NONCOMPACT SETS WITH CONVEX SECTIONS

MAU-HSIANG SHIH AND KOK KEONG TAN
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Two further generalizations of Ky Fan's generalizations of his well-known intersection theorem concerning sets with convex sections are obtained.

1. Introduction. Let \( I \) be an index set; in the case when \( I \) is finite, it is always assumed that \( I \) contains at least two indices. Let \( \{ X_i \}_{i \in I} \) be a family of topological spaces and \( X := \prod_{i \in I} X_i \). For each \( i \in I \), set

\[
X^i := \prod_{j \neq i} X_j \quad \text{(so that } X = X_i \times X^i),
\]

and let \( p_i : X \to X_i \) and \( p^i : X \to X^i \) be the projections. For each \( x \in X \), we write \( p_i(x) = x_i \) and \( p^i(x) = x^i \). For any non-empty subset \( K \) of \( X \), we let \( p_i(K) = K_i \) and \( p^i(K) = K^i \).

Our aim in this paper is to give two generalizations of the following intersection theorem of Ky Fan [2] concerning sets with convex sections.

**Theorem 1.** (Ky Fan.) Let \( X_1, X_2, \ldots, X_n \) be \( n \) \((\geq 2)\) non-empty compact convex sets each in a Hausdorff topological vector space. Let \( X := \prod_{i=1}^n X_i \) and \( A_1, A_2, \ldots, A_n \) be \( n \) subsets of \( X \) such that

(a) For each \( i = 1, 2, \ldots, n \) and any \( x_i \in X_i \), the section

\[
A_i(x_i) := \{ x^i \in X^i : (x_i, x^i) \in A_i \}
\]

is open in \( X^i \).

(b) For each \( i = 1, 2, \ldots, n \) and any \( x^i \in X^i \), the section

\[
A_i(x^i) := \{ x_i \in X_i : (x_i, x^i) \in A_i \}
\]

is convex and non-empty.

Then the intersection \( \bigcap_{i=1}^n A_i \) is non-empty.

Theorem 1 is a unified account of game-theoretic results for arbitrary \( n \)-person games and has several applications [2], [3]. In particular, Tychonoff's fixed point theorem [11], Sion's generalization [10] of von Neumann's minimax principle [8] and Nash's equilibrium point theorem [7] are immediate consequences of Theorem 1.
2. **Infinite system.** Ma [6] extended Theorem 1 to an arbitrary system \( \{ X_i \}_{i \in I} \) of compact convex sets. In a recent paper, Ky Fan [5] extends Ma's result by introducing an auxiliary family \( \{ B_i \}_{i \in I} \). Ky Fan's theorem can be further generalized to non-compact convex sets as follows:

**THEOREM 2.** Let \( \{ E_i \}_{i \in I} \) be a family of Hausdorff topological vector spaces. For each \( i \in I \), let \( X_i \) be a non-empty convex set in \( E_i \). Let \( X := \prod_{i \in I} X_i \). Suppose \( \{ A_i \}_{i \in I} \) and \( \{ B_i \}_{i \in I} \) are two families of subsets of \( X \) satisfying the following conditions:

(a) For each \( i \in I \) and any \( x_i \in X_i \), the section 
\[
A_i(x_i) := \{ x' \in X^i : (x_i, x') \in A_i \}
\]
is open in \( X^i \).

(b) For each \( i \in I \) and any \( x^i \in X^i \), the section 
\[
B_i(x^i) := \{ x_i \in X_i : (x_i, x^i) \in B_i \}
\]
contains the convex hull of the section 
\[
A_i(x^i) := \{ x_i \in X_i : (x_i, x^i) \in A_i \}.
\]

(c) There exists a non-empty compact convex subset \( K \) of \( X \) such that 
(c') for each \( i \in I \) and any \( x^i \in K^i \), the section 
\[
A_i(x^i) := \{ x_i \in X_i : (x_i, x^i) \in A_i \} \neq \emptyset
\]
and 
(c'') \( K \cap \prod_{i \in I} A_i(y^i) \neq \emptyset \) for each \( y \in X \setminus K \).

Then the intersection \( \bigcap_{i \in I} B_i \) is non-empty.

