \lambda + \frac{\epsilon_1}{r_1 \lambda + \epsilon_2} + \frac{\epsilon_2}{r_2 \lambda + \epsilon_3} + \cdots$$

where, in general, for fixed $q$, $\lambda = 2 \cos(\pi/q)$, $q \in Z^+$ and $q \geq 3$, $\epsilon_i = \pm 1$, and $r_i \in Z^+$, $i \geq 1$, $r_0 \in Z$. The continued fraction is developed by a nearest integer algorithm. If $\xi$ is a real number we seek the nearest integral multiple of $\lambda$. This means, if $\{\}$ denotes the nearest integer, then we write $\{\xi/\lambda\} = r_0$, where we specify $-1/2 < r_0 - \xi/\lambda < 1/2$; i.e. $r_0$ is uniquely determined by the inequality.

$$r_0 \lambda - \frac{\lambda}{2} < \xi \leq r_0 \lambda + \frac{\lambda}{2}. \tag{3}$$

Hence $\xi = r_0 \lambda + \epsilon_1/\xi_1$, where it is seen that $\xi_1 = \epsilon_1/(\xi - r_0 \lambda) > 0$, since $\epsilon_1 > 0$ if $r_0 \lambda < \xi$ and $\epsilon_1 < 0$ if $r_0 \lambda > \xi$. If $\xi = n \lambda + \lambda/2 = (n + 1) \lambda - \lambda/2$, then because of inequality (3) $r_0 = n$ and $\epsilon_1 = 1$. Then $r_0 \lambda - \lambda/2 < \xi \leq r_0 \lambda + \lambda/2$ implies $\xi_1 \geq 2/\lambda > 1 > \lambda/2$ and hence $r_1 = \{\xi_1/\lambda\} \geq 1$. Continuing in this way we find that $\xi_m > \lambda/2$ which implies that $r_m \geq 1$ ($m \geq 1$). Henceforth, $\lambda$-fraction will refer to $\lambda_5$-fraction. The $\lambda$-fraction is unique provided that the following few simple rules indicated in [4] are obeyed.
(i) If \( \lambda - 1/r\lambda \) occurs, then \( r \geq 2 \).

(ii) If

\[
\frac{\varepsilon_1}{\lambda - \frac{1}{2\lambda} - \frac{1}{\lambda + \varepsilon_2}} \quad \ldots
\]

occurs, then \( \varepsilon_1 = \varepsilon_2 = 1 \).

We point out that in \( Q(\sqrt{5}) \),

\[
\lambda - \frac{1}{2\lambda} - \frac{1}{\lambda} = \frac{2}{\lambda}.
\]

(iii) If the \( \lambda \)-fraction terminates as

\[
\frac{\varepsilon}{\lambda - \frac{1}{\lambda}} \quad \ldots
\]

then \( \varepsilon = 1 \). In \( Q(\sqrt{5}) \), \( \lambda - 1/\lambda = 1 \), which yields the equation

\[
\lambda^2 - \lambda - 1 = 0.
\]

A \( \lambda \)-fraction satisfying these criteria is called a reduced \( \lambda \)-fraction. Similar criteria will yield unique \( \lambda \)-fractions. Because of (4) the rolled up finite continued fraction produces the quotient of two polynomials in \( \lambda \) which can be reduced to the form

\[
(a + b\lambda)/(c + d\lambda), \quad a, b, c, d \in \mathbb{Z}.
\]

This in turn can be put in the form

\[
(a' + b'\lambda)/c'
\]

by multiplying numerator and denominator by the conjugate of \( c + d\lambda \), which is \((c + d) - d\lambda\). One finds that \( a' = ac + ad - bc, b' = bc - ad, c' = c^2 + cd - d^2 \)—the norm of \( c + d\lambda \).

