THE NUMBER OF EQUATIONS DEFINING POINTS IN GENERAL POSITION

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Bounds are established for the number of generators of the graded homogeneous ideal of a set of points in generic or in uniform position in the projective plane. For $n \leq 11$, $n$ points in uniform position must have the "general" number of generators. It is shown by example that this fails for $n = 12$.

Introduction. Let $Z$ be a set of points in $\mathbb{P}^2_k$, $k$ algebraically closed. We say the points of $Z$ lie in generic position if $Z$ imposes independent conditions on curves containing it. If this holds for all subsets of $Z$ we say that $Z$ lies in uniform position. Given a set $Z$ in one of these types of "general position", one would like to count the number of equations needed to cut out $Z$, or more precisely, the minimal number of generators $\nu$ of the graded homogeneous ideal $I(Z)$.

This question has arisen most recently in calculations of the Cohen-Macaulay-type of singularities. For example, it is shown in [7] that if $A$ is the local ring at a curve singularity $P$ in $\mathbb{A}^3_k$, and if the lines of the tangent cone at $P$ correspond to a set of distinct points $Z$ in generic position in $\mathbb{P}^2_k$, then the Cohen-Macaulay-type of $A$ is equal to $\nu(I(Z)) - 1$. It is then natural to look for geometric conditions on $Z$ which will allow the Cohen-Macaulay-type to be computed.

Let $s$ denote the number of points belonging to $Z$, $d$ the integer such that $\left( \frac{d+1}{2} \right) \leq s < \left( \frac{d+2}{2} \right)$, and define

$$N(s) = \begin{cases} 
  d + 1 - s + \binom{d+1}{2} & \text{if } \binom{d+1}{2} \leq s \leq \frac{d(d+2)}{2} \\
  d + 2 + s - \binom{d+2}{2} & \text{if } \frac{d(d+2)}{2} \leq s < \binom{d+2}{2}.
\end{cases}$$

Geramita and Maroscia [6] have shown that almost all sets of $s$ points in $\mathbb{P}^2$ are defined by exactly $N(s)$ equations. We give a new proof of this fact (1.7). However, $\nu(I(Z))$ is not constant on the sets of $s$ points in generic position. It follows from a theorem of Dubreil [4] that the best one can say is that if $Z$ lies in generic position, then $N(s) \leq \nu(I(Z)) \leq d + 1$. 

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Uniform position was introduced in [7] as a more stringent condition on $Z$; it is known that $v(I(Z)) = N(s)$ if $Z$ is a set of $s$ points in uniform position and $s \leq 11$. If $s = 12$, then $N(s) = 3$, but we construct in §2 a set $Z$ of 12 points in uniform position for which $v(I(Z)) = 4$. Therefore, in a sense, some sets of points in uniform position are "more general" than others. We also establish an upper bound for $v(I(Z))$, where $Z$ lies in uniform position:

$$N(s) \leq v(I(Z)) \leq d \quad \text{if} \quad \binom{d + 1}{2} < s < \binom{d + 2}{2} - 1$$

$$N(s) = v(I(Z)) = d + 1 \quad \text{if} \quad s = \binom{d + 1}{2} \text{ or } \binom{d + 2}{2} - 1.$$

For example, if $s = 12$, this result shows that $3 \leq v \leq 4$.

The author is grateful to W. C. Brown for many valuable conversations.

1. Bounds on the number of generators.

**Definition.** Let $Z$ be a set of $s$ points contained in $\mathbb{P}^2 = \mathbb{P}_k$, $k$ an algebraically closed field, and $\mathcal{I}_Z$ its sheaf of ideals. We say $Z$ lies in **generic position** if for every nonnegative integer $m$, $\dim H^0(\mathcal{I}_Z(m)) = \max\{0, (m + 2)^2 - s\}$, where $\mathcal{I}_Z(m)$ denotes $\mathcal{I}_Z$ twisted $m$ times by the hyperplane line bundle. We say $Z$ lies in **uniform position** if each subset of $Z$ (including $Z$ itself) lies in generic position.

**Remark 1.0.1.** Roughly speaking, $Z$ lies in generic position if $Z$ imposes independent conditions on curves containing it. The sets of $s$ points in generic position in $\mathbb{P}^2$ form a Zariski-open subset of the Hilbert scheme $\text{Hilb}^s(\mathbb{P}^2)$ parametrizing subschemes of $\mathbb{P}^2$ of length $s$. The sets of $s$ points in uniform position form an open subset of the sets of $s$ points in generic position.

Let $Z$ be a zero-dimensional subscheme of $\mathbb{P}^2$ of length $s$. Because the projective dimension of an ideal sheaf in $\mathbb{P}^n$ is at most $n - 1$, $I_Z$ has a minimal projective resolution

$$0 \rightarrow \sum_{i=1}^{\nu-1} \mathcal{O}_{\mathbb{P}^2}(-t_i) \rightarrow \sum_{j=1}^{\nu} \mathcal{O}_{\mathbb{P}^2}(-r_j) \rightarrow \mathcal{I}_Z \rightarrow 0$$

where $A$ is a $(\nu - 1) \times \nu$ "relations matrix" of homogeneous forms of degrees $\alpha_{ij} = t_i - r_j$, and we arrange $t_1 \leq \cdots \leq t_{\nu-1}$ and $r_1 \leq \cdots \leq r_\nu$. A standard Chern class calculation shows $\Sigma_{i=1}^{\nu-1} t_i = \Sigma_{j=1}^{\nu} r_j$ and $2s = \Sigma_{i=1}^{\nu-1} t_i^2 - \Sigma_{j=1}^{\nu} r_j^2$. Further, the minimality of the resolution implies that
the entries of $A$ are in the irrelevant ideal, so we have $t_1 > r_1$ and $t_{\nu - 1} > r_{\nu}$.