**Proof.** Let \( i \in I \). For any \( x^i \in K^i \), we can find \( x_i \in X_i \) such that \( x_i \in A_i(x^i) \) by (c'), so that \( x^i \in A_i(x_i) \); thus \( K^i \subset \bigcup_{x_i \in X_i} A_i(x_i) \). Since each \( A_i(x_i) \) is open in \( X^i \) by (a), by the compactness of \( K^i \) (since each projection \( p_i \) is continuous), there is a finite subset \( \{ x_{i1}, x_{i2}, \ldots, x_{in_i} \} \) of \( X_i \) such that

\[
K^i \subset \bigcup_{k=1}^{n_i} A_i(x_{ik}).
\]

Let \( \Omega_i \) be the convex hull of \( K_i \cup \{ x_{i1}, x_{i2}, \ldots, x_{in_i} \} \). Define \( \Omega := \prod_{i \in I} \Omega_i \) and \( \tilde{A}_i := A_i \cap \Omega \) and \( \tilde{B}_i := B_i \cap \Omega \) for each \( i \in I \). Since the projection \( p_i \) is continuous and affine, \( K_i \) is compact convex for each \( i \in I \); it follows that \( \Omega_i \) is a nonempty compact convex set in \( E_i \) for each \( i \in I \). Furthermore, we have:

(i) For each \( i \in I \) and any \( x_i \in \Omega_i \), the section 
\[
\tilde{A}_i(x_i) := \{ x^i \in \Omega^i : (x_i, x^i) \in \tilde{A}_i \}
\]
is open in \( \Omega^i \) by (a).
(ii) For each \( i \in I \) and any \( x^i \in \Omega^i \), the section
\[
\tilde{B}_i(x^i) := \{ x_i \in \Omega_i : (x_i, x^i) \in \tilde{B}_i \}
\]
contains the convex hull of the section
\[
\tilde{A}_i(x^i) := \{ x_i \in \Omega_i : (x_i, x^i) \in \tilde{A}_i \}
\]
by (b).

(iii) For each \( i \in I \) and any \( x^i \in \Omega^i \), the section
\[
\tilde{A}_i(x^i) := \{ x_i \in \Omega_i : (x_i, x^i) \in \tilde{A}_i \} \neq \emptyset.
\]

Let \( f_{i_1}, f_{i_2}, \ldots, f_{i_{m_i}} \) be a continuous partition of unity subordinated to the covering \( \{ \tilde{A}_i(y_{i_1}), \tilde{A}_i(y_{i_2}), \ldots, \tilde{A}_i(y_{i_{m_i}}) \} \) of \( \Omega^i \). Then
\[
\begin{align*}
& \left\{ \begin{array}{l}
 f_{i k}(x^i) = 0 \quad \text{for } x^i \in \Omega^i \setminus \tilde{A}_i(y_{i k}), k = 1, 2, \ldots, m_i, \\
 \sum_{k=1}^{m_i} f_{i k}(x^i) = 1 \quad \text{for each } x^i \in \Omega^i.
\end{array} \right.
\end{align*}
\]
Define a continuous map \( \phi_i : \Omega^i \to \Omega_i \) by setting
\[
\phi_i(x^i) = \sum_{k=1}^{m_i} f_{i k}(x^i) y_{i k} \quad \text{for } x^i \in \Omega^i.
\]
Since \( f_{i k}(x^i) \neq 0 \) implies \( x^i \in \tilde{A}_i(y_{i k}) \), i.e. \( y_{i k} \in \tilde{A}_i(x^i) \), and since \( \tilde{B}_i(x^i) \)
contains the convex hull of \( \tilde{A}_i(x^i) \) by (ii), we have
\[
(2) \quad \phi_i(x^i) \in \tilde{B}_i(x^i) \quad \text{for each } x^i \in \Omega^i.
\]
Let \( C_i \) be the convex hull of \( \{ y_{i_1}, y_{i_2}, \ldots, y_{i_{m_i}} \} \); then \( C_i \subseteq \Omega_i \). Denote by \( F_i \)
the vector subspace of \( E_i \) generated by \( C_i \); then \( F_i \) is locally convex since it is finite dimensional.

Now let \( C = \prod_{i \in I} C_i \), then \( C \) is a non-empty compact convex subset in the Hausdorff locally convex space \( \prod_{i \in I} F_i \). Note that for each \( i \in I \), we have \( C^i \subseteq \Omega^i \). Define \( \psi : C \to C \) as follows: For each \( x \in C \) and each \( i \in I \), write \( x = (x_i, x^i) \in C_i \times C^i \), then \( \psi(x) := \{ y_i \}_{i \in I} \) is determined by \( y_i := \phi_i(x^i) \) for each \( i \in I \). Clearly \( \psi \) is continuous. By Tychonoff's
fixed point theorem [11], \( \psi \) has a fixed point \( z := \{ z_i \}_{i \in I} \) in \( C \), so that for each \( i \in I \), we have \( z_i = \phi_i(z^i) \in \tilde{B}_i \{ z_i \} \), by (2); it follows that \( z = (z_i, z^i) \in \tilde{B} \subset B_i \) for each \( i \in I \). Hence \( z \in \cap_{i \in I} B_i \). This concludes the proof of our theorem.