As observed on p. 550 of [4] consecutive convergents \( P_{n-1}/Q_{n-1} \) and \( P_n/Q_n \) of a \( \lambda \)-fraction satisfy a determinant relation similar to (2):

\[
P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1.
\]

Finally we remark that the units in \( Q(\sqrt{5}) \) are \( \lambda^n \) which can be written in terms of consecutive Fibonnaci numbers. If \( F_n \) is the \( n \)th Fibonnaci...
number, then \( \lambda^n = F_{n-1} + F_n \lambda \). This can be proved as follows:

Let \( F_0 = 0, F_1 = 1, F_2 = 1 \) then \( \lambda^1 = 0 + \lambda, \lambda^2 = F_1 + F_2 \lambda = \lambda + 1, \)
which is a consequence of (4). By induction then if \( \lambda^k = F_{k-1} + F_k \lambda \), then

\[
\lambda^{k+1} = F_{k-1} \lambda + F_k \lambda^2 = F_k + (F_{k-1} + F_k) \lambda = F_k + F_{k+1} \lambda,
\]
as desired. If \( n < 0 \) one determines first from (4) that \( 1/\lambda = \lambda - 1 \); hence \( \lambda^{-2} = (\lambda - 1)^2 = 2 - \lambda \). By induction, one determines that \( \lambda^{-n} = -F_{n+1} + F_n \lambda \) if \( n \) is odd and \( \lambda^{-n} = F_{n+1} - F_n \lambda \) if \( n \) is even. To show that \( \lambda^n \) is a unit, we observe that the norm of \( F_k + F_{k+1} \lambda \) is \( F_k^2 + F_k F_{k+1} - F_{k+1}^2 \). But the last expression is precisely the determinant relation (2) for the consecutive convergents. \( F_k/F_{k+1}, F_{k+1}/F_{k+2} \) of the regular continued fraction \( (1; 1, 1 \cdots) = \lambda \). Thus each \( \lambda^n, n > 0 \), is indeed a unit. For \( n \) negative \( = -m \), the norm \( N(1/\lambda^m) = 1/N(\lambda^m) = \pm 1 \) too, so \( \lambda^n \) is a unit for all integers \( n \). We now state and prove the main theorem.

**Theorem 1.** Let \( p, q, r \in \mathbb{Z}(\sqrt{5}) \), and suppose that, except for units, \( p, q, r \) are relatively prime. Then the diophantine equation \( px + qy = r \) has integer solutions in \( Q(\sqrt{5}) \). If \( x_0, y_0 \) is a particular solution, then any other solution has the form \( x = x_0 + qt, y = y_0 - pt \). If \( (p, q) = d \) and \( d|r \), then

\[
\frac{p}{d} x + \frac{q}{d} y = \frac{r}{d}
\]
is solvable in \( Q(\sqrt{5}) \).

**Proof.** As in the rational integer case, we first solve \( px + qy = 1 \). This is done by expanding \( p/q \) in its unique \( \lambda \)-fraction. The penultimate convergent will supply the values for \( x \) and \( y \). To solve \( px + qy = r \) multiply the \( x \) and \( y \) values by \( r \).

As in the rational case we note that if a particular solution is \( x_0, y_0 \) then an infinity of solutions is obtained using the usual trick namely putting \( x = x_0 + qt, y = y_0 - pt \), which satisfies the equation for all \( t \in \mathbb{Z}(\lambda) \). Moreover if \( a \) and \( b \) is any solution \( \in \mathbb{Z}(\sqrt{5}) \), i.e., \( pa + qb = r \) then \( a = x_0 + qt, b = y_0 - pt \), for some \( t \). This is clear because from \( pa + qb = r \) and \( px_0 + qy_0 = r \) we obtain \( p(x_0 - a) + q(y_0 - b) = 0 \). Hence \( p(x - a) = -q(y_0 - b) \). Since \( (p, q) = 1 \), it follows that \( p|(y_0 - b) \). Thus \( pl = y_0 - b \). But now \( p(x - a) = -qpl \), hence \( x - a = -ql \). This result has a bearing on the Hecke group \( \Gamma(\lambda) \) in determining which solutions to \( px + qy = 1 \) provide a substitution that belongs to \( \Gamma(\lambda) \).