**Lemma 1.1 (Burch, [3]).** If $I$ is an ideal of projective dimension one in a regular local ring $(R, \mathfrak{m})$, then given a minimal resolution

$$0 \to R^{n-1} \xrightarrow{A} R^n (f_1, \ldots, f_n) \to I \to 0$$

there exists $r \in R$ such that $f_i = r\Delta_i$, $i = 1, \ldots, n$, where the $\Delta_i$ are the maximal minors of the matrix $A$.

The lemma applies in our case with $R = k[x_0, x_1, x_2]_{(x_0, x_1, x_2)}$ and $I(Z) = \sum_{m=0}^{\infty} H^0(\mathcal{I}_Z(m))$ localized at $(x_0, x_1, x_2)$ for $I$. Since $ht I = 2$, $r$ must be a unit, and we may assume in the following that $f_i = \Delta_i$.

**Example 1.1.1.** If $Z$ is a complete intersection of curves of degrees $r_1$ and $r_2$, then the minimal resolution is the familiar Koszul complex

$$0 \to \mathcal{O}_{P^2}(-r_1 - r_2) \to \mathcal{O}_{P^2}(-r_1) \oplus \mathcal{O}_{P^2}(-r_2) \to \mathcal{I}_Z \to 0$$

and the relations matrix is $(-f_2, f_1)$. It is easy to check that the complete intersection of $f_1$ and $f_2$ lies in generic position if and only if $\deg f_i \leq 2$. More generally:

**Proposition 1.2.** (a) A subscheme $Z$ of length $s$ in $P^2$ lies in generic position if and only if $0 \leq \alpha_{ij} \leq 2$ for a minimal resolution (1). (b) Moreover, in this situation, $r_1 \leq r_i \leq r_1 + 1$ for all $1 \leq i \leq \nu$.

**Proof.** (a) Consider the long exact sequence of cohomology

$$0 \to H^0(\mathcal{I}_Z(m)) \to H^0(\mathcal{O}_{P^2}(m)) \to H^0(\mathcal{O}_Z(m))$$

$$\to H^1(\mathcal{I}_Z(m)) \to 0.$$ 

Denote $\dim H^i$ by $h^i$. Because $h^0(\mathcal{O}_{P^2}(m)) = \binom{m+2}{2}$ and $h^0(\mathcal{O}_Z(m)) = s$, it is clear that $Z$ lies in generic position if and only if $\mathcal{I}_Z$ has "seminatural cohomology", i.e. for each $m \geq 0$, at most one of $H^i(\mathcal{I}_Z(m))$, $i = 0, 1, 2$, is nonzero. Since $H^0(\mathcal{I}_Z(m)) \neq 0$ if and only if $m \geq r_1$, $\mathcal{I}_Z$ has seminatural cohomology if and only if $H^1(\mathcal{I}_Z(m)) = 0$ for $m \geq r_1$. By (1) and the fact that $t_{\nu - 1} > r_{\nu}$, this condition is equivalent to $t_{\nu - 1} \leq r_1 + 2$.

(b) In this situation, since $t_{\nu - 1} > r_i$ for all $i$, $r_1 \leq r_i < t_{\nu - 1} \leq r_1 + 2$ for all $i$.

**Remark 1.2.1.** Fact (b) was proved in [7].
One of the advantages of generic position is that it is preserved under linkage. We say that two closed subschemes $Z, Z'$ of $\mathbb{P}^n$, equidimensional and without embedded components, are linked via the complete intersection $X$ containing $Z$ and $Z'$ if $I(Z) = I(X)$: $I(Z') = I(X)$: $I(Z)$, where $I(Z)$ denotes the homogeneous graded ideal of $Z$, and so forth. The next proposition was essentially known to Apéry (see [5]) and has an elementary proof, which is provided in an appendix for lack of a reference.

**Proposition 1.3.** Let $Z$ be a projectively Cohen-Macaulay subscheme of $\mathbb{P}^n$ of codimension two, and let
\[
0 \rightarrow \sum_{i=1}^{\nu-1} \mathcal{O}_{\mathbb{P}^n}(-t_i) \xrightarrow{A} \sum_{j=1}^{\nu} \mathcal{O}_{\mathbb{P}^n}(-r_j)^{(f_1, \ldots, f_\nu)} \rightarrow \mathcal{I}_Z \rightarrow 0
\]
be a minimal projective resolution for the ideal sheaf, where $f_i$ is the $i$th maximal minor of $A$. Suppose $\{f_1, f_\nu\}$ forms a regular sequence of length two in $k[X_0, \ldots, X_n]$, and let $Z'$ be the (projectively Cohen-Macaulay) subscheme of $\mathbb{P}^n$ linked to $Z$ via $\{f_1, f_\nu\}$. Then a relations matrix for a minimal resolution for $\mathcal{I}_{Z'}$ is obtained by deleting columns $i$ and $j$ from $A$ and transposing.