Similar to [2], Theorem 2 has the following analytic formulation:

**Theorem 3.** Let \( \{ E_i \}_{i \in I} \) be a family of Hausdorff topological vector spaces. For each \( i \in I \), let \( X_i \) be a non-empty convex set in \( E_i \). Let \( X := \prod_{i \in I} X_i \) and \( \{ t_i \}_{i \in I} \) be a family of real numbers. Suppose that \( \{ f_i \}_{i \in I} \) and \( \{ g_i \}_{i \in I} \) are two families of real-valued functions defined on \( X \), satisfying the following conditions:

(a) For each \( i \in I \) and any \( x_i \in X_i \), \( f_i(x_i, x^i) \) is a lower semi-continuous function of \( x^i \in X^i \).

(b) For each \( i \in I \) and any \( x^i \in X^i \), the set

\[ \{ x_i \in X_i : g_i(x_i, x^i) > t_i \} \]

contains the convex hull of the set

\[ \{ x_i \in X_i : f_i(x_i, x^i) > t_i \} \].

(c) There exists a non-empty compact convex subset \( K \) of \( X \) such that

\( (c') \) for each \( i \in I \) and any \( x^i \in K^i \), there exists \( x_i \in X_i \) with \( f_i(x_i, x^i) > t_i \) and

\( (c'') \) for any \( y \in X \setminus K \), there exists \( x \in K \) with \( f_i(x_i, y^i) > t_i \) for all \( i \in I \).

Then there exists a point \( \hat{y} \in X \) such that \( g_i(\hat{y}) > t_i \) for all \( i \in I \).

**3. Finite system.** By relaxing the compactness condition for \( X_i \)'s and the convexity condition for the sections of the \( A_i \)'s in Theorem 1, Ky Fan [5] generalizes Theorem 1 as follows:

**Theorem 4.** (Ky Fan) Let \( X_1, X_2, \ldots, X_n \) be \( n \) \((\geq 2)\) convex sets each in a Hausdorff topological vector space. Let \( X := \prod_{i=1}^{n} X_i \) and \( A_1, A_2, \ldots, A_n \) be \( n \) subsets of \( X \) such that

(a) For each \( i = 1, 2, \ldots, n \) and any \( x_i \in X_i \), the section

\[ A_i(x_i) := \{ x^i \in X^i : (x_i, x^i) \in A_i \} \]

is open in \( X^i \),

(b) For each \( i = 1, 2, \ldots, n \) and any \( x^i \in X^i \), the section

\[ A_i(x^i) := \{ x_i \in X_i : (x_i, x^i) \in A_i \} \]

is non-empty.
(c) For any $x \in X$, at least $q$ of the sections $A_1(x^1), A_2(x^2), \ldots, A_n(x^n)$ are convex; where $q$ is a given integer with $2 \leq q \leq n$.

(d) There exists a non-empty compact convex subset $K$ of $X$ such that
\[ K \cap \prod_{i=1}^{n} A_i(y^i) \neq \emptyset \quad \text{for each } y \in X \setminus K. \]

Then at least $q$ of the sets $A_1, A_2, \ldots, A_n$ have a non-empty intersection.

Theorem 4 can be improved as follows:

**Theorem 5.** Let $X_1, X_2, \ldots, X_n$ be $n$ ($\geq 2$) convex sets each in a Hausdorff topological vector space. Let $X := \prod_{i=1}^{n} X_i$ and $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ be $2n$ subsets of $X$ such that

(a) $A_i \subset B_i$ for $i = 1, 2, \ldots, n$.

(b) For each $i = 1, 2, \ldots, n$ and any $x_i \in X_i$, the section
\[ A_i(x_i) := \{ x^i \in X^i : (x_i, x^i) \in A_i \} \]
is open in $X^i$.

(c) For any $x \in X$, at least $q$ of the sections $B_1(x^1), B_2(x^2), \ldots, B_n(x^n)$ are convex; where $q$ is a given integer with $2 \leq q \leq n$.

(d) There exists a non-empty compact convex subset $K$ of $X$ such that

(d') For each $i = 1, 2, \ldots, n$ and for each $x \in K$, the section
\[ A_i(x^i) := \{ x_i \in X^i : (x_i, x^i) \in A_i \} \]
is non-empty and
\[ (d'') K \cap \prod_{i=1}^{n} A_i(y^i) \neq \emptyset \quad \text{for each } y \in X \setminus K. \]

Then at least $q$ of the sets $B_1, B_2, \ldots, B_n$ have a non-empty intersection.