Finally, the last statement of the theorem follows easily from the first statement since \( p/d, q/d, r/d \) are relatively prime.
There is one wrinkle in this method which does not arise in the rational case. The \( \lambda \)-fraction when rolled up and reduced to the form (5) may not be identical with the original fraction unless a suitable unit is factored out from numerator and denominator.

Consider the following example: Solve

\[(8) \quad (3 + 7\lambda)x + (5 - 2\lambda)y = 6 + 5\lambda.\]

One can verify that

\[
\frac{3 + 7\lambda}{5 - 2\lambda} = 5\lambda + \frac{1}{20\lambda - 1} = \frac{\lambda - 1}{3\lambda}.
\]

The right side, when rolled up and reduced using (4), becomes

\[
\frac{487 + 788\lambda}{97\lambda + 60}.
\]

The numerator is \((34 + 55\lambda)(3 + 7\lambda)\) and the denominator is

\[(34 + 55\lambda)(5 - 2\lambda), \quad (55\lambda + 34 = \lambda^{10}).\]

The penultimate convergent is

\[
5\lambda + \frac{1}{20\lambda - 1} = \frac{196\lambda + 100}{20\lambda + 19}.
\]

Hence \(x = (20\lambda + 19)\) and \(y = -(196\lambda + 100)\) solves \((487 + 788\lambda)x + (97\lambda + 60)y = 1\). It follows that \(x' = (20\lambda + 19)(5\lambda + 6) = 214 + 315\lambda\) and \(y' = -(196\lambda + 100)(5\lambda + 6) = -(2656\lambda + 1580)\) solves \((487 + 788\lambda)x' + (97\lambda + 60)y' = 6 + 5\lambda\). Thus to solve (8) we incorporate the common unit factor \((34 + 55\lambda)\) with \(x'\) and \(y'\). Then \((3 + 7\lambda)x'' + (5 - 2\lambda)y'' = 6 + 5\lambda\) has as solution

\[
x'' = (214 + 315\lambda)(34 + 55\lambda) = 24601 + 39805\lambda
\]
\[
y'' = -(1580 + 2656\lambda)(34 + 55\lambda) = -(199800 + 3223284\lambda).
\]

Knowing one solution thus gives all solutions; \(x = x'' + qt, \quad y = y'' - pt\) where \(t \in \mathbb{Z}(\sqrt{5})\) and we assume that \((p, q) = 1)\.

It is interesting to observe here that solving one diophantine equation automatically solves a class of equations. Recalling that the units \(\lambda^n\) can be written as integers in \(\mathbb{Z}(\sqrt{5})\) and noting that

\[
\lambda^n = (F_{n-1} + F_n\lambda) \text{ times } \lambda^{-n} \quad (= F_{n+1} - F_n\lambda \text{ or } -F_{n+1} + F_n\lambda) = 1
\]
then a solution to $px + qy = n$ provides a solution to $(F_{n-1} + F_n\lambda)px' + (F_{n-1} + F_n\lambda)qy' = n$. Clearly, the solution is $x' = (F_{n+1} - F_n\lambda)x$, $y' = (F_{n+1} - F_n\lambda)y$ or $x' = (-F_{n+1} + F_n\lambda)x$, $y' = (-F_{n+1} + F_n\lambda)y$, depending on the parity of $n$. As an example, the equation

$$(7 + 10\lambda)x' + (-2 + 3\lambda)y' = 6 + 5\lambda,$$

which is

$$\lambda(3 + 7\lambda)x + \lambda(5 - 2\lambda)y = 6 + 5\lambda,$$

is solved by $x' = 15204 + 24601\lambda$, $y' = -(123484 + 199800\lambda)$. This solution is obtained from (9) by dividing $x''$ and $y''$ by $\lambda$, i.e., multiplying by $\lambda - 1$.