**Remark 1.3.1.** The liaison theorem ([9, Thm. 3.2]) follows directly from (1.3) and the fact that if $I$ is an ideal in $k[x_0, \ldots, x_n]$, a graded quotient of a polynomial ring, and $I$ contains a non-zero-divisor, then there exists a non-zero-divisor $x \in I$ belonging to a set of minimal generators for $I$.

**Remark 1.3.2.** If we assume $i = \nu - 1, j = \nu$ in (1.3), then $\alpha'_{kl} = \alpha_{lk}$ for $1 \leq k \leq \nu - 2, 1 \leq l \leq \nu - 1$.

**Corollary 1.4.** Let $Z$ be a set of points in $\mathbb{P}^2$ lying in generic position and $f_1, f_2$ two homogeneous forms with no common factor which belong to a minimal set of generators of the ideal $I(Z) = \sum_{m=0}^{\infty} H^0(\mathcal{I}_Z(m))$. Let $Z'$ be the scheme linked to $Z$ via $\{f_1, f_2\}$. Then $Z'$ lies in generic position.

**Proof.** Follows from (1.2) and (1.3.2).

Next we find a lower bound on the number of generators $\nu$ of the graded ideal $I(Z)$, where $Z$ is a length $s$ subscheme of $\mathbb{P}^2$ lying in generic position. A set of generators must contain a basis for $H^0(\mathcal{I}_Z(r))$, where $r$
is the least degree of a curve containing $Z$. Generic position implies that $r$ is the least integer such that $(r+2)^2 > s$, and we have $v \geq k$ where $k = h^0(I_Z(r+s)) = (r+2)^2 - s$. Using (1.2) we see that a minimal resolution for $I_Z$ has the form

$$0 \rightarrow \sum_{i=1}^{r-l^{-1}} \mathcal{O}_{P^2}(-r - 1) \oplus \sum_{i=1}^{l} \mathcal{O}_{P^2}(-r - 2) \rightarrow \sum_{i=1}^{k} \mathcal{O}_{P^2}(-r) \oplus \sum_{i=1}^{v-k} \mathcal{O}_{P^2}(-r - 1) \rightarrow I_Z \rightarrow 0$$

where $l = r + 1 - k$. Since $v - k$ and $v - l - 1$ are nonnegative, $v \geq \max\{k, r - k + 2\}$. Set $N(s) = \max\{k, r - k + 2\}$ (clearly $r$ and $k$ depend only on $s$). Then we can compute

$$N(s) = \max\{k, r - k + 2\}$$

$$= \max\left\{\left(\frac{r + 2}{2}\right) - s, r - \left(\frac{r + 2}{2}\right) + s + 2\right\}$$

$$= \begin{cases} 
  r + 1 - s + \left(\frac{r + 1}{2}\right) & \text{if } \left(\frac{r + 1}{2}\right) \leq s \leq \frac{r(r + 2)}{2} \\
  r + 2 + s - \left(\frac{r + 2}{2}\right) & \text{if } \frac{r(r + 2)}{2} \leq s < \left(\frac{r + 2}{2}\right).
\end{cases}$$

Therefore, $N(s)$, a number depending only on $s$, is a lower bound on the number of generators for the ideal $I(Z)$. To show this bound is sharp, we will exhibit for each integer $s$ a zero-dimensional scheme of length $s$ in generic position with exactly $N(s)$ ideal generators. By the following Lemma 1.5, having $N(s)$ ideal generators is an open condition on the set $Z$. It follows (1.7) that the general set of $s$ distinct points in $P^2$ has $N(s)$ ideal generators.

**Lemma 1.5.** Let $U$ be the open dense subset of $\text{Hilb}^r(P^2)$ corresponding to schemes in generic position. For any integer $N$, the subset of $U$ corresponding to schemes defined by at most $N$ equations is open in $\text{Hilb}^r(P^2)$.

**Proof.** The sheaf of differentials $\Omega_{P^2}$ fits in the following exact sequence:

$$0 \rightarrow \Omega_{P^2} \rightarrow 3\mathcal{O}_{P^2}(-1) \rightarrow \mathcal{O}_{P^2} \rightarrow 0$$

where $P^2 = \text{Proj} k[x, y, z]$. Let $Z \in \text{Hilb}^r(P^2)$, lying in generic position. Tensor the above sequence with $\mathcal{I}_Z(r+1)$, where $r$ is the least integer
such that \((r^+2)^2 > s\). The long exact sequence of cohomology yields

\[
H^0(3\mathcal{I}_Z(r)) \to H^0(\mathcal{I}_Z(r + 1)) \to H^1(\Omega_{\mathbb{P}^2} \otimes \mathcal{I}_Z(r + 1)) \to 0
\]

since \(H^1(3\mathcal{I}_Z(r)) = 0\) by the seminatural cohomology of \(\mathcal{I}_Z\). The dimension of \(H^1(\Omega_{\mathbb{P}^2} \otimes \mathcal{I}_Z(r + 1))\) measures the number of generators of \(I(Z)\) of degree \(r + 1\). Thus

\[
v(I(Z)) = \left( r + 2 \right) - s + h^1(\Omega_{\mathbb{P}^2} \otimes \mathcal{I}_Z(r + 1)).
\]

By the semicontinuity theorem ([8], Thm. III.12.8), \(v(I(Z))\) is an upper semicontinuous function of \(Z\) in \(\mathcal{U}\).