For $n = 2$, Theorem 5 was given in [9] together with an application to von Neumann type minimax inequalities. The proof of Theorem 5 is a slight modification of that in Ky Fan [5], hence we need the following further generalization of the KKM mapping principle due to Ky Fan [5]:

**Theorem 6.** (Ky Fan) Let $Y$ be a convex set in a Hausdorff topological vector space and let $X$ be a non-empty subset of $Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of $Y$ such that the convex hull of every finite subset $\{ x_1, x_2, \ldots, x_n \}$ of $X$ is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a non-empty subset $X_0$ of $X$ such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and $X_0$ is contained in a compact convex subset of $Y$, then $\bigcap_{x \in X} F(x) \neq \emptyset$. 
Proof of Theorem 5. For each \( x \in X \), let
\[
F(x) := \{ y \in X : (x, y^i) \not\in A_i \text{ for at least one index } i \},
\]
then \( F(x) \) is relative closed in \( X \) by (b). By (d'), for each \( y \in K \), for each \( i = 1, 2, \ldots, n \), there exists \( x_i \in A_i(y^i) \), so that by setting \( x = (x_1, x_2, \ldots, x_n) \in X \), we have \( y \not\in F(x) \) and it follows that \( K \cap \bigcap_{x \in X} F(x) = \emptyset \). On the other hand, by (d''), for each \( y \in X \setminus K \), there exists \( x \in K \) such that \( (x_i, y^i) \in A_i \) for all \( i = 1, 2, \ldots, n \), so that \( y \not\in F(x) \); it follows that \( (X \setminus K) \cap \bigcap_{x \in K} F(x) = \emptyset \). Hence \( \bigcap_{x \in X} F(x) = \emptyset \) and \( \bigcap_{x \in K} F(x) \) is compact, being a closed subset of the compact set \( K \).

According to Theorem 6, there exist \( x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in X \), and non-negative real numbers \( \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)} \) with \( \sum_{k=1}^{m} \alpha^{(k)} = 1 \) such that \( \sum_{k=1}^{m} \alpha^{(k)} x^{(k)} \not\in \bigcup_{k=1}^{m} F(x^{(k)}) \). Let \( z := \sum_{k=1}^{m} \alpha^{(k)} x^{(k)} \), then \( (x^{(i)}, z^i) \in A_i \) for all \( 1 \leq i \leq n \) and \( 1 \leq k \leq m \), or \( x^{(i)} \in A_i(z^i) \) for all \( 1 \leq i \leq n \) and \( 1 \leq k \leq m \). By (a), we have
\[
(3) \quad x^{(i)} \in B_i(z^i) \quad \text{for all } 1 \leq i \leq n \text{ and } 1 \leq k \leq m.
\]
By (c), at least \( q \) of the sections \( B_1(z_1), B_2(z_2), \ldots, B_n(z_n) \) are convex. Since \( z_i = \sum_{k=1}^{m} \alpha^{(k)} x^{(k)} \) for \( i = 1, 2, \ldots, n \), (3) implies that \( z_i \in B_i(z^i) \) holds for at least \( q \) indices \( i \). Thus \( z \) is a point common to at least \( q \) of the sets \( B_1, B_2, \ldots, B_n \). This completes the proof. \( \square \)

The following is an analytic formulation of Theorem 5:

**Theorem 7.** Let \( X_1, X_2, \ldots, X_n \) be \( n \) (\( \geq 2 \)) convex sets each in a Hausdorff topological vector space. Let \( X := \prod_{i=1}^{n} X_i \) and \( \{ t_i \}_{i=1}^{n} \) be a set of \( n \) real numbers. Let \( \{ f_i \}_{i=1}^{n} \) and \( \{ g_i \}_{i=1}^{n} \) be \( 2n \) real-valued functions defined on \( X \) satisfying the following conditions:

(a) \( f_i \leq g_i \) on \( X \) for each \( i = 1, 2, \ldots, n \).

(b) For each \( i = 1, 2, \ldots, n \) and any \( x_i \in X_i \), \( f_i(x_i, x^i) \) is a lower semi-continuous function of \( x^i \in X^i \).

(c) For any \( x \in X \), at least \( q \) of the functions \( g_i(y_i, x^i) \) are quasi-concave functions of \( y_i \in X_i \).

(d) There exists a non-empty compact convex subset \( K \) of \( X \) such that

(d') For each \( i = 1, 2, \ldots, n \) and any \( x_i \in K_i \), there exists \( x_i \in X_i \) such that \( f_i(x_i, x^i) > t_i \), and

(d'') for each \( y \in X \setminus K \), there exists \( x \in K \) such that \( f_i(x, y^i) > t_i \) for all \( i = 1, 2, \ldots, n \).

Then there exists a point \( \hat{y} \in X \) such that \( g_i(\hat{y}) > t_i \) for at least \( q \) indices \( i \) in \( \{1, 2, \ldots, n\} \).
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REFERENCES


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