The above procedures could be extended to other number fields if a suitable continued fraction representation were available. A continued fraction representation for the number fields $Q(2\cos(\pi/q))$ similar to the foregoing was developed in [4], but as Wolfart showed [5] the only possible $q$'s for which all the rational elements in $Q(\lambda_q)$ have a finite $\lambda_q$-fraction are $q = 3, 5, 9$. It appears therefore that it is true only for the fields $q = 3$ and $q = 5$; while for $q = 9$ the questions is still open. For other values of $q$, equation (1) can be solved in $Z(\lambda_q)$ provided $a/b$ has a finite $\lambda_q$-fraction. The formal statement is:

**Theorem 2.** If $\lambda_q = 2\cos(\pi/q)$, $q$ an integer $\geq 4$, then if $a, b \in Z(\lambda_q)$, then the diophantine equation $ax + by = 1$ has solutions in $Z(\lambda_q)$ if and only if $(a, b) = d$ and $d|c$, $d$ is not a unit; and if $a/b$ has a finite $\lambda_q$-fraction representation.

For $q = 4$, $\lambda_4 = \sqrt{2}$, and for $q = 6$, $\lambda_6 = \sqrt{3}$. The finite $\lambda_4$- and $\lambda_6$-fractions when rolled up have the form $a\sqrt{r}/b$ or $a/b\sqrt{r}$, $r = 2, 3$. Thus not all elements of $Q(\sqrt{r})$ are realizable as finite $\lambda_4$ or $\lambda_6$ continued fractions. However, consider

$$7x + 3\sqrt{2}y = 4 + 9\sqrt{2}.$$}

We find the $\lambda_4$ continued fraction for $7/3\sqrt{2}$ which turns out to be $7/3\sqrt{2} = \sqrt{2} + 1/3\sqrt{2}$. Clearly

$$\frac{p_2}{q_2} = \frac{7}{3\sqrt{2}}, \quad \frac{p_1}{q_1} = \frac{\sqrt{2}}{1},$$

and $7 \cdot 1 - \sqrt{2} \cdot 3\sqrt{2} = 1$ so $x = 1$ and $y = -\sqrt{2}$ solves $7x + 3\sqrt{2}y = 1$.

Hence $x' = 4 + 9\sqrt{2}$, $y' = -\sqrt{2}(4 + 9\sqrt{2}) = -(18 + 4\sqrt{2})$ solves the original equation and of course there are an infinite of solutions of the
form \( x'' = 4 + 9\sqrt{2} + (18 + 4\sqrt{2})t, \quad y'' = -(18 + 4\sqrt{2}) + (4 + 9\sqrt{2})t, \) 
\( t \in \mathbb{Z}(\lambda_4). \)

This same procedure will work for any of the algebraic fields \((2\cos(\pi/q)).\) Examples can be easily found by first taking a finite \(\lambda_q\)-fraction and using the numerator and denominator for the coefficients. For example in \(\lambda_7,\) compute

\[
2\lambda + \frac{1}{\lambda - 1} = 2\lambda + \frac{3\lambda}{3\lambda^2 - 1} = \frac{6\lambda^3 + \lambda}{3\lambda^2 - 1}.
\]

In \(\lambda_7,\)

\[
\lambda - \frac{1}{\lambda - 1} = 1
\]

so the rational elements will be of the form

\[
\frac{a\lambda^2 + b\lambda + c}{d\lambda^2 + e\lambda + f}
\]

The equation \((6\lambda^3 + \lambda)x + (3\lambda^2 - 1)y = 1\) is solved by \(x = 2\lambda, \ y = -(2\lambda^2 + 1),\) since

\[
(6\lambda^3 + \lambda)2\lambda + (3\lambda^2 - 1) - (2\lambda^2 + 1) = 6\lambda^4 + \lambda^2 - (6\lambda^4 + \lambda^2 - 1) = 1.
\]

We remark that there are other ways of solving the linear diophantine equation in \(Q(\sqrt{5})\), but the algorithm presented above bears such a striking similarity to the usual algorithm for the rational case that it gives \(Q(\sqrt{5})\) a special status. The author knows of no other algebraic field in which a continued fraction can be similarly developed.

It seems that Pell's equation \((x^2 - dy^2 = 1)\) should also be solvable in \(Q(\sqrt{5})\) but there are still some difficulties in showing that \(\sqrt{d}\) is a periodic \(\lambda_5\)-function. However, if \(\sqrt{d}\) is periodic then Pell's equations can be solved as in the rational case

References


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