**Lemma 1.6** (Buchsbaum, Eisenbud [2]). Let \(R\) be a commutative noetherian ring, \(F_j\) free \(R\)-modules. The complex

\[
0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} F_{n-2} \to \cdots \to F_0
\]

is exact if and only if for \(j = 1, 2, \ldots, n,\)

(a) rank \(\varphi_j + \) rank \(\varphi_{j-1} = \) rank \(F_{j-1},\)

(b) the ideal of maximal minors of \(\varphi_j\) contains a regular sequence of length \(j\).

**Proposition 1.7** (Geramita, Marosci ([6]). If \(Z\) is a set of \(s\) points in generic position in \(\mathbb{P}^2\), then \(v(I(Z)) \geq N(s)\). Moreover, equality holds for all \(Z\) in an open dense subset of \(\text{Hilb}^s(\mathbb{P}^2)\).

**Proof.** The first statement is proved above. For the second, it remains to construct for each \(s\) a length \(s\) scheme \(Z\) in generic position with \(v(I(Z)) = N(s)\).

**Case 1.** Suppose \(\left( \frac{r+1}{2} \right) \leq s \leq r(r + 2)/2\) for some integer \(r\). Set \(m = r(r + 2) - 2s, N = N(s) = r + 1 - s + \left( \frac{r+1}{2} \right) > 0\). An easy calculation shows \(0 \leq m \leq N - 1\). Consider the \((N - 1) \times N\) matrix

\[
A = \begin{pmatrix}
I_{11} & \cdots & I_{1n} \\
\vdots & & \vdots \\
I_{m1} & \cdots & I_{mN} \\
c_{m+1,1} & \cdots & c_{m+1,N} \\
\vdots & & \vdots \\
c_{N-1,1} & \cdots & c_{N-1,N}
\end{pmatrix}
\]
where \( \deg l_{ij} = 1 \) and \( \deg c_{ij} = 2 \). If the entries of the matrix are chosen generally it is clear that at least two of the maximal minors \( f_1, \ldots, f_N \) will share no common factor, and grade \( \langle f_1, \ldots, f_N \rangle \geq 2 \). Now (1.6) applies to show that the corresponding sequence (1) is exact with \( \nu = N \), so that we have constructed a zero-dimensional scheme \( Z \) with \( A \) as relations matrix.

Counting degrees, we have \( r_1 = \cdots = r_n = 2N - m - 2 \), \( t_1 = \cdots = t_m = 2N - m - 1 \), and \( t_{m+1} = \cdots = t_{N-1} = 2N - m \). A calculation shows the length of \( Z \) is \( s \), and by (1.2) \( Z \) lies in generic position. The resolution is minimal, so \( \nu(I(Z)) = N(s) \).

**Case 2.** Suppose \( r(r + 2)/2 \leq s < (r + 2)^2 \) for some integer \( r \). Set \( m = (r + 2) - s \) and \( N = N(s) = s - (r + 2) + r + 2 > 0 \). Note that \( 0 \leq m \leq N \). Consider the \((N - 1) \times N\) matrix

\[
A = \begin{pmatrix}
c_{11} & \cdots & c_{1m} & l_{1,m+1} & \cdots & l_{1N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_{N-1,1} & \cdots & c_{N-1,m} & l_{N-1,m+1} & \cdots & l_{N-1,N}
\end{pmatrix}
\]

where \( \deg l_{ij} = 1 \) and \( \deg c_{ij} = 2 \). As before, if the entries are chosen generally there exists a minimal projective resolution of an ideal sheaf \( I_Z \) with \( A \) as relations matrix. We have

\[
r_1 = \cdots = r_m = N + m - 2, \quad r_{m+1} = \cdots = r_N = N + m - 1,
\]

\[
t_1 = \cdots = t_{N-1} = N + m,
\]

showing that \( Z \) consists of \( s \) points in generic position, and \( \nu(I(Z)) = N(s) \).

**Proposition 1.8.** Let \( Z \) be a set of \( s \) points in generic position in \( \mathbb{P}^2 \), \( r \) the least integer such that \( (r + 2)^2 > s \). Then \( \nu(I(Z)) \leq r + 1 \).

**Proof.** Let \( Z \) be a length \( s \) subscheme of \( \mathbb{P}^2 \), and \( r \) the least degree of a curve containing \( Z \). Let \( A \) be a relations matrix for a minimal resolution (1) of \( \mathcal{F}_Z \). We may arrange \( A \) such that \( a_{11} \geq a_{12} \geq \cdots \geq a_{1r} \); then \( \deg f_1 = r \) and \( f_1 \) is the determinant of a \((v - 1) \times (v - 1)\) matrix whose nonzero entries are of degree at least one. Therefore \( r = \deg f_1 \geq \nu(I(Z)) - 1 \), i.e. \( \nu(I(Z)) \leq r + 1 \). This is a theorem of Dubreil ([4], Thm. I).

For points in uniform position, the upper bound of (1.8) can be improved:

**Proposition 1.9.** Let \( Z \) be a set of \( s \) points in uniform position in \( \mathbb{P}^2 \), such that \( (r + 2)^2 + 1 \leq s \leq (r + 2)^2 - 2 \) for some \( r \). Then \( \nu(I(Z)) \leq r \).
Proof. Let \( k = (r^2 - 2) - s \); so that \( 2 \leq k \leq r \). Suppose \( v(I(Z)) = r + 1 \), and let \( A \) be the relations matrix of a minimal resolution (1) of \( J_Z \).

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1,r+1} \\
  \vdots & & \vdots \\
  a_{r,1} & \cdots & a_{r,r+1}
\end{pmatrix}
\]

By (1.2), \( 0 \leq \alpha_{ij} \leq 2 \), and \( \alpha_{ij} = \deg a_{ij} \). We may assume \( \alpha_{11} \geq \alpha_{12} \geq \cdots \geq \alpha_{1,r+1} \); then we have

\[
(2) \quad \alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{ik} = \alpha_{i,k+1} + 1 = \cdots = \alpha_{i,r+1} + 1.
\]

We may also assume \( \alpha_{11} \leq \alpha_{21} \leq \cdots \leq \alpha_{rl} \).

Since \( \deg f_{r+1} = r + 1 \), we have

\[
r + 1 = \alpha_{11} + \alpha_{22} + \cdots + \alpha_{rr}
= \alpha_{11} + \cdots + \alpha_{kl} + (\alpha_{k+1,1} - 1) + \cdots + (\alpha_{r1} - 1)
= \alpha_{11} + \cdots + \alpha_{r1} - r + k
\]

and thus

\[
(3) \quad 2r - k + 1 = \alpha_{11} + \cdots + \alpha_{r1}.
\]

By (2), \( \alpha_{i1} \geq 1 \) for all \( i \).

Suppose \( \alpha_{k-1,1} \geq 2 \). Then \( 2 \leq \alpha_{k-1,1} \leq \alpha_{k1} \leq \cdots \leq \alpha_{r1} \) and so \( \alpha_{11} + \cdots + \alpha_{r1} \geq k - 2 + 2(r - k + 2) = 2r - k + 2 \), contradicting (3). Therefore, \( 1 = \alpha_{k-1,1} = \alpha_{k-2,1} = \cdots = \alpha_{11} \).

By (2), \( a_{ij} = 0 \) for \( 1 \leq i \leq k - 1, k + 1 \leq j \leq r + 1 \). Since all entries of \( A \) lie in the maximal homogeneous ideal \( (X_0, X_1, X_2) \) of \( k[X_0, X_1, X_2] \), \( a_{ij} = 0 \) for \( 1 \leq i \leq k - 1, k + 1 \leq j \leq r + 1 \). \( A \) has the following form:

\[
\begin{pmatrix}
  * & A_1 & 0 \\
  \vdots & & \vdots \\
  * & A_2 & A_3 \\
  1 & k - 1 & r - k + 1
\end{pmatrix}
\]

Case 1. If \( s = (r^2 + 1) + 1 \), then \( k = r \), and the form of \( A \) is

\[
\begin{pmatrix}
  B & 0 \\
  \vdots & \vdots \\
  0 & 0 \\
  a_{r,r+1}
\end{pmatrix}
\]

where \( \deg a_{r,r+1} = 1 \). The complete intersection of the curves \( \det B = 0 \) and \( a_{r,r+1} = 0 \) lies in \( Z \). Since \( r \geq 2 \), \( \deg(\det B) \geq 3 \), so \( Z \) contains three
collinear points (counted with multiplicity), a contradiction to the uniform position assumption.

Case 2. If \((r+1) + 2 \leq s \leq (r+2) - 2\), then \(f_1 = (\det A_1) \cdot (\det A_3)\), so \(f_1\) is the composite of curves of degrees \(k-1 \geq 1\) and \(v-k \geq 1\). This contradicts the following fact:

**Lemma 1.10** (Geramita, Maroscia [6], Thm. 3.4). Let \(Z\) be a set of \(s\) points in uniform position in \(\mathbf{P}^2\) where \((r+1) + 2 \leq s \leq (r+2) - 1\). Then every curve of degree \(r\) containing \(Z\) is irreducible.

**Proposition 1.11.** Let \(Z\) be a set of \(s\) points in \(\mathbf{P}^2\), and set

\[
N(s) = \begin{cases} 
    r + 1 - s + \binom{r+1}{2} & \text{if } \binom{r+1}{2} \leq s \leq \frac{r(r+2)}{2} \\
    r + 2 + s - \binom{r+2}{2} & \text{if } \frac{r(r+2)}{2} \leq s < \binom{r+2}{2} 
\end{cases}
\]

(a) If \(Z\) lies in generic position, then \(N(s) \leq \nu(I(Z)) \leq r + 1\).
(b) If \(Z\) lies in uniform position, then

\[
N(s) = \nu(I(Z)) = r + 1 \quad \text{if } s = \binom{r+1}{2} \text{ or } \binom{r+2}{2} - 1.
\]

**Proof.** Follows from (1.7), (1.8) and (1.9).

**Corollary 1.12.** (Geramita, Maroscia, Orecchia [6], [7]). (a) If \(Z\) lies in generic position and \(s = \binom{r+1}{2}\) or \(\binom{r+2}{2} - 1\), then \(\nu(I(Z)) = r + 1\). (b) If \(Z\) lies in uniform position and \(s = \binom{r+1}{2} + 1\) or \(\binom{r+2}{2} - 2\), then \(\nu(I(Z)) = r\).

**Corollary 1.13.** If \(Z\) lies in uniform position and \(s = \binom{r+2}{2} - 3\), then \(r - 1 \leq \nu(I(Z)) \leq r\).

**Remark 1.13.1.** It follows from (1.11) that if \(s \leq 11\), \(s\) points in uniform position have exactly \(N(s)\) ideal generators. In the next section we show that this statement fails for \(s = 12\).

2. A counterexample. The following result is related to the classical Cayley-Bacharach Theorem.

**Lemma 2.1.** Let \(S, T\) be zero-dimensional subschemes of \(\mathbf{P}^2\) linked by two curves \(C, D\) of degrees \(c\) and \(d\), respectively, having no common components. Suppose \(d \leq c + 2\). Then \(h^0(I_S(d-3)) = h^1(I_T(c))\).
Proof. Suppose $\mathcal{I}_T$ has minimal resolution

$$0 \rightarrow \sum_{i=1}^{\nu-1} \mathcal{O}_{P_2}(-t_i) \rightarrow \sum_{j=1}^{\nu} \mathcal{O}_{P_2}(-r_j) \rightarrow \mathcal{I}_T \rightarrow 0.$$  

Then linkage implies ([9], Prop. 2.5)

$$0 \rightarrow \sum_{j=1}^{\nu} \mathcal{O}_{P_2}(r_j - c - d)$$

$$\rightarrow \sum_{i=1}^{\nu-1} \mathcal{O}_{P_2}(t_i - c - d) \oplus \mathcal{O}_{P_2}(-c) \oplus \mathcal{O}_{P_2}(-d) \rightarrow \mathcal{I}_S \rightarrow 0$$

is a projective resolution for $\mathcal{I}_S$. By the long exact sequences of cohomology,

$$h^0(\mathcal{I}_S(d - 3)) = \sum_{i=1}^{\nu-1} h^0(\mathcal{O}_{P_2}(t_i - c - 3)) - \sum_{j=1}^{\nu} h^0(\mathcal{O}_{P_2}(r_j - c - 3))$$

$$= \sum_{i=1}^{\nu-1} h^2(\mathcal{O}_{P_2}(c - t_i)) - \sum_{j=1}^{\nu} h^2(\mathcal{O}_{P_2}(c - r_j))$$

$$= h^1(\mathcal{I}_T(c)).$$

**Lemma 2.2.** Suppose $P_1, \ldots, P_{13}$ are points in $\mathbb{P}^2$ such that no three are collinear, and suppose $\{Q_1, Q_2, Q_3\}$ is linked to $\{P_1, \ldots, P_{13}\}$ via two quartic curves. Then every subset of $\{Q_1, Q_2, Q_3, P_1, \ldots, P_{10}\}$ imposes independent conditions on quartics.

Proof. Set $T = \{Q_1, Q_2, Q_3, P_1, \ldots, P_{10}\}$ and $S = \{P_{11}, P_{12}, P_{13}\}$ in (2.1). Since $P_{11}, P_{12}, P_{13}$ are not collinear, $h^1(\mathcal{I}_T(4)) = 0$, i.e. $T$ imposes independent conditions on quartics. Since each point imposes at most one condition, it follows that every subset of $T$ imposes independent conditions on quartics.

According to (1.13) a set of 12 points in uniform position must have either 3 or 4 ideal generators. We know almost all have 3 ideal generators (1.7). If $\nu(I(Z)) = 3$, then the minimal resolution for $\mathcal{I}_Z$ must be

$$0 \rightarrow 2 \mathcal{O}_{P_2}(-6) \rightarrow A \mathcal{O}_{P_2}(-4) \rightarrow \mathcal{I}_Z \rightarrow 0$$

and $A$ has form

$$\begin{pmatrix}
    c_{11} & c_{12} & c_{13} \\
    c_{21} & c_{22} & c_{23}
\end{pmatrix}$$

where deg $c_{ij} = 2$. 
Suppose there are two quartic curves with no common component containing \(Z\). We may assume they are given by \(f_1 = 0\) and \(f_2 = 0\). By (1.3), \(Z\) is linked via \(f_1\) and \(f_2\) to the complete intersection of the two conics \(c_{13} = 0\) and \(c_{23} = 0\).

**Example 2.3.** A set of 12 points in uniform position in \(\mathbb{P}^2\) with \(\nu(I(Z)) = 4\).

We will construct a set \(Z\) in uniform position that is linked via two quartics to four points, three of which are collinear. By the preceding argument, we conclude \(\nu(I(Z)) = 4\).

The following criterion for uniform position will be used below.

**Lemma 2.4 (Brun, [1]).** A set of points \(Z\) in \(\mathbb{P}^2\) is in uniform position if and only if

(a) at most \(\left(\binom{r+2}{2}\right) - 1\) points of \(Z\) lie on a degree \(r\) curve, for each \(r\), and

(b) for each \(z \in Z\), there exists a degree \(d\) curve \(Y\) such that \(Y \cap Z = Z \setminus \{z\}\), where \(d\) is the smallest integer such that \(\binom{d+2}{2} \geq s\).

**Construction.** Fix a nonsingular quartic \(C\) in \(\mathbb{P}^2\) and three distinct collinear points \(Q_1, Q_2, Q_3\) on \(C\). We will show that the general quartic passing through \(Q_1, Q_2, Q_3\), intersects \(C\) in \(Q_1, Q_2, Q_3, P_1, \ldots, P_{13}\) such that any twelve of \(P_1, \ldots, P_{13}\) lie in uniform position.

The family of quartics in \(\mathbb{P}^2\) is parametrized by \(\mathbb{P}^{14}\), and an 11-dimensional subfamily \(S\) passes through \(Q_1, Q_2, Q_3\), since it is clear that none of these three points is a base point for quartics through the other two.

The quartics in \(S\) missing the point \(Q_4\), the residual intersection of \(C\) and the line \(L\) connecting \(Q_1, Q_2, Q_3\), form again an 11-dimensional family \(S'\). Suppose \(C' \in S'\) and \(C \cap C' = \{Q_1, Q_2, Q_3, P_1, \ldots, P_{13}\}\), counting multiplicities, and that \(P_1, P_2, P_3\) are collinear. None of the \(P_i\) lie on \(L\). \(P_3\) is not a base point for quartics through \(\{Q_1, Q_2, Q_3, P_1, P_2\}\): for example, set \(L_1 = \overline{P_iQ_i}\), where we choose \(Q_i \notin P_1 P_2\), and \(L_2, L_3, L_4\) lines containing \(P_2, Q_j, Q_k\), respectively, where \(\{i, j, k\} = \{1, 2, 3\}\). Then there exists a quartic \(L_1 L_2 L_3 L_4\) containing \(\{Q_1, Q_2, Q_3, P_1, P_2\}\) but not \(P_3\).

Similar arguments show that none of the six points \(\{Q_1, Q_2, Q_3, P_1, P_2, P_3\}\) is a base point for quartics through the other five, so that the six points induce independent conditions on quartics.

Thus there are 8 dimensions of quartics through \(\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}\). The choice of a set of collinear points \(\{P_1, P_2, P_3\}\) on \(C \setminus L\) is 2-dimensional (since there is a finite choice (four) of three-point sets
associated with each line in \( \mathbf{P}^2 \)). Therefore there is a 10-dimensional set of quartics \( C' \) belonging to \( S' \) such that the remaining 13 points of the complete intersection \( C \cap C' \) contain three collinear points. We conclude that the general quartic through \( Q_1, Q_2, Q_3 \) is such that the residual intersection with \( C \) does not contain three collinear points.

Showing the general residual intersection contains no six on a conic and no ten on a cubic requires similar arguments; we argue only the latter. Suppose \( C' \) is a general quartic such that \( C \cap C' = \{ Q_1, Q_2, Q_3, P_1, \ldots, P_{13} \} \) where \( \{ P_1, \ldots, P_{10} \} \) are situated on a cubic. Since no three of \( P_1, \ldots, P_{13} \) are collinear, (2.2) applies to show \( \{ Q_1, \ldots, P_{10} \} \) imposes independent conditions on quartics, so that the family of quartics through \( \{ Q_1, \ldots, P_{10} \} \) is one-dimensional. The family of 10-point sets on \( C \) lying on a cubic is at most 9-dimensional, therefore the family of quartics through \( Q_1, Q_2, Q_3 \) such that the residual 13 points of \( C \cap C' \) contain 10 cocubic points is at most 10-dimensional. We conclude that the general quartic through \( Q_1, Q_2, Q_3 \) intersects \( C \) in 13 residual points that have no three collinear, no six on a conic, and no ten on a cubic.

Let \( C' \) be such a general quartic. We will show that any twelve of \( P_1, \ldots, P_{13} \) satisfy the criterion of (2.4). Condition (a) is verified; for (b) we reason as follows. Given any point \( z \) among a subset of twelve, choose a cubic containing nine of the rest and a line containing the other two. The point \( z \) cannot lie on this reducible quartic \( Y \), therefore \( Y \cap Z = Z \setminus \{ z \} \). Such a set \( Z \) of twelve points lies in uniform position and is linked via \( C \) and \( C' \) to a set of four points, three of which are collinear, so \( v(I(Z)) = 4 \).

**Remark 2.4.1.** It is clear, at least if \( \text{char } k = 0 \), that by Bertini's theorem the general such \( C' \) intersects \( C \) transversally, i.e. in the terminology of [6], provides sets of 12 points in "transversal uniform position" and \( v = 4 \).

**Appendix.** We present a proof of (1.3). Let \( R \) be a commutative ring and

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1\nu} \\
  \vdots & \ddots & \vdots \\
  a_{\nu-1,1} & \cdots & a_{\nu-1,\nu}
\end{pmatrix}
\]

be a matrix whose entries lie in \( R \). Set

\[
f_i = (-1)^{i+1} \det \left( \begin{pmatrix} (\nu - 1) \times (\nu - 1) \text{ matrix} \end{pmatrix} \right)
\]

obtained by deleting column \( i \).
\[ \Gamma_{ij}^k = (-1)^{i+j+k+1} \det \left( (\nu - 2) \times (\nu - 2) \text{ matrix obtained by deleting columns } i,j \text{ row } k \right). \]

**Lemma 1.** If \( i < j \), then

\[ \left( \Gamma_{ij}^1, \ldots, \Gamma_{ij}^{\nu-1} \right) \cdot A = \begin{pmatrix} 0, \ldots, 0, f_j, 0, \ldots, 0, -f_i, 0, \ldots, 0 \end{pmatrix}_i^j. \]

**Lemma 2.** If \( 1 \leq i < j \leq \nu, l \neq i \text{ or } j \), then

\[ \Gamma_{ij}^k f_l = \Gamma_{il}^k f_j - \Gamma_{jl}^k f_i \text{ for } k = 1, \ldots, \nu - 1. \]

**Proof.** For \( l \neq i \text{ or } j \), multiply the equation in Lemma 1 on the right by the matrix

\[ B_l = \begin{pmatrix} \Gamma_{1l}^1 & \Gamma_{1l}^2 & \cdots & \Gamma_{1l}^{\nu-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{l-1,l}^1 & \Gamma_{l-1,l}^2 & \cdots & \Gamma_{l-1,l}^{\nu-1} \\ 0 & 0 & \cdots & 0 \\ -\Gamma_{l+1,l}^1 & -\Gamma_{l+1,l}^2 & \cdots & -\Gamma_{l+1,l}^{\nu-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\Gamma_{\nu,l}^1 & -\Gamma_{\nu,l}^2 & \cdots & -\Gamma_{\nu,l}^{\nu-1} \end{pmatrix}. \]

**Lemma 3.** Let \( (R, \mathcal{M}) \) be a regular local ring, \( A \) a \((\nu - 1) \times \nu\) matrix with entries in \( \mathcal{M} \). If \( \{f_1, f_2\} \) is a regular sequence in \( \mathcal{M} \), then \((f_1, f_2, \ldots, f_\nu) = (\Gamma_{ij}^1, \ldots, \Gamma_{ij}^{\nu-1})\).

**Proof.** By Lemma 2, \((\Gamma_{ij}^1, \ldots, \Gamma_{ij}^{\nu-1}) \subseteq (f_1, f_2) : (f_3, \ldots, f_\nu)\). On the other hand, let \( r \in (f_1, f_2) : (f_3, \ldots, f_\nu)\). Then

\[ \begin{pmatrix} f_1 \\ \vdots \\ f_\nu \end{pmatrix} r = C \begin{pmatrix} f_i \\ f_j \end{pmatrix}, \]

where

\[ C = \begin{pmatrix} c_{11} & c_{12} \\ \vdots & \vdots \\ c_{\nu1} & c_{\nu2} \end{pmatrix}, \quad c_{ij} \in R, \]
and we may assume $c_{i1} = r, c_{i2} = 0, c_{j1} = 0, c_{j2} = r$. Multiply on the left by $A$:

$$0 = AC \begin{pmatrix} f_i \\ f_j \end{pmatrix}.\]

$AC$ is a $(v - 1) \times 2$ matrix whose first column is divisible by $f_j$ and whose second column is divisible by $f_i$, since $\{ f_i, f_j \}$ form a regular sequence. So write

$$AC = \begin{pmatrix} d_1 f_j & -d_1 f_i \\ \vdots & \vdots \\ d_{v-1} f_j & -d_{v-1} f_i \end{pmatrix}$$

where $d_k \in R$ for all $k = 1, \ldots, v - 1$. Multiplying on the left by the matrix $(\Gamma^1_{ij}, \ldots, \Gamma^{r-1}_{ij})$ and using Lemma 1, we get

$$r = \sum_{k=1}^{v-1} d_k \Gamma^k_{ij} \in \left( \Gamma^1_{ij}, \ldots, \Gamma^{r-1}_{ij} \right).$$

Proposition (1.3) follows from

**Lemma 4.** Let $(R, \mathcal{M})$ be a regular local ring, $I$ an ideal with a minimal resolution

$$0 \to R^{v-1} \overset{A}{\to} R^v \to I \to 0,$$

$f_i$ the maximal minors of $A$. Suppose $\{ f_i, f_j \}$ is a regular sequence in $R$, and let $J = (f_i, f_j): (f_1, \ldots, f_v)$ be the ideal linked to $I$ by $\{ f_i, f_j \}$. Then $J$ has a minimal resolution

$$0 \to R^{v-2} \overset{B}{\to} R^{v-1} \to J \to 0$$

where $B$ is the transpose of the matrix obtained by eliminating columns $i$ and $j$ from $A$.

**Proof.** Easy to check that

$$B \begin{pmatrix} \Gamma^{r-1}_{ij} \\ \vdots \\ \Gamma^1_{ij} \end{pmatrix} = 0.$$
Therefore

\[
\begin{pmatrix}
\Gamma_{ij}^{-1} \\
\vdots \\
\Gamma_{ij}^1
\end{pmatrix}
\]

\[0 \rightarrow R^{v-2} \overset{B}{\rightarrow} R^{v-1} \rightarrow J \rightarrow 0\]

is a complex. Rank \( A = v - 1 \Rightarrow \text{rank } B = v - 2 \), so the complex is exact on the left. It is exact on the right by Lemma 3. The grade of \( J \) is 2 by ([9], Prop. 1.3), so (1.6) applies to prove exactness in the middle. Since the entries of \( B \) lie in \( \mathcal{M} \), this exact sequence is a minimal resolution of \( J \).

\textit{Note added in proof.} An example with the properties of Example 2.3, constructed by different techniques, is included in a later version of [6].

\section*{References}


Received April 6, 1984.

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